Direct Linear Transformation & Computer Vision Models

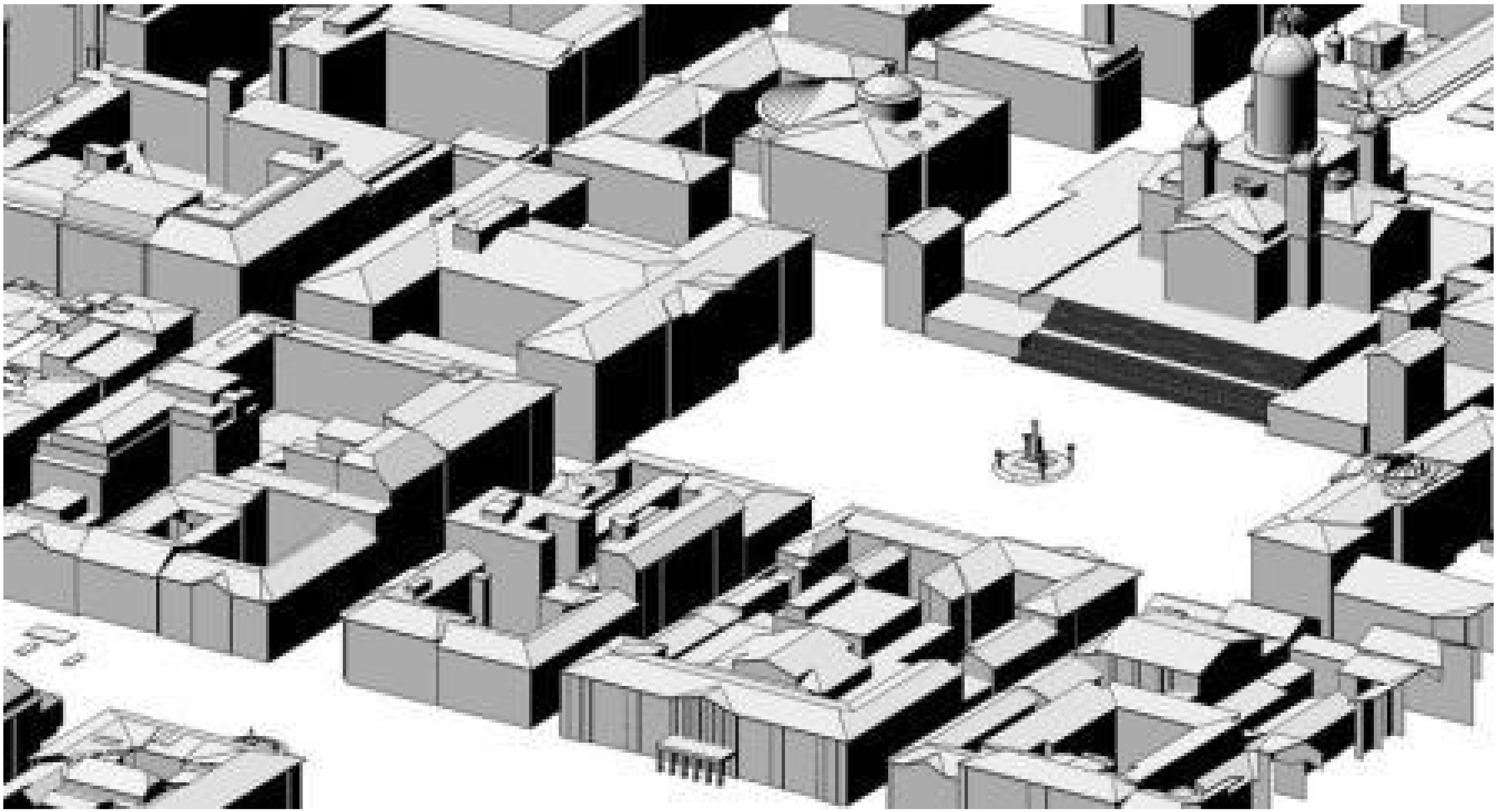
Chapter 7-A4

Photogrammetry Vs. Computer Vision

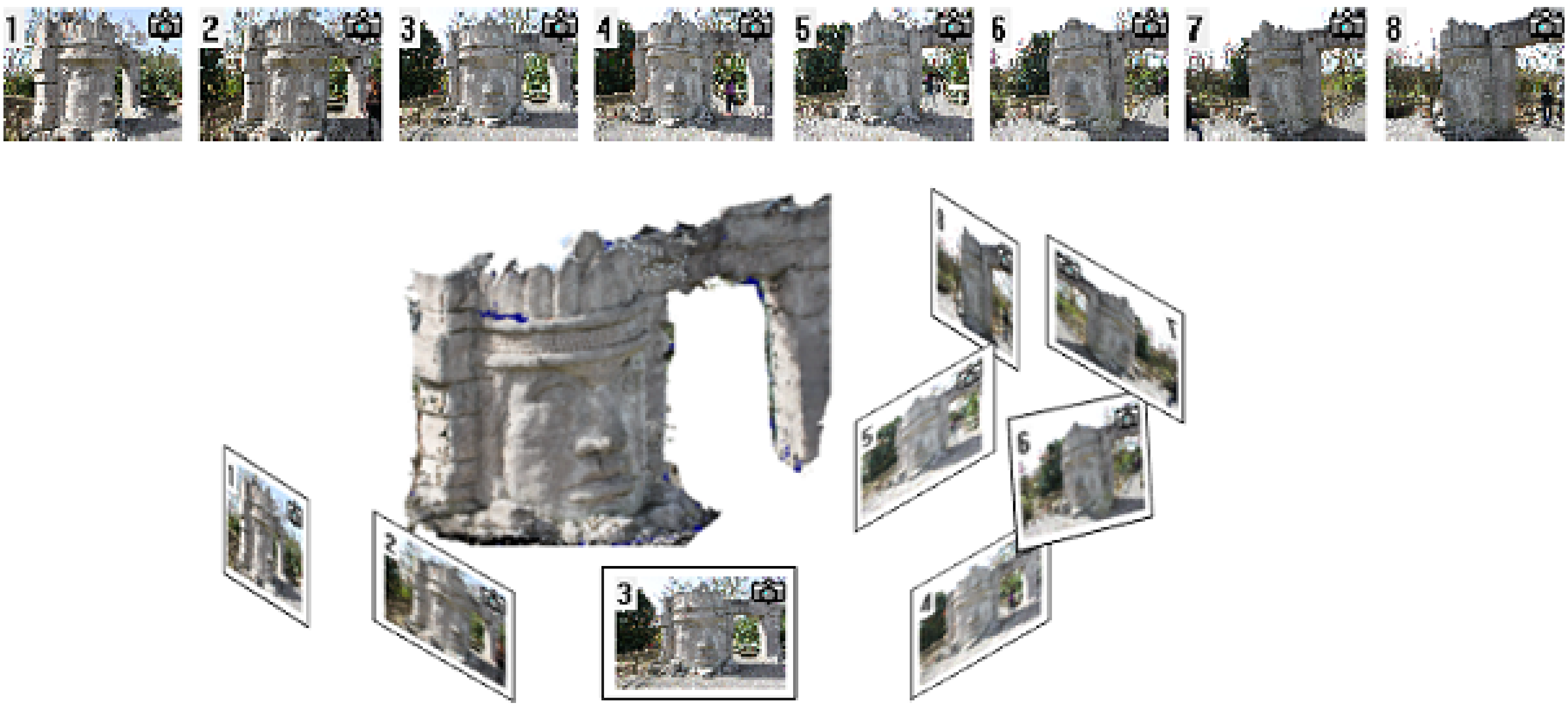
- Conventional Photogrammetry is focusing on **precise** geometric information extraction from imagery.
  - Topographic mapping from space borne and airborne imagery
  - Metrological information extraction through close-range photogrammetry (terrestrial photogrammetry)
    - Object-to-camera distance is less than 100meter
- Computer Vision (CV) is mainly concerned with **automated** image understanding:
  - Object recognition,
  - Navigation and obstacle avoidance, and
  - **Object modeling**

