Projective Transformation

Chapter 7-A3

Projective Transformation

- Projective transformation is a plane to plane perspective transformation.
- The projective transformation involves eight parameters.
- The next few slides will establish the parametric relationship between the projective transformation parameters and the IOPs of the implemented camera and the EOPs of the involved image.

Collinearity Equations – Projective Trans.

Collinearity Equations $$\begin{bmatrix} x - x_p \\ y - y_p \\ -c \end{bmatrix} = \lambda R_m^c \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -x_p \\ 0 & 1 & -y_p \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ 0 & 0 & Z - Z_o \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Object Plane Equation: $Z = AX + BY + C$

$$\begin{bmatrix} 1 & 0 & -x_p \\ 0 & 1 & -y_p \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ 0 & 0 & AX + BY + C - Z_o \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Collinearity Equations – Projective Trans.

$$\begin{bmatrix} 1 & 0 & -x_p \\ 0 & 1 & -y_p \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ 0 & 0 & AX + BY + C - Z_o \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -x_p \\ 0 & 1 & -y_p \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ A & B & C - Z_o \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & -{}^{x_p}/_{c} \\ 0 & 1 & -{}^{y_p}/_{c} \\ 0 & 0 & -{}^{1}/_{c} \end{bmatrix} R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ A & B & C - Z_o \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Collinearity Equations – Projective Trans.

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & -{}^{x_p}/_{c} \\ 0 & 1 & -{}^{y_p}/_{c} \\ 0 & 0 & -{}^{1}/_{c} \end{bmatrix} R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ A & B & C - Z_o \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda C_{33} \begin{bmatrix} {}^{C_{11}}/_{C_{33}} & {}^{C_{12}}/_{C_{33}} & {}^{C_{13}}/_{C_{33}} \\ {}^{C_{21}}/_{C_{33}} & {}^{C_{22}}/_{C_{33}} & {}^{C_{23}}/_{C_{33}} \\ {}^{C_{31}}/_{C_{33}} & {}^{C_{32}}/_{C_{33}} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Collinearity Equations – Projective Trans.

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda C_{33} \begin{bmatrix} C_{11}/C_{33} & C_{12}/C_{33} & C_{13}/C_{33} \\ C_{21}/C_{33} & C_{22}/C_{33} & C_{23}/C_{33} \\ C_{31}/C_{33} & C_{32}/C_{33} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda C_{33} \begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \lambda` \begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$x = \frac{C_1 X + C_2 Y + C_3}{C_7 X + C_8 Y + 1}$$

$$y = \frac{C_4 X + C_5 Y + C_6}{C_7 X + C_8 Y + 1}$$

Collinearity Equations – Projective Trans.

$$\begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & 1 \end{bmatrix} = \gamma \begin{bmatrix} 1 & 0 & -{x_p}/{c} \\ 0 & 1 & -{y_p}/{c} \\ 0 & 0 & -{1}/{c} \end{bmatrix} R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ A & B & C - Z_o \end{bmatrix}$$

- Can we derive some of the camera parameters & EOP from the projective transformation parameters?
  - Yes
- For example (just to simplify the derivation), we can assume the following:
  - The principal point coordinates (x_p, y_p) are (0,0), and
  - The object plane equation is $Z = 0$.

Collinearity Equations – Projective Trans.

- Given the above assumptions, the projective transformation parameters are expressed as follows:

$$\begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & 1 \end{bmatrix} = \gamma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -{}^{1}/_{c} \end{bmatrix} R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ 0 & 0 & -Z_o \end{bmatrix}$$

$$\begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & 1 \end{bmatrix} = \gamma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -{}^{1}/_{c} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ 0 & 0 & -Z_o \end{bmatrix}$$

$$\begin{bmatrix} C_1 & C_2 \\ C_4 & C_5 \\ C_7 & C_8 \end{bmatrix} = \gamma \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ -{}^{r_{31}}/_{c} & -{}^{r_{32}}/_{c} & -{}^{r_{33}}/_{c} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Collinearity Equations – Projective Trans.

