4.4 Poisson Random Variable
Poisson Random Variable

Definition
Let \( X \) be a **Poisson** random variable. Then, the PMF of \( X \) is

\[
p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 1, 2, \ldots,
\]

where \( \lambda > 0 \) is the Poisson rate. We write

\( X \sim \text{Poisson}(\lambda) \)

to say that \( X \) is drawn from a Poisson distribution with a parameter \( \lambda \).

**Example.** Telephone arrivals. Photon arrivals. Passenger arrivals.
PMF of Poisson

![Graph showing PMF of Poisson distributions with λ = 1, λ = 20, and λ = 50.]
Example 1

Photon Arrivals.

- Let $x$ be the intensity of a pixel. Assume $x \in [0, 1]$.
- Let $\lambda \overset{\text{def}}{=} \alpha x$ be the photon flux, i.e., number of photons per unit time. $\alpha > 1$ is a gain factor.

Let $Y$ be a random variable representing the number of photons. Then, the probability of getting $k$ photons is

$$P[Y = k] = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{(\alpha x)^k e^{-\alpha x}}{k!}$$

$\alpha = 10$  $\alpha = 100$  $\alpha = 1000$
Small $\lambda$.  $x_i \in [0,1]$.

Poisson

Large $\lambda$.

$X_i \sim \text{Poisson}(\lambda x_i)$

$i = 1, 2, 3, \ldots, N$

$\lambda_i = \alpha x_i$

$\text{scale} = \frac{\lambda i}{x_i}$

$\lambda_i \sim \text{Poisson}(\lambda x_i)$
Example 2

Energy efficient escalators in airport.
  - These escalators have two modes: ON, or STAND-BY.
  - When no pedestrian for \( t_0 \) seconds, turn to STAND-BY.
  - On average people arrives at a rate of \( \lambda \) people per second.
  - How much energy is saved?

Partial Solution.
Let \( N \) be the number of pedestrians for \( t \) seconds.
\[
\text{R.V.} \quad \mathbb{P}[N = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}.
\]

Let \( T \) be the inter-arrival time.
\[
\mathbb{P}[T > t] = 
\mathbb{P}[	ext{no arrival in } t] = 
\mathbb{P}[N = 0] = e^{-\lambda t}.
\]
(Full answer in Ch. 4 when we introduce function of R.V.)
\[ P(T > t) = P(\text{inter-arrival time} > t) \]
\[ = P(\text{within} \ t \ \text{no one comes}) \]
\[ = P(N = 0) \]
\[ = (\lambda t)^0 e^{-\lambda t} \]
\[ \frac{0!}{0!} = e^{-\lambda t} \]
Moments of Poisson

Property

If $X \sim \text{Poisson}(\lambda)$, then

$$\mathbb{E}[X] = \lambda,$$

$$\mathbb{E}[X^2] = \lambda + \lambda^2,$$

$$\text{Var}[X] = \lambda.$$
Poisson Binomial Approximation

Theorem (Poisson Approximation to Binomial)

For small $p$ and large $n$, and let $\lambda \overset{\text{def}}{=} np$,

$$
\binom{n}{k} p^k (1 - p)^{n-k} \approx \frac{\lambda^k}{k!} e^{-\lambda}
$$
Example

Problem.
- Data arrival rate: $n = 10^9$ bits per second.
- Probability of having one error bit: $p = 10^{-9}$.
- In one second, how likely will we get $k = 5$ error bits?

Solution.
If you stick to binomial:
- Binomial: Flip coin $10^9$ times. Get 5 heads.
- $\binom{10^9}{5}(10^{-9})^5(1 - 10^{-9})^{10^9-5}$.
- Numerical problem!!

If we use Poisson approximation:
- $\lambda = np = (10^9)(10^{-9}) = 1$.
- $\frac{1^5}{5!} e^{-1}$.
- Much easier!
Proof (not required)

Let $\lambda = np$. Then,

\[
\binom{n}{k} p^k (1 - p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda} \cdot \frac{n!}{k!(n-k)!} \left( \frac{\lambda}{n} \right)^k \left( 1 - \frac{\lambda}{n} \right)^{n-k}
\]

\[
= \frac{\lambda^k}{k!} \frac{n(n-1) \cdots (n-k+1)}{n \cdot n \cdots n} \left( 1 - \frac{\lambda}{n} \right)^{n-k}
\]

\[
= \frac{\lambda^k}{k!} \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) \left( 1 - \frac{\lambda}{n} \right)^{-k} \left( 1 - \frac{\lambda}{n} \right)^n
\]

\[
\to 1 \text{ as } n \to \infty \quad \to 1 \text{ as } n \to \infty
\]

\[
= \frac{\lambda^k}{k!} \left( 1 - \frac{\lambda}{n} \right)^n.
\]
Proof (not required)

We claim that \((1 - \frac{\lambda}{n})^n \to e^{-\lambda}\). This can be proved by noting that

\[
\log(1 + x) \approx x, \quad x \ll 1.
\]

It then follows that \(\log \left(1 - \frac{\lambda}{n}\right) \approx -\frac{\lambda}{n}\). Hence, \((1 - \frac{\lambda}{n})^n \approx e^{-\lambda}\)

\[
l - \frac{\lambda}{n} \approx e^{-\frac{\lambda}{n}}
\]

\[
(1 - \frac{\lambda}{n})^n \approx \left(e^{-\frac{\lambda}{n}}\right)^n = e^{-\lambda}.
\]