3. Cumulative Distribution Function

PMF
\[ P(x) \]

\[ x \]

\[ \frac{d}{dx} \]

CDF
Cumulative Distribution Function

Definition

The **cumulative distribution function** (CDF) of a discrete random variable \( X \) is

\[
F_X(x) \overset{\text{def}}{=} \mathbb{P}[X \leq x] = \sum_{x' \leq x} p_X(x').
\]

**Interpretation:**
- CDF is the “integration” of PMF
- CDF is *well-defined* whereas PMF is not quite
- CDF works for both discrete and continuous random variables

**PMF**

\[
P_X(a) = \mathbb{P}(X = a)
\]

**CDF**

\[
F_X(a) = \mathbb{P}(X \leq a)
\]  

*cumulative sum*
Example. Consider a random variable $X$ with PMF

$$p_X(0) = \frac{1}{4}, \quad p_X(1) = \frac{1}{2}, \quad p_X(2) = \frac{1}{4}.$$ 

Find and sketch CDF.

$$F_X(0) = \mathbb{P}(X \leq 0) = \frac{1}{4},$$

$$F_X(1) = \mathbb{P}(X \leq 1) = \frac{1}{4} + \frac{1}{2},$$

$$F_X(2) = \mathbb{P}(X \leq 2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4},$$

$$F_X(0.5) = \mathbb{P}(X \leq 0.5) = \mathbb{P}(X=0) = \frac{1}{4}.$$
Properties of CDF

1. The CDF is a sequence of **increasing** numbers.

2. \( F_X(\infty) = 1 \Rightarrow P(X \leq \infty) \)

3. \( F_X(-\infty) = 0 \Rightarrow P(X \leq -\infty) \)

4. At positions where \( p_X(x) > 0 \), there is always a **jump**.
Properties of CDF

5. The height of each jump is $P(X = a)$.

6. The solid dot is always on the left hand side.
Generating Arbitrary Random Numbers from CDF

**Question:** How to generate random number from a PMF $p_X(k) = [0.1 \ 0.4 \ 0.2 \ 0.3]$?

**Procedure:**

1. Compute $F_X(k)$
2. Draw $U \sim \text{Uniform}(0, 1)$
3. Check which bin in $F_X(k)$ does $U$ fall into.
4. Common Discrete Random Variables
Common Discrete Random Variables

- Bernoulli Random Variable
  \[ p_X(1) = p, \quad p_X(0) = 1 - p. \]

- Binomial Random Variable
  \[ p_X(k) = \binom{n}{k} p^{n-k} (1 - p)^k \]

- Geometric Random Variable
  \[ p_X(k) = (1 - p) p^{k-1} \]

- Poisson Random Variable
  \[ p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!} \]
4.1 Bernoulli Random Variable
Bernoulli Random Variable

Definition
Let $X$ be a Bernoulli random variable. Then, the PMF of $X$ is

$$p_X(0) = 1 - p, \quad p_X(1) = p,$$

where $0 < p < 1$ is called the Bernoulli parameter. We write

$$X \sim \text{Bernoulli}(p)$$

to say that $X$ is drawn from a Bernoulli distribution with a parameter $p$.

Example. Coin flip.

$$\mu = \mathbb{E}[X] = (0)(1-p) + (1)(p) = p$$

$$\text{Var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= p - p^2 = p(1-p).$$
Bernoulli Example

**Randomized Algorithm.** Consider a large linear system:

\[
\begin{bmatrix}
y_1 \\
\vdots \\
y_N
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1N} \\
\vdots & \vdots & \ddots & \vdots \\
a_{M1} & a_{M2} & \ldots & a_{MN}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_N
\end{bmatrix}
\]

where \( M \) and \( N \) are very large.

**Example.**
- Large network analysis (million-node network)
- Large-scale inverse problem (giga-pixel deconvolution)
- Large-scale routing (air traffic control)
- Large-scale decomposition (genome analysis)
Bernoulli Example

**Brute-force computation:**

\[ y_i = \sum_{j=1}^{N} a_{ij} x_j. \]

**Randomized computation:** Let \( l_j \sim \text{Bernoulli}(p_j) \). Then,

\[ \hat{y}_i = \sum_{j=1}^{N} a_{ij} x_j l_j / p_j, \]

\[ \mathbb{E}[\hat{y}_i] = y_i. \]

We can prove that with extremely high probability, the deviation between \( \hat{y}_i \) and \( y_i \) is very small: As \( N \to \infty \),

\[ \mathbb{P}[|\hat{y}_i - y_i| > \epsilon] \to 0. \]

**Bernoulli Example**

**Erdos-Renyi Graph**
- A very famous (and simple) model for large networks.
- Is used nowadays to study network structures, e.g. Facebook, Google.

Graph

Matrix

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