$$\begin{bmatrix} C_1 & C_2 \\ C_4 & C_5 \\ C_7 & C_8 \end{bmatrix} = \gamma \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ -{}^{r_{31}}/_{c} & -{}^{r_{32}}/_{c} & -{}^{r_{33}}/_{c} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} C_1 & C_2 \\ C_4 & C_5 \\ C_7 & C_8 \end{bmatrix} = \gamma \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \\ -{}^{r_{31}}/_{c} & -{}^{r_{32}}/_{c} \end{bmatrix}$$

- The first and second columns of R_m^c are orthogonal to each other:

$$C_1C_2 + C_4C_5 + c^2C_7C_8 = r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} = 0$$

$$\boldsymbol{c = \sqrt{-{}^{(C_1C_2 + C_4C_5)}/_{C_7C_8}}}$$

Collinearity Equations – Projective Trans.

$$\begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & 1 \end{bmatrix} = \gamma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -{}^{1}/_{c} \end{bmatrix} R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ 0 & 0 & -Z_o \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & 1 \end{bmatrix} = \gamma R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ 0 & 0 & -Z_o \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & 1 \end{bmatrix}$$

$$\mathrm{B} = \gamma R_m^c \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ 0 & 0 & -Z_o \end{bmatrix}$$

Collinearity Equations – Projective Trans.

$$A^T A = B^T B$$

$$A^T A = \begin{bmatrix} C_1 & C_4 & C_7 \\ C_2 & C_5 & C_8 \\ C_3 & C_6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c^2 \end{bmatrix} \begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & 1 \end{bmatrix}$$

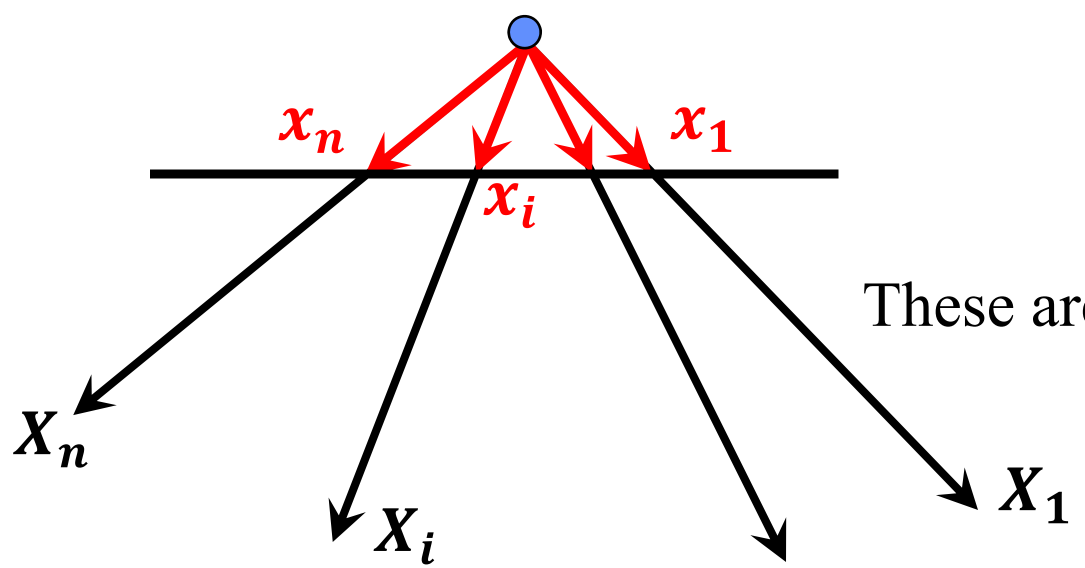
$$B^T B = \gamma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -X_o & -Y_o & -Z_o \end{bmatrix} \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ 0 & 0 & -Z_o \end{bmatrix}$$

$$A^T A(1,1) = \gamma^2$$

$$X_0 = -\frac{(A^T A)_{13}}{(A^T A)_{11}}, Y_0 = -\frac{(A^T A)_{23}}{(A^T A)_{11}}, Z_0 = \sqrt{\frac{(A^T A)_{33}}{(A^T A)_{11}} - X_0^2 - Y_0^2}$$

Collinearity Equations – Projective Trans.