Airborne Photogrammetric Mapping

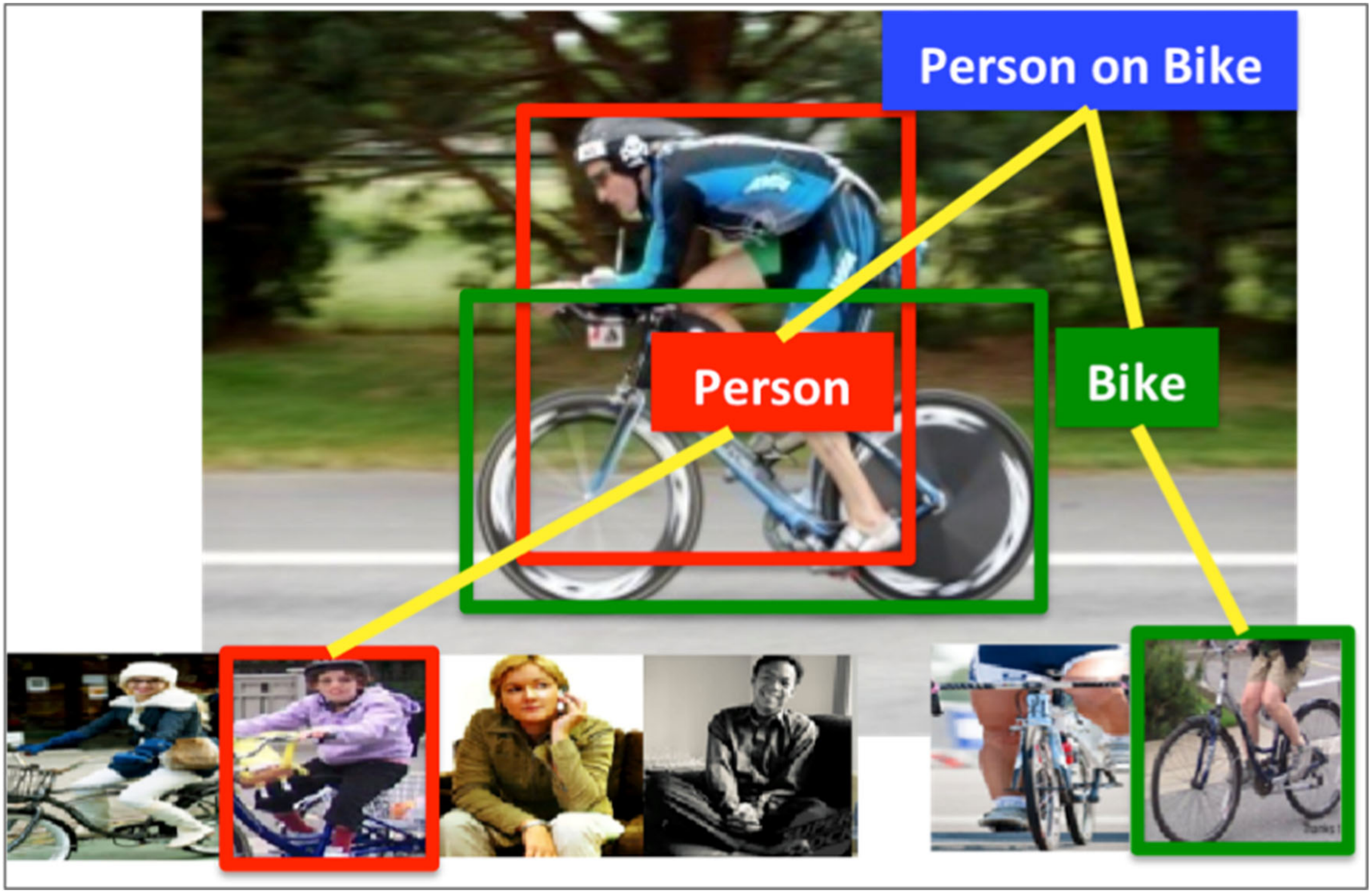
Airborne Photogrammetric Mapping



Close-Range Photogrammetric Mapping



CV: Object Recognition



CV: Navigation & Obstacle Avoidance

Photogrammetry Vs. Computer Vision

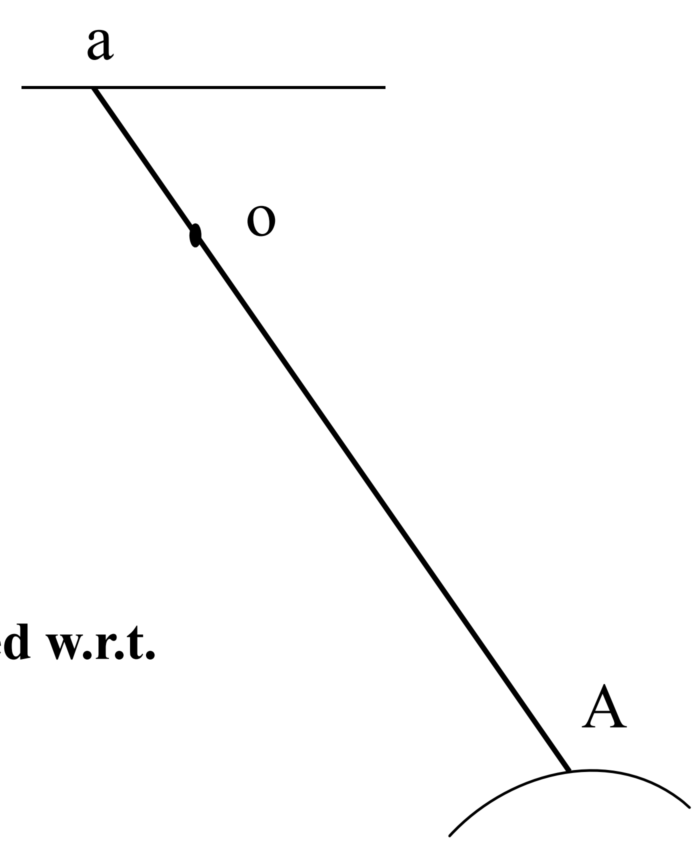
- Photogrammetry is always concerned with **precise** geometric information extraction.
  - Photogrammetric mapping considers potential deviations from the assumed perspective projection.
- For Computer Vision (CV):
  - Focus is always on **automation**.
  - Object recognition and navigation applications do not require precise derivation of geometric information.
  - Depending on the application, object modeling might require precise geometric information extraction.
  - CV usually assumes that the collinearity of the object point, perspective center, and corresponding image point is maintained, even for un-calibrated cameras.

Object-to-Image Coordinate Transformation in Photogrammetry

Collinearity Equations

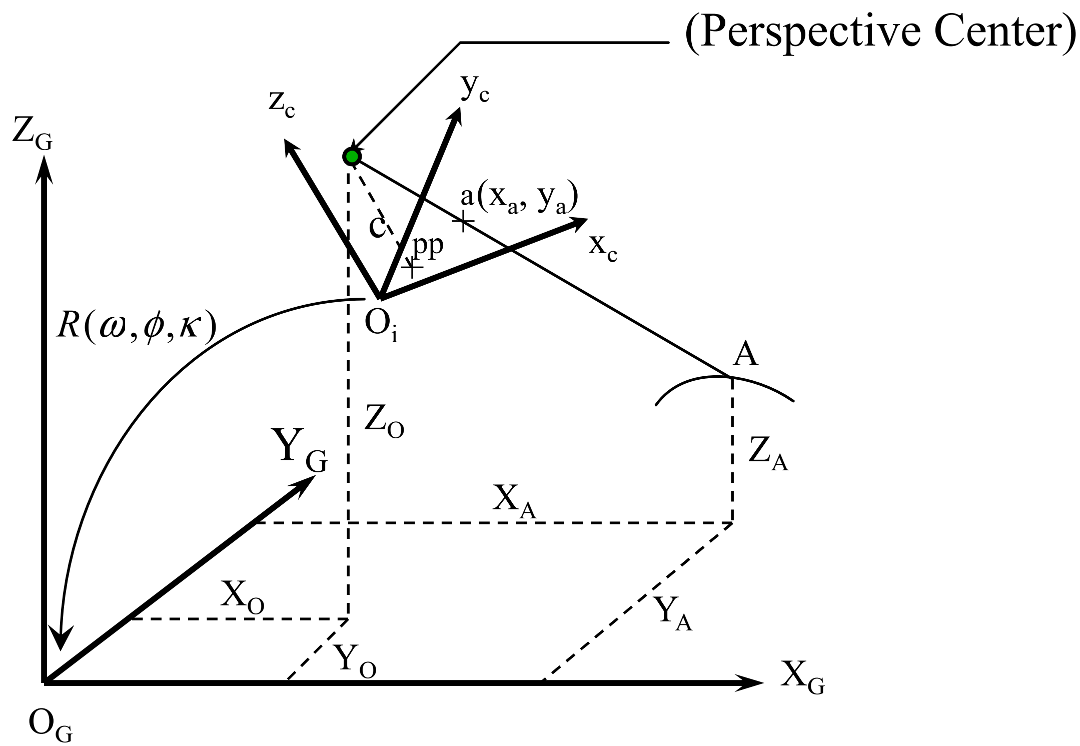
Collinearity Equations

$$\vec{oa} = \lambda \ \vec{oA}$$



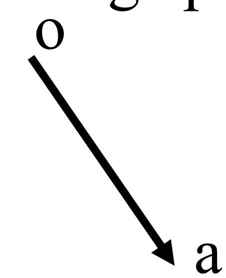
These vectors should be defined w.r.t. the same coordinate system.

