Basic Principles of Linear Algebra

Matrices and Vectors

- A Matrix is a rectangular array of numbers arranged in rows and columns.
- For example, the dimensions of the matrix below are *2 × 3* (read "two by three"), because there are two rows and three columns.

$$\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{8} & \mathbf{6} & \mathbf{9} \end{bmatrix}$$

- A column vector or column matrix is an *m × 1* matrix (i.e., a matrix consisting of a single column of m elements).

$$\begin{bmatrix} \mathbf{3} \\ \mathbf{5} \\ \mathbf{9} \end{bmatrix}$$

Transpose of a Matrix

- Given a matrix A: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ with m rows and n columns
- The transpose of the matrix is expressed as follows:
  - $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$ with n rows and m columns

Symmetric and Skew Matrices

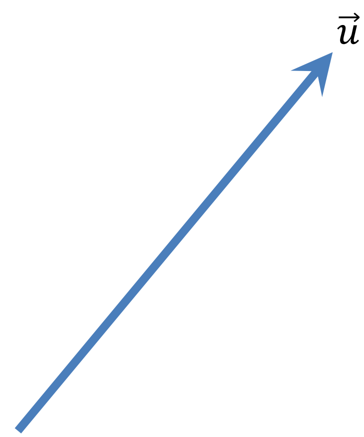
- A symmetric matrix is a square matrix where $a_{ij} = a_{ji}$
  - $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 5 \end{bmatrix}$
  - $A = A^T$
- A skew symmetric matrix is a square matrix where $a_{ij} = -a_{ji}$ & diagonal elements are zeros
  - $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 6 \\ 3 & -6 & 0 \end{bmatrix}$
  - $A^T = -A$

Inverse of a Matrix

- The inverse of an invertible square matrix A is represented as A^{-1}
- $A\,A^{-1} = A^{-1}A = Identity\ Matrix$
- $I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{n \times n}$
- Inverse of a symmetric matrix is a symmetric matrix

Magnitude of a Vector

- Given a vector: $\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$
- Its magnitude (norm) $\|\vec{u}\| = \sqrt{\left(u_x^2 + u_y^2 + u_z^2\right)}$



Matrix Addition

- Matrix addition is the operation of adding two matrices by adding the corresponding entries together.
- Two matrices must have an equal number of rows and columns to be added. The sum of two matrices A and B will be a matrix, which has the same number of rows and columns as do A and B.

$$A + B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Matrix Multiplication

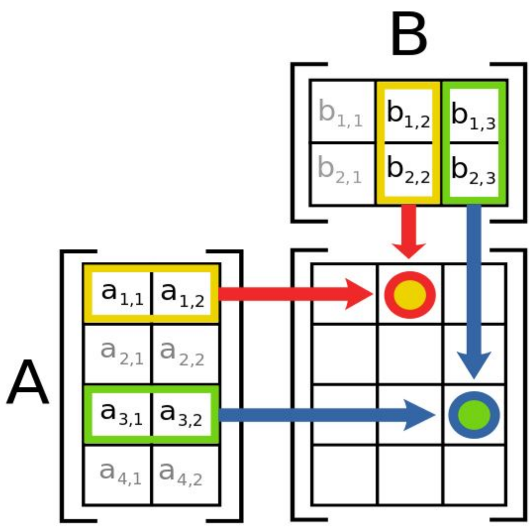
- Matrix multiplication or matrix product is a binary operation that produces a matrix from two matrices.

4 x 2 matrix

2x3 matrix

4 x 3 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ . & . \\ a_{31} & a_{32} \\ . & . \end{bmatrix} + \begin{bmatrix} . & b_{12} & b_{13} \\ . & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} . & x_{12} & x_{13} \\ . & . & . \\ . & x_{32} & x_{33} \\ . & . & . \end{bmatrix}$$



$$x_{12} = a_{11}b_{12} + \ a_{12}b_{22}$$
$$x_{33} = a_{31}b_{13} + \ a_{32}b_{23}$$

