2. Probability Model
What is Probability?

- It is a **number**.
- Always between **0** and **1**.
- Always the probability of an **event**.

**Example.** The probability of getting a Head when tossing a coin:

\[ P(\text{"H"}) = \frac{\#}{2} = \frac{1}{2} \]
Three Elements of a Probability Model

1. Sample Space
2. Event
3. Probability Law
Sample Space

Definition (Sample Space)

A sample space $\Omega$ is the collection of all possible outcomes.

We denote $\omega$ as an element in $\Omega$.

Example.

- Coin flip:
  $\Omega = \{ H, T \}$

- Throw a dice:
  $\Omega = \{ 1, \ldots, 6 \}$

- Waiting time for a bus in West Lafayette:
  $\Omega = \{ t \mid 0 \leq t \leq 30 \text{min} \}$
Event

Definition (Event)
An event $F$ is a subset in the sample space $\Omega$.

Outcome VS Event:
- Single element
- Collection of outcomes

Example. Throw a dice. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- $F_1 = \{\text{even numbers}\} = \{2, 4, 6\}$. 
- $F_2 = \{\text{less than 3}\} = \{1, 2\}$.

Example. Wait a bus. Let $\Omega = \{0 \leq t \leq 30\}$.
- $F_1 = \{0 \leq t < 10\}$
- $F_2 = \{0 \leq t < 5\} \cup \{20 < t \leq 30\}$.
Event Space

Definition (Event Space)

The collection of all possible events is called the Event Space or \( \sigma \)-field (not needed) denoted as \( \mathcal{F} \). \( \mathcal{F} \) satisfies the following two properties:

- If \( F \in \mathcal{F} \), then \( F^c \in \mathcal{F} \).
  
  \[ F = \{1, 2\} \quad F^c = \{3, 4, 5, 6\} \]

- If \( F_1, F_2, \ldots \in \mathcal{F} \), then \( F_i \cap F_j \in \mathcal{F} \) and \( F_i \cup F_j \in \mathcal{F} \).
  
  \[ F_1 = \{1, 2\} \quad F_1 \cap F_2 = \{2\} \]
  \[ F_2 = \{2, 3\} \quad F_1 \cup F_2 = \{1, 2, 3\} \]

Example. \( \Omega = \{H, T\} \), the event space is \( \{\emptyset, H, T, \Omega\} \).
Probability Law

Definition

A **probability law** is a function \( P : \mathcal{F} \to [0, 1] \) that maps an event \( A \) to a real number in \([0, 1]\). The function must satisfy three axioms known as **Probability Axioms**.

I. Non-negativity:

\[
P(A) \geq 0, \quad \text{for any } A \in \mathcal{F}
\]

II. Normalization:

\[
P(\Omega) = 1.
\]
III. Additivity:

For any disjoint subsets \( \{A_1, A_2, \ldots \} \), it holds that

\[
P \left[ \bigcup_{n=1}^{\infty} A_n \right] = \sum_{n=1}^{\infty} P[A_n].
\]

\[
P(A_1 \cup A_2) = P(A_1) + P(A_2).
\]
Properties of Probability

1. \( P(A^c) = 1 - P(A). \)
   \[ A = \{1, 2, 3\} \]
   \[ A^c = \{4, 5, 6\} \]

2. For any \( A \subseteq \Omega, P[A] \leq 1. \)

3. \( P(\emptyset) = 0. \)
Properties of Probability

- For any $A$ and $B$,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B].$$