Collinearity Equations



Collinearity Equations

The vector connecting the perspective center to the image point



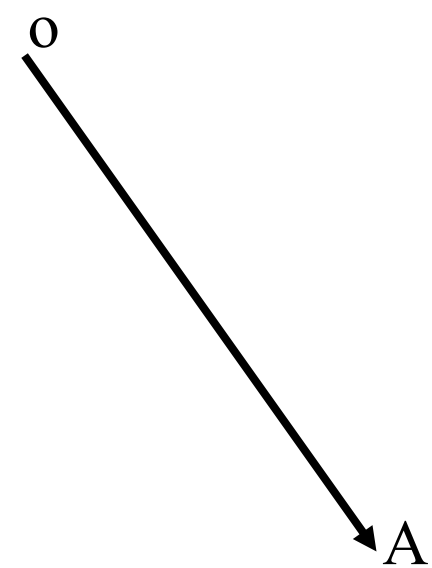
$$\vec{v}_i = r_{oa}^{c} = \begin{bmatrix} x_a - dist_x \\ y_a - dist_y \\ 0 \end{bmatrix} - \begin{bmatrix} x_p \\ y_p \\ c \end{bmatrix} = \begin{bmatrix} x_a - x_p - dist_x \\ y_a - y_p - dist_y \\ -c \end{bmatrix}$$

w.r.t. the image coordinate system

Collinearity Equations

The vector connecting the perspective center to the object point

$$\vec{V}_o = r_{oA}^{m} = \begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} - \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} = \begin{bmatrix} X_A - X_o \\ Y_A - Y_o \\ Z_A - Z_o \end{bmatrix}$$



w.r.t. the ground coordinate system

Collinearity Equations

$$\overrightarrow{oa} = \lambda \overrightarrow{oA}$$

$$\vec{v}_i = r^c_{oa} = \lambda \quad M(\omega, \varphi, \kappa) \quad \vec{V}_O = \lambda \quad R^c_m \, r^m_{oA}$$

$$\begin{bmatrix} x_a - x_p - dist_x \\ y_a - y_p - dist_y \\ -c \end{bmatrix} = \lambda \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} X_A - X_o \\ Y_A - Y_o \\ Z_A - Z_o \end{bmatrix}$$

Where: λ is a scale factor (+ve).