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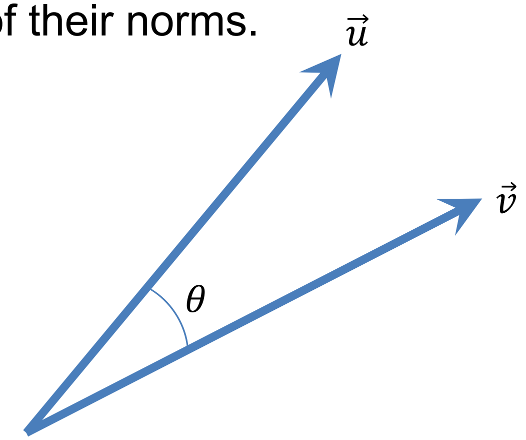
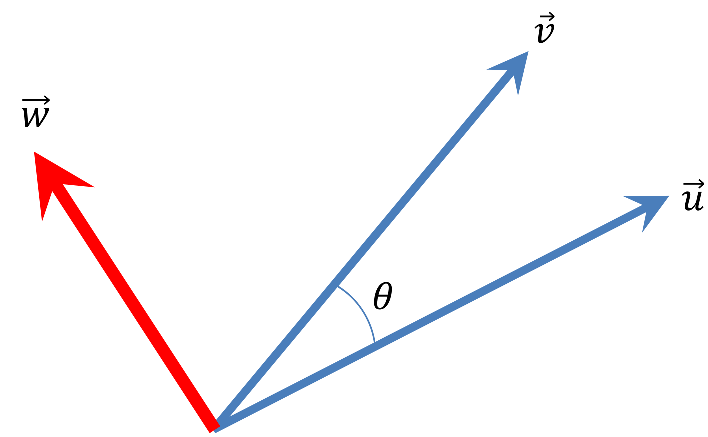
Dot Product of Two Vectors

- Given two vectors: $\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$
- The dot product of these two vectors is expressed as follows:
  - $\vec{u} \odot \vec{v} = u_x * v_x + u_y * v_y + u_z * v_z = \|\vec{u}\| \|\vec{v}\| \, cos\theta$
  - If the two vectors are orthogonal, the dot product is zero.
  - If the two vectors are parallel, the dot product reduces to the product of their norms.



Cross Product of Two Vectors

- Given two vectors: $\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$
- The cross product of the two vectors is expressed as follows:
  - $\vec{w} = \vec{u} \times \vec{v} = \begin{bmatrix} i & j & k \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} u_y v_z - u_z v_y \\ -u_x v_z + u_z v_x \\ u_x v_y - u_y v_x \end{bmatrix}$, where $i, j, and\ k$ are the unit vectors along the x, y, and z axes, respectively.
  - $\vec{w} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} u_y v_z - u_z v_y \\ -u_x v_z + u_z v_x \\ u_x v_y - u_y v_x \end{bmatrix} = \hat{\vec{u}}\ \vec{v}$
  - $\vec{w} = \begin{bmatrix} 0 & v_z & -v_y \\ -v_z & 0 & v_x \\ v_y & -v_x & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_y v_z - u_z v_y \\ -u_x v_z + u_z v_x \\ u_x v_y - u_y v_x \end{bmatrix} = \hat{\vec{v}}^T\ \vec{u}$
  - $\|\vec{w}\| = \|\vec{u}\|\ \|\vec{v}\| \sin\theta$
  - $\vec{w} \perp \vec{u}$ &$\vec{w} \perp \vec{v}$



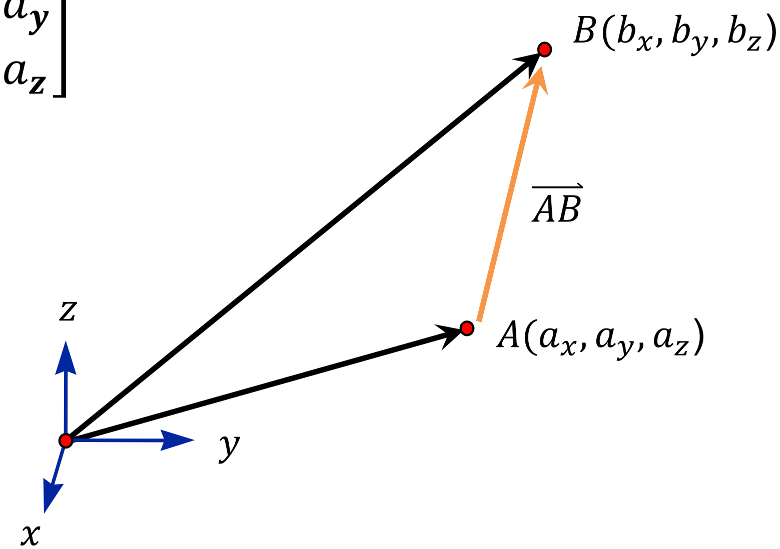
Vector Connecting Two Points

- Given two points in 3D space, which are defined by the following vectors ($\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$):

