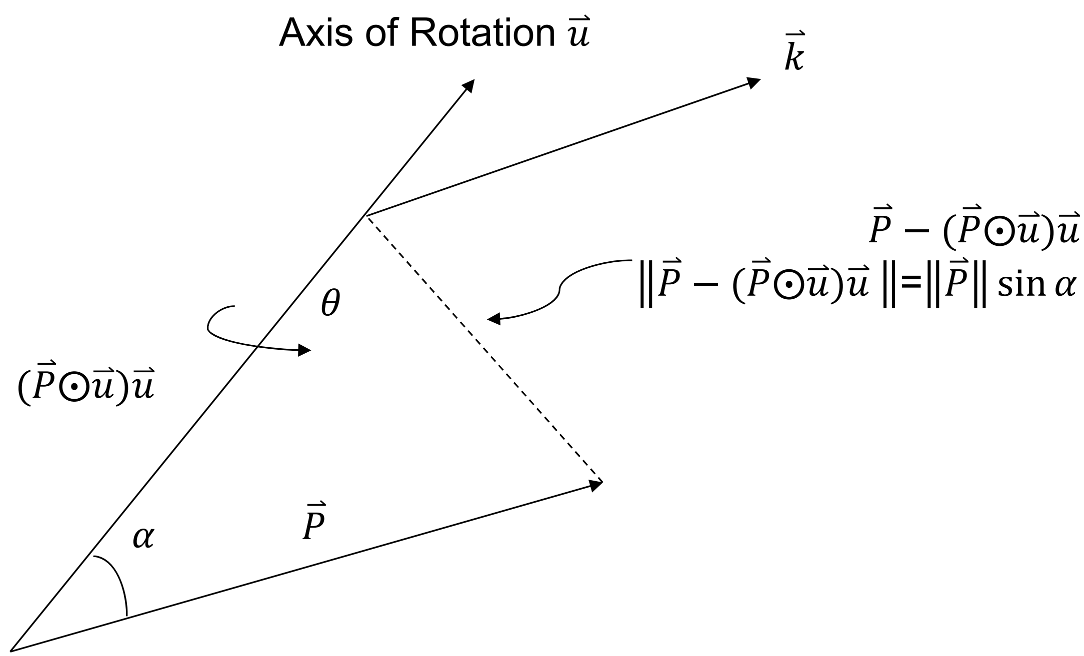
Rodriguez Approach for Representing Rotation

Chapter 7 – A2

Rodriguez Approach for Representing Rotation



Rodriguez Approach for Representing Rotation

$\vec{k} = \vec{u} \times \left(\vec{P} - (\vec{P} \odot \vec{u})\vec{u}\right) = \vec{u} \times \vec{P}$

$\|\vec{k}\| = \|\vec{P}\| \sin\alpha$

Then $\|\vec{k}\|$ & $\|\vec{P} - (\vec{P} \odot \vec{u})\vec{u}\|$ are equivalent in size ($\|\vec{P}\| \sin\alpha$)

Thus

$\overrightarrow{P'}$ = rotated vector around $\vec{u}$ with angle an angle θ

$\overrightarrow{P'} = (\vec{P} \odot \vec{u})\vec{u} + \cos\theta \{\vec{P} - (\vec{P}.\vec{u})\vec{u}\} + \sin\theta \ \vec{u} \times \vec{P} = \ (1 - \cos\theta)(\vec{P} \odot \vec{u})\vec{u} + \cos\theta \, \vec{P} + \sin\theta \, \hat{U}\vec{P}$

3 × 3 matrix

$$\hat{U} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

Rodriguez Approach for Representing Rotation

- $\overrightarrow{P'} = (1 - \cos\theta)(\vec{P} \odot \vec{u})\vec{u} + \cos\theta\, \vec{P} + \sin\theta\, \hat{U}\vec{P}$
- Now, let's focus on $(\vec{P} \odot \vec{u})\vec{u}$
- First, let's consider $\vec{u} \times (\vec{u} \times \vec{P}) = \hat{U}\hat{U}\vec{P} = \hat{U}^2\vec{P}$
- $\vec{u} \times (\vec{u} \times \vec{P}) = \vec{u}(\vec{u} \odot \vec{P}) - \vec{P}(\vec{u} \odot \vec{u})$
- Thus $\vec{u} \times (\vec{u} \times \vec{P}) = \hat{U}^2\vec{P} = (\vec{u} \odot \vec{P})\vec{u} - \vec{P} = (\vec{P} \odot \vec{u})\vec{u} - \vec{P}$
- $(\vec{P} \odot \vec{u})\vec{u} = \hat{U}^2\vec{P} + \vec{P}$
- Finally: $\overrightarrow{P'} = (1 - \cos\theta)\hat{U}^2\vec{P} + \vec{P} + \sin\theta\, \hat{U}\vec{P}$

Rodriguez Approach for Representing Rotation

- In other words:
- $\overrightarrow{P'} = \left[I_3 + \sin\theta\,\widehat{U} + (1-\cos\theta)\widehat{U}^2\right]\vec{P}$
- Therefore, the rotation matrix $R = I_3 + \sin\theta\,\widehat{U} + (1-\cos\theta)\widehat{U}^2$
- Claim: $R = e^{\widehat{U}\theta}$
- In conclusion, Rodriguez Rotation Formula
- $\overrightarrow{P'} = \left[I_3 + \sin\theta\,\widehat{U} + (1-\cos\theta)\widehat{U}^2\right]\vec{P}$