Collinearity Equations

$$M = R_m^c$$

$$x_a = x_p - c\frac{m_{11}(X_A - X_o) + m_{12}(Y_A - Y_o) + m_{13}(Z_A - Z_o)}{m_{31}(X_A - X_o) + m_{32}(Y_A - Y_o) + m_{33}(Z_A - Z_o)} + dist_x$$

$$y_a = y_p - c\frac{m_{21}(X_A - X_o) + m_{22}(Y_A - Y_o) + m_{23}(Z_A - Z_o)}{m_{31}(X_A - X_o) + m_{32}(Y_A - Y_o) + m_{33}(Z_A - Z_o)} + dist_y$$

$$R = R_c^m$$

$$x_a = x_p - c\frac{r_{11}(X_A - X_o) + r_{21}(Y_A - Y_o) + r_{31}(Z_A - Z_o)}{r_{13}(X_A - X_o) + r_{23}(Y_A - Y_o) + r_{33}(Z_A - Z_o)} + dist_x$$

$$y_a = y_p - c\frac{r_{12}(X_A - X_o) + r_{22}(Y_A - Y_o) + r_{32}(Z_A - Z_o)}{r_{13}(X_A - X_o) + r_{23}(Y_A - Y_o) + r_{33}(Z_A - Z_o)} + dist_y$$

Object-to-Image Coordinate Transformation

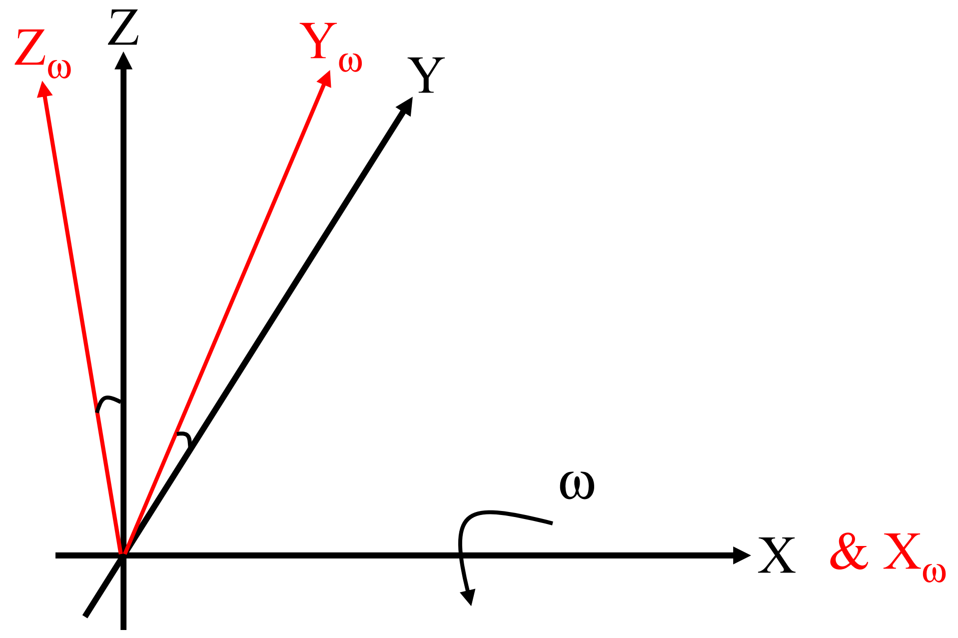
Direct Linear Transformation

Computer Vision Model

DLT & Computer Vision Models

- The DLT and computer vision models encompass:
  - Collinearity Equations,
  - Non-orthogonality (α) between the axes of the image/camera coordinate system, and
  - Two scale factors (S_x, S_y) along the axes of the image coordinate system.
- DLT & CV models can directly deal with pixel coordinates.
- We will start with modifying the rotation matrix to consider the impact of the non-orthogonality (α).
  - Primary rotation ω @ the X-axis of the ground coord. system
  - Secondary rotation φ @ the Y_ω-axis
  - Tertiary rotation κ & $(\kappa + \alpha)$ @ the$Z_{\omega\varphi}$-axis

Primary Rotation (ω)