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \text{ and } \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix},$$

- Then, the vector connecting the two points will be defined as:

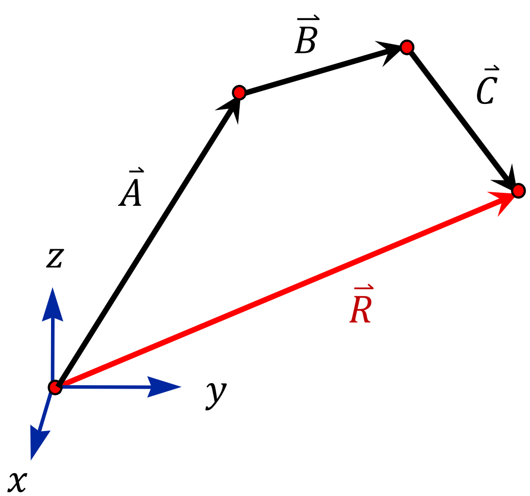
$$\overrightarrow{\boldsymbol{AB}} = \vec{\boldsymbol{B}} - \vec{\boldsymbol{A}} = \begin{bmatrix} b_x - a_x \\ b_{\boldsymbol{y}} - a_{\boldsymbol{y}} \\ b_{\boldsymbol{z}} - a_{\boldsymbol{z}} \end{bmatrix}$$



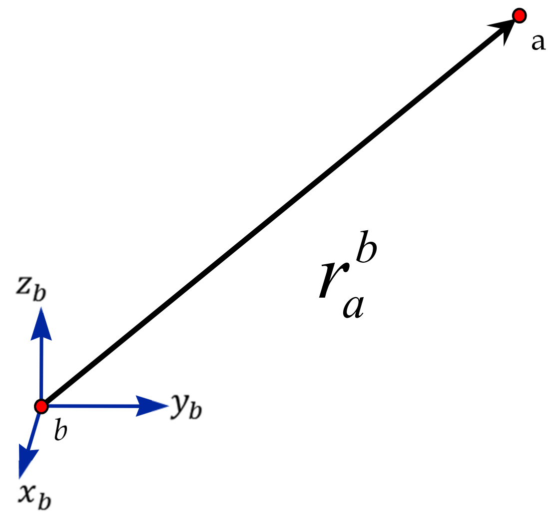
Resultant and Vector Summation

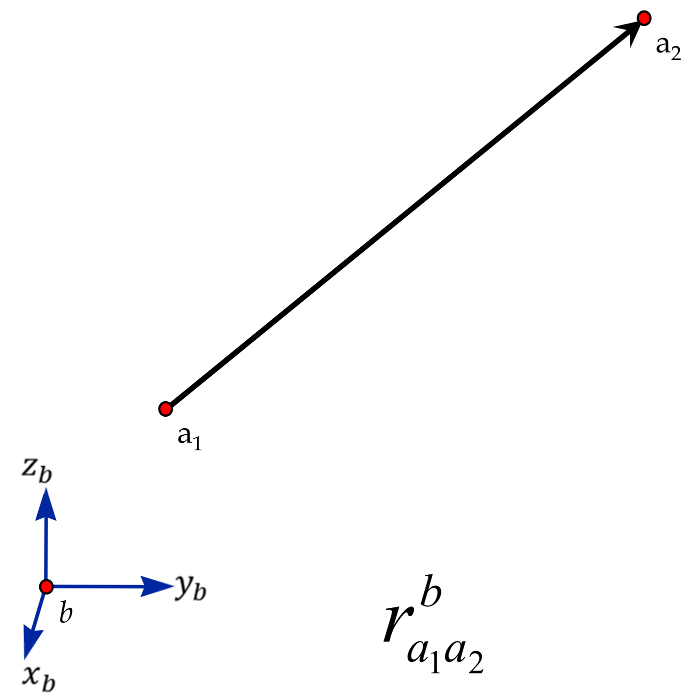
- The resultant is the vector sum of two or more vectors.
- It is the result of adding two or more vectors together.
- If displacement vectors $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ are added together, the result will be vector $\vec{\boldsymbol{R}}$.

