Example 2: Communication Channel

Consider a communication channel shown below. The probability of sending a 1 is $p$ and the probability of sending a 0 is $1 - p$. Given that 1 is sent, the probability of receiving 1 is $1 - \eta$. Given that 0 is sent, the probability of receiving 0 is $1 - \varepsilon$.

- Find $P[\text{Receive 1}]$
- Find $P[\text{Send 1}]$
- Find $P[\text{Receive 1} \mid \text{Send 1}]$
- Find $P[\text{Send 1} \mid \text{Receive 1}]$
\[ P(S = 1) = p \]
\[ P(S = 0) = 1 - p \]
\[ P(R = 1) = P(R = 1 \cap S = 1) + P(R = 1 \cap S = 0) \]
\[ = \frac{P(R = 1 | S = 1) P(S = 1)}{P} + \frac{P(S = R = 1 | S = 0)}{P} \]
\[ = P(1 - \eta) + \varepsilon (1 - p) \]
\[ P(S = 1 \mid R = 1) = \frac{P(R = 1 \mid S = 1) P(S = 1)}{P(R = 1)} \]

\[ = \frac{(1 - \gamma) P}{(1 - \gamma) P + \varepsilon (1 - P)} \]
Properties of Conditional Probability

Theorem

Let $P[B] > 0$. The conditional probability $P[A | B]$ satisfies Axiom I to Axiom III.

Proof.

- Axiom I:
  $$P(A | B) \geq 0$$

- Axiom II:
  $$P(\Omega | B) = 1$$

- Axiom III:
  $$P(A_1 \cup A_2 | B) = P(A_1 | B) + P(A_2 | B)$$
  $$A_1 \cap A_2 = \emptyset.$$
Independence

\[ P(A \cap B) = \frac{P(A|B) P(B)}{P(B|A) P(A)} = P(A) P(B) \]

Definition

Two events \( A \) and \( B \) are statistically **independent** if

\[ P(A \cap B) = P(A) P(B) \]

Disjoint VS Independent.

Disjoint: \( A \cap B = \emptyset \)

Independent: \( P(A \cap B) = P(A) P(B) \)

Disjoint \( \Rightarrow P(A \cap B) = 0 \) \( \neq P(A \cap B) = P(A) P(B) \)

Independent \( \Rightarrow P(A \cap B) = P(A) P(B) \neq A \cap B = \emptyset \).
Examples of Independence

Example 1. Throw a dice twice. Let

\[ A = \{\text{1st dice is 3}\} \quad \text{and} \quad B = \{\text{2nd dice is 4}\}. \]

Are \( A \) and \( B \) independent?

\[ P(A) = \frac{1}{6}, \quad P(B) = \frac{1}{6}, \]

\[ P(A \cap B) = \frac{1}{36}. \]

\[ P(A \cap B) = P(A)P(B). \quad \checkmark \quad \text{independent} \]
Examples of Independence

Example 2. Throw a dice twice. Let

\[ A = \{\text{1st dice is 1}\} \quad \text{and} \quad B = \{\text{sum is 7}\}. \]

Are \( A \) and \( B \) independent?

- Think about \( \mathbb{P}[A|B] \).
- If you know the sum is 7, then the pair has to be \((1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\).
- The chance of getting first dice = 1 is still 1/6. It has been not been changed by \( B \).
- So independent.
- How about you change \( B = \{\text{sum is 8}\} \)?
\[ P(A) = \frac{1}{6} \]
\[ P(B) = \frac{6}{36} = \frac{1}{6} \]
\[ P(A \cap B) = \frac{1}{36} \]
\[ P(A \cap B) = P(A) \cdot P(B) \]

Independent.
Examples of Independence

Example 3. Throw a dice twice. Let

$$A = \{\text{1st dice is 2}\} \quad \text{and} \quad B = \{\text{sum is 8}\}.$$ 

Are $A$ and $B$ independent?

- Think about $\mathbb{P}[A|B]$.
- If you know the sum is 8, then the pair has to be $(2,6), (3,5), (4,4), (5,3), (6,2)$.
- The chance of getting first dice $= 2$ is no longer $1/6$. It has been changed by $B$.
- So dependent.
Examples of Independence

Example 4. Throw a dice twice. Let

\[ A = \{ \text{max is 2} \} \quad \text{and} \quad B = \{ \text{min is 2} \}. \]

Are $A$ and $B$ independent?
\[ P(A) = \frac{3}{36} \]
\[ P(B) = \frac{9}{36} \]
\[ P(A \cap B) = \frac{1}{36} \]

\[ P(A \cap B) \neq P(A) \cdot P(B) \]

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>V</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>V</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>V</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td>V</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

dependent.
Independence Via Conditional Probability

- Recall that $P[A \mid B] = \frac{P[A \cap B]}{P[B]}$.
- If $A$ and $B$ are independent, then $P[A \cap B] = P[A]P[B]$.

Therefore,

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]P[B]}{P[B]} = P[A].$$

Interpretation.

Having $B$ has no influence to probability of getting $A$. 
$$\iiint_{A \cap B} dV = (\int_{x_1}) (\int_{x_2}) (\int_{x_3}) (\int_{x_4}) = \frac{\text{IP}(A \cap B)}{\text{IP}(B)} = \frac{\text{IP}(A \cap B)}{\text{IP}(B)} = \text{IP}(A).$$
Prisoner’s Dilemma

- Three Prisoners: A, B, C. The King decides to release 2 and kill 1.
- You were A.
- Your chance of release is 2/3.
- Suppose you know the guard well. You can ask him about which of B or C will be released.
- But if you find out B (or C) is released, your chance becomes 1/2.
- How come!!