Rodriguez Approach for Representing Rotation

$$\vec{U} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$ unit vector: $u_x^2 + u_y^2 + u_z^2 = 1$

$$\hat{U} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

$$\hat{U}^2 = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}\begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} = \begin{bmatrix} -u_z^2 - u_y^2 & u_x u_y & u_x u_z \\ u_x u_y & -u_x^2 - u_z^2 & u_y u_z \\ u_x u_z & u_y u_z & -u_x^2 - u_y^2 \end{bmatrix}$$

$$= \begin{bmatrix} u_x^2 - 1 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 - 1 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 - 1 \end{bmatrix}$$

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Rodriguez Approach for Representing Rotation

$$\hat{U}^3 = \begin{bmatrix} u_x^2 - 1 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 - 1 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 - 1 \end{bmatrix} \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_x u_y u_z - u_x u_y u_z & -u_x^2 u_z + u_z + u_x^2 u_z & u_x^2 u_y - u_y - u_x^2 u_y \\ u_y^2 u_z - u_z - u_y^2 u_z & -u_x u_y u_z + u_x u_y u_z & u_x u_y^2 - u_x u_y^2 + u_x \\ u_y u_z^2 - u_y u_z^2 + u_y & -u_x u_z^2 + u_x u_z^2 - u_x & u_x u_y u_z - u_x u_y u_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & u_z & -u_y \\ -u_z & 0 & u_x \\ u_y & -u_x & 0 \end{bmatrix} = -\hat{U}$$

$$\hat{U}^4 = -\hat{U}^2 \quad \hat{U}^5 = \hat{U} \quad \hat{U}^6 = \hat{U}^2 \cdots \qquad \begin{matrix} \hat{U} & \hat{U}^2 & -\hat{U} & -\hat{U}^2 & \hat{U} & \hat{U}^2 & -\hat{U} & -\hat{U}^2 \\ \updownarrow, & \updownarrow, & \updownarrow, & \updownarrow, & \updownarrow, & \updownarrow, & \updownarrow, & \updownarrow \\ \hat{U}^1 & \hat{U}^2 & \hat{U}^3 & \hat{U}^4 & \hat{U}^5 & \hat{U}^6 & \hat{U}^7 & \hat{U}^8 \end{matrix}$$

Rodriguez Approach for Representing Rotation

Now, let's evaluate $e^{\widehat{U}\theta}$

$$e^{\widehat{U}\theta} = I_3 + \frac{\theta\widehat{U}}{1!} + \frac{\theta^2\widehat{U}^2}{2!} + \frac{\theta^3\widehat{U}^3}{3!} + \frac{\theta^4\widehat{U}^4}{4!} + \frac{\theta^5\widehat{U}^5}{5!} + \frac{\theta^6\widehat{U}^6}{6!} + \cdots$$

$$= I_3 + \theta\widehat{U} + \frac{\theta^2}{2!}\widehat{U}^2 - \frac{\theta^3}{3!}\widehat{U} - \frac{\theta^4}{4!}\widehat{U}^2 + \frac{\theta^5}{5!}\widehat{U} + \frac{\theta^6}{6!}\widehat{U}^2 - \frac{\theta^7}{7!}\widehat{U} + \cdots$$

$$= I_3 + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)\widehat{U} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \frac{\theta^8}{8!} + \cdots\right)\widehat{U}^2$$

$$= I_3 + \sin\theta\,\widehat{U} + (1 - \cos\theta)\widehat{U}^2$$ we need to prove that

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According to Taylor Series expansion

$$sin\,\theta = sin(0) + cos(0)\theta - sin(0)\frac{\theta^2}{2!} - cos(0)\frac{\theta^3}{3!} + sin(0)\frac{\theta^4}{4!} + cos(0)\frac{\theta^5}{5!}$$

$$- sin(0)\frac{\theta^6}{6!} - cos(0)\frac{\theta^7}{7!} + \cdots$$

$$= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}$$

$$cos\,\theta = cos(0) - sin(0)\,\theta - cos(0)\frac{\theta^2}{2!} + sin(0)\frac{\theta^3}{3!} + cos(0)\frac{\theta^4}{4!} - sin(0)\frac{\theta^5}{5!}$$

$$- cos(0)\frac{\theta^6}{6!} + sin(0)\frac{\theta^7}{7!} + cos(0)\frac{\theta^8}{8!}\cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!}$$

Rodriguez Approach for Representing Rotation

So, finally

$$R = I_3 + \sin\theta\,\widehat{U} + (1-\cos\theta)\widehat{U}^2 = e^{\widehat{U}\theta}$$

Rodriguez Rotation Formula

$$\overrightarrow{P'} = \left[I_3 + \sin\theta\,\widehat{U} + (1-\cos\theta)\widehat{U}^2\right]\vec{P}$$

where $\widehat{U} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$ & $\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ is unit direction vector