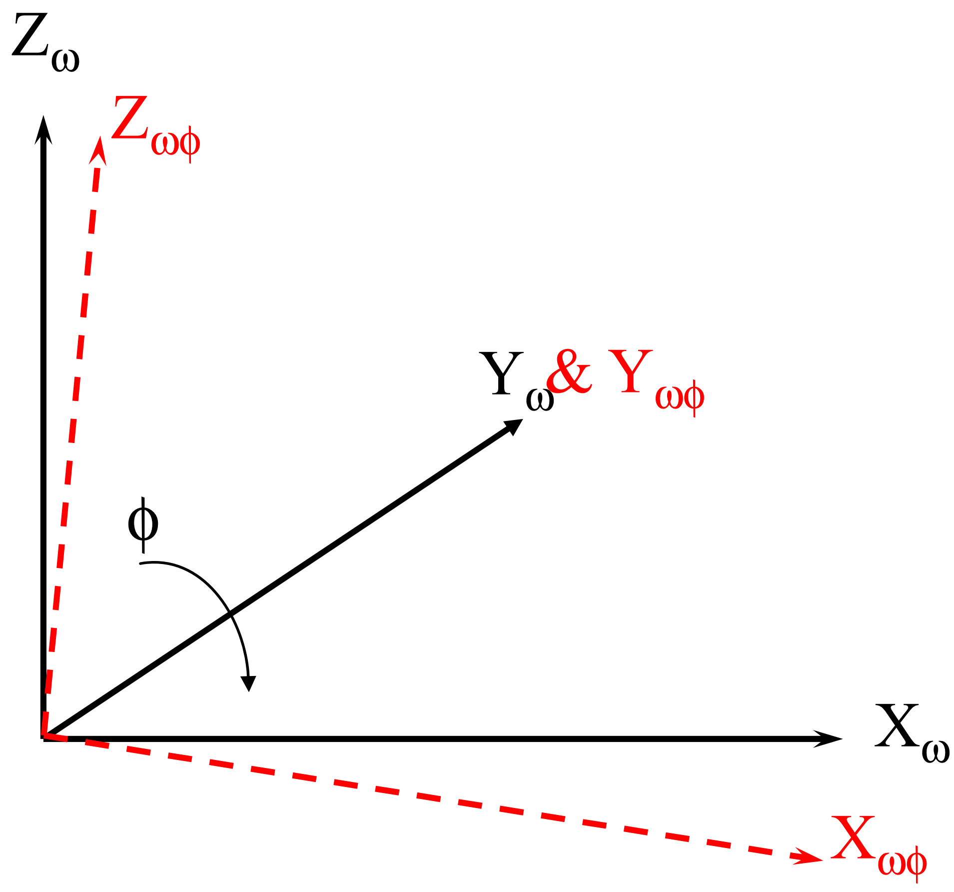


Primary Rotation (ω)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x_\omega \\ y_\omega \\ z_\omega \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_\omega \begin{bmatrix} x_\omega \\ y_\omega \\ z_\omega \end{bmatrix}$$

Secondary Rotation (ϕ)

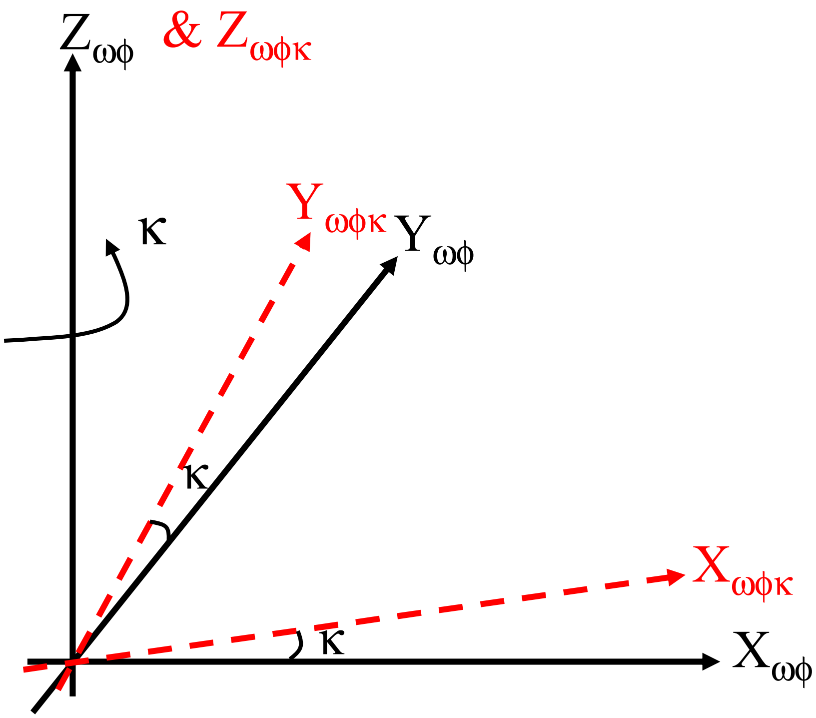


Secondary Rotation (ϕ)

$$\begin{bmatrix} x_\omega \\ y_\omega \\ z_\omega \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix}$$

$$\begin{bmatrix} x_\omega \\ y_\omega \\ z_\omega \end{bmatrix} = R_\phi \begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix}$$

Tertiary Rotation (κ)



Tertiary Rotation (κ)

$$\begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\omega\phi\kappa} \\ y_{\omega\phi\kappa} \\ z_{\omega\phi\kappa} \end{bmatrix}$$

$$\begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix} = R_{\kappa} \begin{bmatrix} x_{\omega\phi\kappa} \\ y_{\omega\phi\kappa} \\ z_{\omega\phi\kappa} \end{bmatrix}$$

Rotation in Space

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_{\omega} R_{\phi} R_{\kappa} \begin{bmatrix} x_{\omega\phi\kappa} \\ y_{\omega\phi\kappa} \\ z_{\omega\phi\kappa} \end{bmatrix}$$

// to the ground coordinate system // to the image coordinate system

Rotation in Space

$$R_\omega R_\phi R_\kappa = R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

where :

$$r_{11} = \cos\phi \cos\kappa$$

$$r_{12} = -\cos\phi \sin\kappa$$

$$r_{13} = \sin\phi$$

$$r_{21} = \cos\omega \sin\kappa + \sin\omega \sin\phi \cos\kappa$$

$$r_{22} = \cos\omega \cos\kappa - \sin\omega \sin\phi \sin\kappa$$

$$r_{23} = -\sin\omega \cos\phi$$

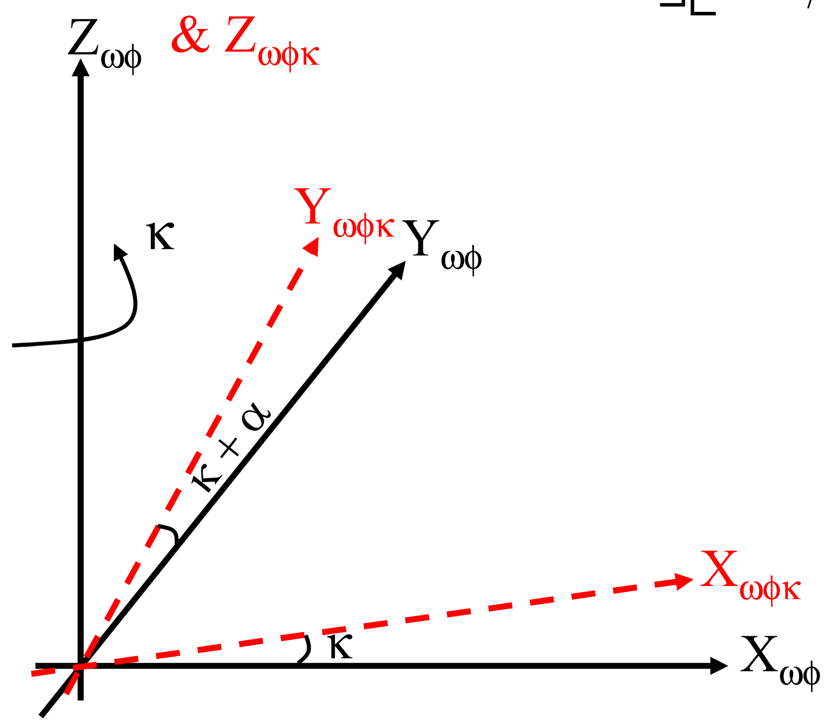
$$r_{31} = \sin\omega \sin\kappa - \cos\omega \sin\phi \cos\kappa$$

