2. Joint Expectation
Joint Expectation

Let $X$ and $Y$ be two random variables. The joint expectation is

$$E[XY] = \sum_{y} \sum_{x} xy p_{X,Y}(x, y)$$

if $X$ and $Y$ are discrete, or

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x, y) \, dx \, dy$$

if $X$ and $Y$ are continuous. Joint expectation is also called correlation.
Joint Expectation for Independent Variables

true when \( X, Y \) independent

\[ f(x, y) = f(x) f(y) \]

If \( X \) and \( Y \) are independent, then

\[ \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]. \]

Proof.

\[
\begin{align*}
\mathbb{E}[XY] &= \int \int xy f(x, y) \, dx \, dy \\
&= \int \int xy f(x) f(y) \, dx \, dy \\
&= \left( \int x f(x) \, dx \right) \left( \int y f(y) \, dy \right)
\end{align*}
\]

The converse is not true. That is, if \( \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \), then \( X \) and \( Y \) may not be independent.
Joint Moment

Let $X$ and $Y$ be two random variables. The joint moment is

$$
\mathbb{E}[X^k Y^\ell] = \sum_y \sum_x x^k y^\ell p_{X,Y}(x, y)
$$

(1)

if $X$ and $Y$ are discrete, or

$$
\mathbb{E}[X^k Y^\ell] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^\ell f_{X,Y}(x, y) \, dx \, dy
$$

(2)

if $X$ and $Y$ are continuous.
Let $X$ and $Y$ be two random variables. The **covariance** is

$$
\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)],
$$

where $\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$.

**if $X = Y$?**

$$
\text{Cov}(X, X) = \mathbb{E}[(X - \mu_X)^2] = \text{Var}(X)
$$

$$
\begin{bmatrix}
\text{Var}(X_1) & \text{Cov}(X_1, X_2) \\
\text{Cov}(X_2, X_1) & \text{Var}(X_2)
\end{bmatrix}
$$
Property of Covariance

Property 1:

\[ \text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \]

\[
\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] \\
= \mathbb{E}[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\
= \mathbb{E}[XY] - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y \\
= \mathbb{E}[XY] - \mathbb{E}(X)\mathbb{E}(Y).
\]
Property of Covariance

Independency $\implies$ uncorrelated.

**Property 2:**
If $X$ and $Y$ are independent, then $\text{Cov}(X, Y) = 0$.

\[
\implies E(xy) - E(x)E(y) = 0
\]

$\iff E(xy) = E(x)E(y)$.

$X$ and $Y$ indep

\[
\implies E(xy) = E(x)E(y).
\]

\[
\implies E(xy) - E(x)E(y) = 0
\]

$\text{Cov}(X, Y)$
Property of Covariance

Property 3:
If $\text{Cov}(X, Y) = 0$, then it is not always true that $X$ and $Y$ are independent.

Here is a counter example:
Consider a discrete random variable $Z$ with PMF

$$p_Z(z) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$ 

Let $X$ and $Y$ be

$$X = \cos \frac{\pi}{2} Z \quad \text{and} \quad Y = \sin \frac{\pi}{2} Z.$$ 

$$X = \{1, 0, -1, 0\} \quad \Rightarrow \quad \mathbb{E}(X) = 0$$

$$Y = \{0, 1, 0, -1\} \quad \Rightarrow \quad \mathbb{E}(Y) = 0$$
Show that $\text{Cov}(X, Y) = 0$.

and show that $X, Y$ are dependent.

$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y] = 0$.

$X$

\[
\begin{array}{c|ccc}
  & -1 & 0 & 1 \\
\hline
-1 & 0 & 4 & 0 \\
0 & 1/4 & 0 & 1/4 \\
1 & 0 & 1/4 & 0 \\
\end{array}
\]

$Y$

$x: 0, 1, -1$

$y: 0, 1, -1$.

$x = -1 \text{ and } y = -1$

$P(-) = 0$.

$Z = 0$, then $X = 1$, and $Y = 0$.

$Z = 1$, then $X = 0$, and $Y = 1$. 
\[ E(XY) = \sum \sum x\ y\ p(x, y) \]
\[ = (0)(-1)(\frac{1}{4}) + (1)(0)(\frac{1}{4}) + (-1)(0)(\frac{1}{4}) + (0)(1)(\frac{1}{4}) \]
\[ = 0. \]

\[ p(x, y) \neq p(x)\ p(y) \]

\[ p(x) = \left[ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right], \]
\[ p(y) = \left[ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right]. \]
Property of Covariance

(continue...)

- Show that $\mathbb{E}[X] = 0$, $\mathbb{E}[Y] = 0$.
- Hence show that $\text{Cov}(X, Y) = 0$. (Uncorrelated)
- Show that $p_{X,Y}(x, y) \neq p_X(x)p_Y(y)$. (Dependent)
Expectation and Variance of $X + Y$

For any $X$ and $Y$ (not necessarily independent),

a. $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

b. $\text{Var}[X + Y] = \text{Var}[X] + 2\text{Cov}(X, Y) + \text{Var}[Y]$.

$$\text{Var}(X + Y) = \mathbb{E}\left[\left((X+Y) - (\mu_X + \mu_Y)\right)^2\right]$$

If $X$, $Y$ independent,

Then

$$\text{Cov}(X, Y) = 0$$

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$
Correlation Coefficient

Let $X$ and $Y$ be two random variables. The correlation coefficient is

$$
\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}[X]\text{Var}[Y]}}
$$

(4)

The correlation coefficient $\rho$ has the properties that:

- $\rho = +1$ when $X = Y$ (fully correlated);
- $\rho = -1$ when $X = -Y$ (negatively correlated);
- $\rho = 0$ when $X$ and $Y$ are independent.
- $-1 \leq \rho \leq 1$.

If $\rho = 0$, then $X, Y$ are uncorrelated.

If $\rho = 0$, it does not mean $X, Y$ are independent.