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} + \vec{\boldsymbol{C}} = \begin{bmatrix} a_x + b_x + c_x \\ a_y + b_y + c_y \\ a_z + b_z + c_z \end{bmatrix}$$



Vector Notations

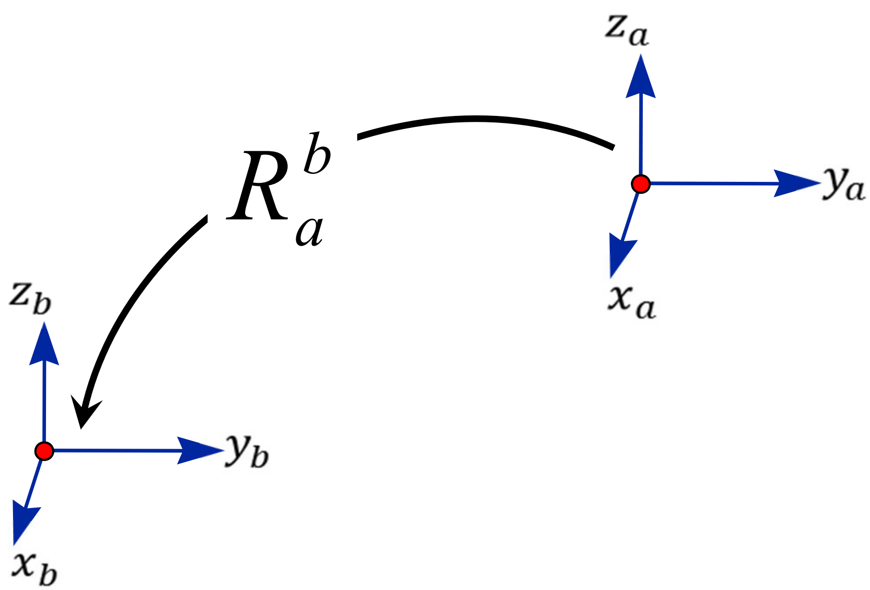




Rotation Matrices

- In Photogrammetry, we are dealing with two distinct coordinate systems:
  - Camera coordinate systems, and
  - Mapping coordinate systems.
- The camera and mapping coordinate systems are not necessarily parallel.
- The derivation of the collinearity equations would require the summation of vectors related to the camera and mapping coordinate systems.
- Vectors summation/subtraction cannot be performed unless they are represented relative to the same coordinate systems.
- Vectors represented in different coordinate systems can be transformed to common coordinate system using a rotation matrix.
- Therefore, we need to establish the rotation matrix relating the camera and mapping coordinate systems.

Rotation Matrix Notations



Collinearity Equations

$$R = R_c^m$$

$$x_a = x_p - c\,\frac{r_{11}(X_A - X_o) + r_{21}(Y_A - Y_o) + r_{11}(Z_A - Z_o)}{r_{13}(X_A - X_o) + r_{23}(Y_A - Y_o) + r_{33}(Z_A - Z_o)} + dist_x$$

$$y_a = y_p - c\,\frac{r_{12}(X_A - X_o) + r_{22}(Y_A - Y_o) + r_{32}(Z_A - Z_o)}{r_{13}(X_A - X_o) + r_{23}(Y_A - Y_o) + r_{33}(Z_A - Z_o)} + dist_y$$

Involved parameters:

- **Image coordinates (x_a, y_a)**
- **Ground coordinates (A_A, Y_A, Z_A)**
- **Exterior Orientation Parameters – EOPs (X_O, Y_O, Z_O, ω, ϕ, κ)**
- **Interior Orientation Parameters – IOPs (x_p, y_p, c, and the coefficients describing $dist_x$ and $dist_y$)**

Collinearity Equations

- **In the collinearity equations, the observed image coordinates are expressed as non-linear function of the:**
  - Ground coordinates (A_A, Y_A, Z_A),
  - Exterior Orientation Parameters – EOP (X_O, Y_O, Z_O, ω, ϕ, κ), and
  - Interior Orientation Parameters – IOP (x_p, y_p, c, and the coefficients describing $dist_x$ and $dist_y$).
- **One should note that the rotation angles ω, ϕ, κ are those that need to be applied to the X, Y, and Z-axes, respectively, of the mapping coordinate system to make it parallel to the camera coordinate system.**
- **Bundle Adjustment is an approach used to solve for the ground coordinates of tie points, EOP of the images, and IOP of the used camera(s).**
- **BASC (Bundle Adjustment with Self Calibration) is provided to solve for such parameters (ground coordinates of tie points, EOP of the images, and IOP of the used camera).**