$$r_{32} = \sin\omega \cos\kappa + \cos\omega \sin\phi \sin\kappa$$

$$r_{33} = \cos\omega \cos\phi$$

Consideration of the Non-Orthogonality (α)

$$\begin{bmatrix} X_{\omega\phi} \\ Y_{\omega\phi} \\ Z_{\omega\phi} \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin(\kappa+\alpha) & 0 \\ \sin\kappa & \cos(\kappa+\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$



Consideration of the Non-Orthogonality (α)

$$\begin{bmatrix} X_{\omega\phi} \\ Y_{\omega\phi} \\ Z_{\omega\phi} \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin(\kappa+\alpha) & 0 \\ \sin\kappa & \cos(\kappa+\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

Assuming small non-orthogonality angle (α)

$$\sin(\kappa+\alpha) = \sin\kappa\ \cos\alpha + \cos\kappa\ \sin\alpha = \sin\kappa + \alpha\cos\kappa$$

$$\cos(\kappa+\alpha) = \cos\kappa\ \cos\alpha - \sin\kappa\ \sin\alpha = \cos\kappa - \alpha\ \sin\kappa$$

$$\begin{bmatrix} X_{\omega\phi} \\ Y_{\omega\phi} \\ Z_{\omega\phi} \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin\kappa - \alpha\cos\kappa & 0 \\ \sin\kappa & \cos\kappa - \alpha\sin\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

Consideration of the Non-Orthogonality (α)

$$\begin{bmatrix} \cos\kappa & -\sin\kappa - \alpha\cos\kappa & 0 \\ \sin\kappa & \cos\kappa - \alpha\sin\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R_\omega R_\phi R_\kappa \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix} = R \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

Consideration of the Non-Orthogonality (α)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

Note: $\begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

// to the image coordinate system // to the ground coordinate system

Consideration of the Non-Orthogonality (α)

- Collinearity Equations while considering the non-orthogonality (α) between the axes of the image coordinate system.

$$\begin{bmatrix} x - x_p \\ y - y_p \\ -c \end{bmatrix} = \lambda \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X - X_O \\ Y - Y_O \\ Z - Z_O \end{bmatrix}$$

Consideration of the Scale Factors

- Collinearity Equations while considering the non-orthogonality (α) between the axes of the image coordinate system & different scale factors.

$$\begin{bmatrix} (x-x_p)/s_x \\ (y-y_p)/s_y \\ -c \end{bmatrix} = \lambda \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

  - Divide both sides by (-c).

$$\begin{bmatrix} -(x-x_p)/(cs_x) \\ -(y-y_p)/(cs_y) \\ 1 \end{bmatrix} = -\lambda/c \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

Consideration of the Scale Factors

- $cs_x \rightarrow c_x$, $cs_y \rightarrow c_y$ & $-\lambda/c \rightarrow \lambda`$.

$$\begin{bmatrix} -(x-x_p)/(c_x) \\ -(y-y_p)/(c_y) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

$$\begin{bmatrix} -1/c_x & 0 & 0 \\ 0 & -1/c_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (x-x_p) \\ (y-y_p) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

$$\begin{bmatrix} (x-x_p) \\ (y-y_p) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} -c_x & 0 & 0 \\ 0 & -c_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

DLT & Computer Vision Models

$$\begin{bmatrix}(x-x_p)\\(y-y_p)\\1\end{bmatrix}=\lambda'\begin{bmatrix}-c_x & -\alpha c_x & 0\\0 & -c_y & 0\\0 & 0 & 1\end{bmatrix}R^T\begin{bmatrix}X-X_O\\Y-Y_O\\Z-Z_O\end{bmatrix}$$

$$\begin{bmatrix}(x-x_p)\\(y-y_p)\\1\end{bmatrix}=\lambda'\begin{bmatrix}-c_x & -\alpha c_x & 0\\0 & -c_y & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}r_{11} & r_{21} & r_{31}\\r_{12} & r_{22} & r_{32}\\r_{13} & r_{23} & r_{33}\end{bmatrix}\begin{bmatrix}X-X_O\\Y-Y_O\\Z-Z_O\end{bmatrix}$$

$$\begin{bmatrix}\boldsymbol{x-x_p}\\\boldsymbol{y-y_p}\\\mathbf{1}\end{bmatrix}=\begin{bmatrix}\mathbf{1} & \mathbf{0} & \boldsymbol{-x_p}\\\mathbf{0} & \mathbf{1} & \boldsymbol{-y_p}\\\mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\mathbf{1}\end{bmatrix} \quad \& \quad \begin{bmatrix}\mathbf{1} & \mathbf{0} & \boldsymbol{-x_p}\\\mathbf{0} & \mathbf{1} & \boldsymbol{-y_p}\\\mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}^{\mathbf{-1}}=\begin{bmatrix}\mathbf{1} & \mathbf{0} & \boldsymbol{x_p}\\\mathbf{0} & \mathbf{1} & \boldsymbol{y_p}\\\mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}$$

DLT & Computer Vision Models

$$\begin{bmatrix} x - x_p \\ y - y_p \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} -c_x & -\alpha c_x & 0 \\ 0 & -c_y & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X - X_O \\ Y - Y_O \\ Z - Z_O \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} 1 & 0 & x_p \\ 0 & 1 & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & -\alpha c_x & 0 \\ 0 & -c_y & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X - X_O \\ Y - Y_O \\ Z - Z_O \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X - X_O \\ Y - Y_O \\ Z - Z_O \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R^T & -R^T \mathbf{X_O} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{X_O} = \begin{bmatrix} X_O & Y_O & Z_O \end{bmatrix}^T$$