- The rotation matric R_m^c can be derived through quaternions.
- This approach is based on conjugate points in the image and object space.



These are unit vectors.

Collinearity Equations – Projective Trans.

$$x_i \text{ is the unit vector corresponding to } \begin{bmatrix} x_i - x_p - dist_x \\ y_i - y_p - dist_y \\ -c \end{bmatrix}$$

$$X_i \text{ is the unit vector corresponding to } \begin{bmatrix} X_i - X_o \\ Y_i - Y_o \\ Z_i - Z_o \end{bmatrix}$$

- These vectors are assumed to be known.

Quaternion-Based Derivation of R_c^m

$$\boldsymbol{X}_i = R_c^m \boldsymbol{x}_i + e_i$$

- We need to derive R_c^m that minimizes the Sum of Squared Errors.

$$\min_{R_c^m} \sum_{i=1}^{n} e_i^T e_i = \min_{R_c^m} \sum_{i=1}^{n} (\boldsymbol{X}_i - R_c^m \boldsymbol{x}_i)^T (\boldsymbol{X}_i - R_c^m \boldsymbol{x}_i)$$

$$= \min_{R_c^m} \sum_{i=1}^{n} \boldsymbol{X}_i^T \boldsymbol{X}_i + \boldsymbol{x}_i^T \boldsymbol{x}_i - 2\boldsymbol{x}_i^T (R_c^m)^T \boldsymbol{X}_i$$

$$\Updownarrow$$

$$\max_{R_c^m} \sum_{i=1}^{n} \boldsymbol{x}_i^T (R_c^m)^T \boldsymbol{X}_i = \max_{R_c^m} \sum_{i=1}^{n} R_c^m \boldsymbol{x}_i . \boldsymbol{X}_i$$

Quaternion-Based Derivation of R_c^m

$$\max_{R_c^m} \sum_{i=1}^{n} \boldsymbol{x}_i^T (R_c^m)^T \boldsymbol{X}_i = \max_{R_c^m} \sum_{i=1}^{n} R_c^m \boldsymbol{x}_i . \boldsymbol{X}_i$$

$$\Updownarrow$$

$$\max_{\dot{q}} \sum_{i=1}^{n} (\dot{q}\dot{\boldsymbol{x}}_i \dot{q}^*) . \dot{\boldsymbol{X}}_i = \max_{\dot{q}} \sum_{i=1}^{n} (\dot{q}\dot{\boldsymbol{x}}_i) . (\dot{\boldsymbol{X}}_i \dot{q})$$

$$= \max_{\dot{q}} \sum_{i=1}^{n} (\bar{C}(\dot{\boldsymbol{x}}_i)\dot{q}) . (C(\dot{\boldsymbol{X}}_i)\dot{q}))$$

$$= \max_{\dot{q}} \sum_{i=1}^{n} \dot{q}^T \bar{C}(\dot{\boldsymbol{x}}_i)^T C(\dot{\boldsymbol{X}}_i)\dot{q}$$

$$= \max_{\dot{q}} \dot{q}^T \left(\sum_{i=1}^{n} \bar{C}(\dot{\boldsymbol{x}}_i)^T C(\dot{\boldsymbol{X}}_i) \right) \dot{q} = \max_{\dot{q}} \ \dot{q}^T \boldsymbol{S} \dot{q}$$

