2. Integration
Even and Odd Functions

Definition
A function $f : \mathbb{R} \to \mathbb{R}$ is **even** if for any $x \in \mathbb{R}$,

$$f(x) = f(-x),$$

and $f$ is **odd** if

$$f(x) = -f(-x),$$
Integration of Odd and Even Function

**Example.** Evaluate the integral

\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx
\]

\[= 0.\]

**Example.** Evaluate the integral

\[
\int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} \, dx
\]

\[= 2 \int_{0}^{\infty} \frac{1}{2} e^{-x} \, dx
\]

\[= \int_{0}^{\infty} e^{-x} \, dx = \left[ -e^{-x} \right]_{0}^{\infty}
\]

\[= [-0] - [-1] = 1.\]
Three Ways of Doing Integration

There are literally only two three ways of doing integration:

- Substitution (freshman calculus)
- By-part (freshman calculus)
- Unit probability (new!)

**Example.** Evaluate the integral

\[
\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} \, dx
\]

Sub. let \( u = x - \mu \).

\[
= \int_{-\infty}^{\infty} (u+\mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du
\]

\[
= \int_{-\infty}^{\infty} u \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du.
\]

\[
= 0 + \mu \frac{1}{\sqrt{2\pi}} \left[ e^{-\frac{u^2}{2}} \right]_{-\infty}^{\infty} = \mu.
\]
Theorem (Fundamental Theorem of Calculus)

Let \( f : [a, b] \to \mathbb{R} \) be a continuous function defined on a closed interval \([a, b]\). Then,

\[
f(x) = \frac{d}{dx} \int_a^x f(t) dt,
\]

for any \( x \in (a, b) \).
Example. Evaluate the integral

\[
\frac{d}{dx} \int_0^{x-\mu} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{t^2}{2\sigma^2} \right\} dt.
\]

Solution. Let \( y = x - \mu \). Then by using the fundamental theorem of calculus, we can show that

\[
\frac{d}{dx} \int_0^{x-\mu} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{t^2}{2\sigma^2} \right\} dt
\]

\[
= \frac{dy}{dx} \cdot \frac{d}{dy} \int_0^{y} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{t^2}{2\sigma^2} \right\} dt
\]

\[
= \frac{d(x - \mu)}{dx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{y^2}{2\sigma^2} \right\}
\]

\[
= \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}.
\]
3. Linear Algebra
Basic Notation

- Vector: \( \mathbf{x} \in \mathbb{R}^n \)
- Matrix: \( \mathbf{A} \in \mathbb{R}^{m \times n} \); Entries are \( a_{ij} \) or \( [\mathbf{A}]_{ij} \).
- Transpose:

\[
\mathbf{A} = \begin{bmatrix}
\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \\
\end{bmatrix}, \quad \text{and} \quad \mathbf{A}^T = \begin{bmatrix}
\mathbf{a}_1^T & \mathbf{a}_2^T & \ldots & \mathbf{a}_n^T \\
\end{bmatrix}.
\]

- Column: \( \mathbf{a}_i \) is the \( i \)-th column of \( \mathbf{A} \)
- Identity matrix \( \mathbf{I} \)
- All-one vector \( \mathbf{1} \) and all-zero vector \( \mathbf{0} \)
- Standard basis \( \mathbf{e}_i \).

\[
4 \times 2 \text{ matrix.}
\]

\[
\mathbf{A} = \begin{bmatrix}
1 & 2 \\
3 & 6 \\
4 & 7 \\
5 & 8 \\
\end{bmatrix} \quad \Rightarrow \quad \mathbf{A}^T = \begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
\end{bmatrix}
\]

\( \Rightarrow \) \( \mathbf{n.p., transpose (A)} \)
Inner Product

Definition

Let \( x = [x_1, x_2, \ldots, x_N]^T \) and \( y = [y_1, y_2, \ldots, y_N]^T \) be two vectors. The inner product \( x^T y \) is

\[
x^T y = x_1 y_1 + x_2 y_2 + \ldots + x_N y_N
= \sum_{n=1}^{N} x_n y_n
\]

Example. Let \( x = [x_1, x_2]^T \). The inner product \( x^T x = \)

\[
\begin{bmatrix} x_1 & x_2 \\ x_2 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2
\]
Weighted Inner Product

Example.

Let \( x = [x_1, x_2]^T \), \( \mu = [\mu_1, \mu_2] \), and \( C = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \). The product \((x - \mu)^T C (x - \mu)\) is

\[
\begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} = a(x_1 - \mu_1)^2 + b(x_2 - \mu_2)^2
\]

\( (\mu_1, \mu_2) \).
The $\ell_2$-norm

Also called the **Euclidean norm**:

$X'X = \| x \|_2^2$

**Definition**

$$\| x \|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$  \hfill (11)

- The set $\Omega = \{ x \mid \| x \|_2 \leq r \}$ defines a circle:

  $$\Omega = \{ x \mid \| x \|_2 \leq r \} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq r^2\}.$$

- $f(x) = \| x \|_2$ is not the same as $f(x) = \| x \|_2^2$.

- Triangle inequality holds:

  $$\| x + y \|_2 \leq \| x \|_2 + \| y \|_2.$$
Cauchy-Schwarz Inequality

Theorem

Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$. Then,

$$|\mathbf{x}^T \mathbf{y}| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2,$$

where the equality holds if and only if $\mathbf{x} = \alpha \mathbf{y}$ for some scalar $\alpha$.

- $\mathbf{x}^T \mathbf{y}/(\|\mathbf{x}\|_2 \|\mathbf{y}\|_2)$ defines the cosine angle between the two vectors $\mathbf{x}$ and $\mathbf{y}$.
- Cosine is always less than 1. So is $\mathbf{x}^T \mathbf{y}/(\|\mathbf{x}\|_2 \|\mathbf{y}\|_2)$.
- The equality holds if and only if the two vectors are parallel.
Determinant and Inverse

Definition

Determinant \( \Sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), the determinant of \( \Sigma \) is

\[
|\Sigma| = ad - bc
\]

Definition (Inverse)

Let \( \Sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), the inverse of \( \Sigma \) is

\[
\Sigma^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}
\]
Visualizing a 2D Gaussian

Definition
A $d$-dimensional Gaussian has a distribution

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right\}.$$