DLT & Computer Vision Models

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R^T & -R^T X_o \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda' K R^T \begin{bmatrix} I_3 & -X_o \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Where:

$$K = \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

DLT & Computer Vision Models

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda' K\, R^T \begin{bmatrix} I_3 & -X_O \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$K = \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \equiv \{Calibration\ \ Matrix\}$$

$$R^T \begin{bmatrix} I_3 & -X_O \end{bmatrix} \equiv \{Exterior\ \ Orientation\ \ Matrix\}$$

$$\{\boldsymbol{Exterior\ Orientation\ Matrix}\} = \boldsymbol{R^T} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & -\boldsymbol{X_O} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & -\boldsymbol{Y_O} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -\boldsymbol{Z_O} \end{bmatrix}$$

DLT & Computer Vision Models

- The **Direct Linear Transformation** (DLT), which has been developed by the photogrammetric community, is an alternative to the collinearity equations that allows for direct transformation between machine/pixel coordinates and corresponding ground coordinates.
  - $x = \frac{L_1X+L_2Y+L_3Z+L_4}{L_9X+L_{10}Y+L_{11}Z+1}$ & $y = \frac{L_5X+L_6Y+L_7Z+L_8}{L_9X+L_{10}Y+L_{11}Z+1}$
- The DLT can be also represented by the following form:
  - $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & L_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$

DLT & Computer Vision Models

DLT: Direct Linear Transformation

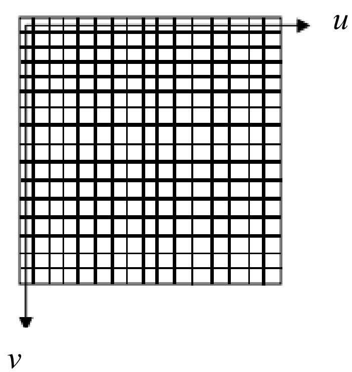
$$\begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & L_{12} \end{bmatrix} = \boldsymbol{\lambda' K R^T} [\boldsymbol{I_3} \quad \boldsymbol{-X_o}]$$

$$D = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda' \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T$$

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\lambda' \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}$$

DLT & CV Models: Pixel Coordinates

- The DLT & CV models can also consider the direct transformation from pixel to ground coordinates.



Pixel Coordinates

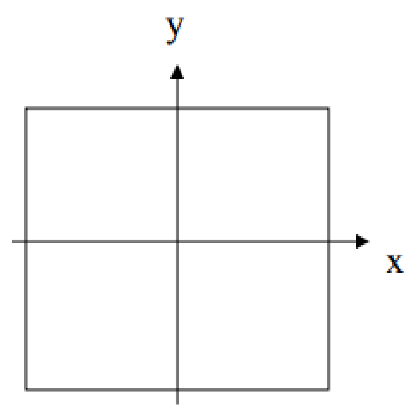


Image Coordinates

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} (u - n_c/2) \times x\_pix\_size \\ (n_r/2 - v) \times y\_pix\_size \\ 1 \end{bmatrix}$$

DLT & CV Models: Pixel Coordinates

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} (u - n_c/2) \times x\_pix\_size \\ (n_r/2 - v) \times y\_pix\_size \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\_pix\_size & 0 & -n_c/2 \times x\_pix\_size \\ 0 & -y\_pix\_size & n_r/2 \times y\_pix\_size \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda' K R^T [I_3 \quad -X_o] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} x\_pix\_size & 0 & -n_c/2 \times x\_pix\_size \\ 0 & -y\_pix\_size & n_r/2 \times y\_pix\_size \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$= \lambda' K R^T [I_3 \quad -X_o] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

DLT & CV Models: Pixel Coordinates

$$\begin{bmatrix} x\_pix\_size & 0 & -n_c/2 \times x\_pix\_size \\ 0 & -y\_pix\_size & n_r/2 \times y\_pix\_size \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \lambda' K R^T [I_3 \quad -X_o] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} x\_pix\_size & 0 & -n_c/2 \times x\_pix\_size \\ 0 & -y\_pix\_size & n_r/2 \times y\_pix\_size \\ 0 & 0 & 1 \end{bmatrix}^{-1} K R^T [I_3 \quad -X_o] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x\_pix\_size & 0 & -n_c/2 \times x\_pix\_size \\ 0 & -y\_pix\_size & n_r/2 \times y\_pix\_size \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/x\_pix\_size & 0 & n_c/2 \\ 0 & -1/y\_pix\_size & n_r/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} 1/x\_pix\_size & 0 & n_c/2 \\ 0 & -1/y\_pix\_size & n_r/2 \\ 0 & 0 & 1 \end{bmatrix} K R^T [I_3 \quad -X_o] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

DLT & CV Models: Pixel Coordinates

- Modified Calibration Matrix:

$$K' = \begin{bmatrix} \mathbf{1/x\_pix\_size} & \mathbf{0} & \mathbf{n_c/2} \\ \mathbf{0} & \mathbf{-1/y\_pix\_size} & \mathbf{n_r/2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{-c_x} & \mathbf{-\alpha c_x} & \mathbf{x_p} \\ \mathbf{0} & \mathbf{-c_y} & \mathbf{y_p} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$K' = \begin{bmatrix} \mathbf{-c_x/x\_pix\_size} & \mathbf{-\alpha c_x/x\_pix\_size} & \mathbf{x_p/x\_pix\_size + n_c/2} \\ 0 & \mathbf{c_y/y\_pix\_size} & \mathbf{-y_p/y\_pix\_size + n_r/2} \\ 0 & 0 & 1 \end{bmatrix}$$