Quaternion-Based Derivation of R_c^m

$$\boldsymbol{S_i} = \bar{C}(\dot{x}_i)^T C(\dot{X}_i)$$

$$\boldsymbol{S_i} = \begin{bmatrix} x_{o_i} & -x_{x_i} & -x_{y_i} & -x_{z_i} \\ x_{x_i} & x_{o_i} & x_{z_i} & -x_{y_i} \\ x_{y_i} & -x_{z_i} & x_{o_i} & x_{x_i} \\ x_{z_i} & x_{y_i} & -x_{x_i} & x_{o_i} \end{bmatrix}^T \begin{bmatrix} X_{o_i} & -X_{x_i} & -X_{y_i} & -X_{z_i} \\ X_{x_i} & X_{o_i} & -X_{z_i} & X_{y_i} \\ X_{y_i} & X_{z_i} & X_{o_i} & -X_{x_i} \\ X_{z_i} & -X_{y_i} & X_{x_i} & X_{o_i} \end{bmatrix}$$

$$\boldsymbol{S_i(1,1) = x_{o_i}X_{o_i} - x_{x_i}X_{x_i} - x_{y_i}X_{y_i} - x_{z_i}X_{z_i}}$$
$$\boldsymbol{S_i(2,2) = -x_{x_i}X_{x_i} + x_{o_i}X_{o_i} + x_{z_i}X_{z_i} + x_{y_i}X_{y_i}}$$
$$\boldsymbol{S_i(3,3) = -x_{y_i}X_{y_i} + x_{z_i}X_{z_i} + x_{o_i}X_{o_i} + x_{x_i}X_{x_i}}$$
$$\boldsymbol{S_i(4,4) = -x_{z_i}X_{z_i} + x_{y_i}X_{y_i} + x_{x_i}X_{x_i} + x_{o_i}X_{o_i}}$$

Remember that $\dot{x}_i$ & $\dot{X}_i$ are pure quaternions.

Trace $(S_i) = 0$

Trace $\boldsymbol{S} = 0$

Quaternion-Based Derivation of R_c^m

$$\max_{\dot{q}} \dot{q}^T \boldsymbol{S} \dot{q} \;, \qquad \|\dot{q}\| = 1$$

$$\max_{\dot{q}} \boldsymbol{\varphi}(\dot{q}) = \dot{q}^T \boldsymbol{S} \dot{q} - \boldsymbol{\lambda}(\dot{q}^T \dot{q} - 1)$$

$$\frac{\partial \boldsymbol{\varphi}}{\partial \dot{q}} = 2\boldsymbol{S}\dot{q} - 2\boldsymbol{\lambda}\dot{q} = 0$$

$$\boldsymbol{S}\dot{q} = \boldsymbol{\lambda}\dot{q}$$

This is case only when $\dot{q}$ is the eigenvector of $\boldsymbol{S}$

$$\dot{q}^T \boldsymbol{S} \dot{q} = \dot{q}^T \boldsymbol{\lambda} \dot{q} = \boldsymbol{\lambda} \dot{q}^T \dot{q} = \boldsymbol{\lambda}$$

$\dot{q}^T \boldsymbol{S} \dot{q}$ will be maximum when $\dot{q}$ is the eigenvector of $\boldsymbol{S}$ that corresponds to the largest eigenvalue.

Quaternion-Based Derivation of R_c^m

$$R_c^m = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$r_{11} = q_x^2 + q_o^2 - q_z^2 - q_y^2$$

$$r_{12} = 2q_x q_y - 2q_o q_z$$

$$r_{13} = 2q_x q_z + 2q_o q_y$$

$$r_{21} = 2q_x q_y + 2q_o q_z$$

$$r_{22} = q_y^2 - q_z^2 + q_o^2 - q_x^2$$

$$r_{23} = 2q_y q_z - 2q_o q_x$$

$$r_{31} = 2q_x q_z - 2q_o q_y$$

$$r_{32} = 2q_y q_z + 2q_o q_x$$

$$r_{33} = q_z^2 - q_y^2 - q_x^2 + q_o^2$$