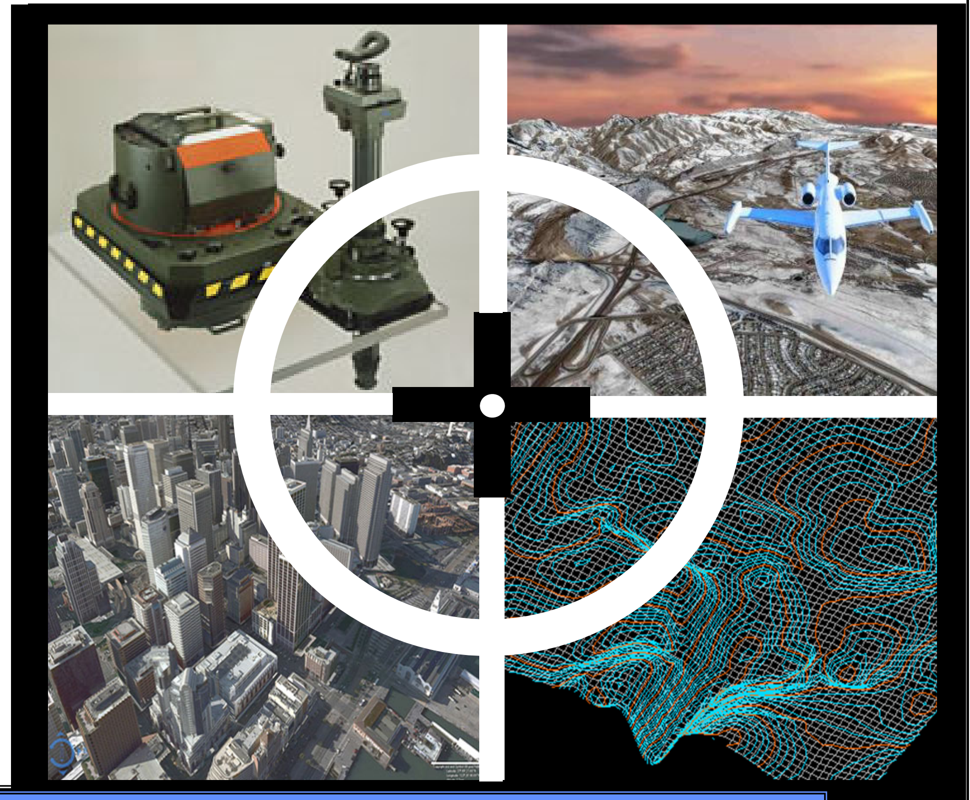
$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{1} \end{bmatrix} = \boldsymbol{\lambda' K' R^T [I_3 \quad -X_o]} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \\ \mathbf{1} \end{bmatrix}$$

DLT & CV Models: Pixel Coordinates

- For DLT when working with pixel coordinates, we have the following model.
  - $\begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & L_{12} \end{bmatrix} = \boldsymbol{\lambda' K' R^T [I_3 \quad -X_o]}$
    - $\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \boldsymbol{\lambda' K' R^T}$
    - $\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\boldsymbol{\lambda' K' R^T} \begin{bmatrix} \boldsymbol{X_o} \\ Y_{\boldsymbol{o}} \\ Z_{\boldsymbol{o}} \end{bmatrix}$

Modern Photogrammetry & Computer Vision

- Modern Photogrammetry and Computer Vision are converging fields.



Art and science of tool development for automatic generation of spatial and descriptive information from multi-sensory data and/or systems

DLT → IOPs & EOPs

Approach # 1

DLT → IOP & EOP

$$D = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T$$

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}$$

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} X_{\boldsymbol{o}} \\ Y_{\boldsymbol{o}} \\ Z_{\boldsymbol{o}} \end{bmatrix}$$

DLT → IOP & EOP

- Given:

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}$$

- Then:

$$\begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix} = -\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix}$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = (\lambda\,K\,R^T)(\lambda\,K\,R^T)^T = \lambda^2\,KK^T = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$D\,D^T = \lambda^2 \begin{bmatrix} -c_x & -\alpha\,c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & 0 & 0 \\ -\alpha\,c_x & -c_y & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

$$(D\,D^T)_{3\times3} = L_9^2 + L_{10}^2 + L_{11}^2 = \lambda^2$$

Then:

$$\lambda = \pm\sqrt{L_9^2 + L_{10}^2 + L_{11}^2} \;\{Sign\;\;Ambiguity\}$$

DLT → IOP & EOP

$$D\,D^T = (\lambda\,K\,R^T)(\lambda\,K\,R^T)^T = \lambda^2\,KK^T = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{3\times1} = L_9\,L_1 + L_{10}\,L_2 + L_{11}\,L_3 = \lambda^2\;x_p$$

Then:

$$x_p = {(L_9\,L_1 + L_{10}\,L_2 + L_{11}\,L_3)}\Big/{(L_9^2 + L_{10}^2 + L_{11}^2)}$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = (\lambda\,K\,R^T)(\lambda\,K\,R^T)^T = \lambda^2\,KK^T = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{3\times2} = L_9\,L_5 + L_{10}\,L_6 + L_{11}\,L_7 = \lambda^2\; y_p$$

Then:

$$y_p = (L_9\,L_5 + L_{10}\,L_6 + L_{11}\,L_7) \big/ (L_9^2 + L_{10}^2 + L_{11}^2)$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = (\lambda\,K\,R^T)(\lambda\,K\,R^T)^T = \lambda^2\,KK^T = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{2\times 2} = L_5^2 + L_6^2 + L_7^2 = \lambda^2\,(y_p^2 + c_y^2)$$

Then:

$$c_y = \left[L_5^2 + L_6^2 + L_7^2 \Big/ (L_9^2 + L_{10}^2 + L_{11}^2) - y_p^2 \right]^{0.5}$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = (\lambda\,K\,R^T)(\lambda\,K\,R^T)^T = \lambda^2\,KK^T = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{1\times 2} = L_1\,L_5 + L_2\,L_6 + L_3\,L_7 = \lambda^2\;(\alpha\,c_x\,c_y + x_p\,y_p)$$

Then:

$$\alpha\,c_x = 1/c_y \left[L_1\,L_5 + L_2\,L_6 + L_3\,L_7 \Big/ (L_9^2 + L_{10}^2 + L_{11}^2) - x_p\,y_p \right]$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = (\lambda\,K\,R^T)(\lambda\,K\,R^T)^T = \lambda^2\,KK^T = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{1\times 1} = L_1^2 + L_2^2 + L_3^2 = \lambda^2\ (c_x^2 + \alpha^2\ c_x^2 + x_p^2)$$

Then:

$$c_x = \left[L_1^2 + L_2^2 + L_3^2 \Big/ (L_9^2 + L_{10}^2 + L_{11}^2) - \alpha^2\ c_x^2 - x_p^2 \right]^{0.5}$$

No Sign Ambiguity

DLT → IOP & EOP

- Given:

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- Then:

$$L_9 = \lambda\, r_{13} = \lambda \sin\phi$$

Sign Ambiguity

Collinearity Equations

- Objective: Resolve the sign ambiguity in λ

$$\begin{bmatrix} x - x_p \\ y - y_p \\ -c \end{bmatrix} = S \begin{bmatrix} r_{11}(X - X_O) + r_{21}(Y - Y_O) + r_{31}(Z - Z_O) \\ r_{12}(X - X_O) + r_{22}(Y - Y_O) + r_{32}(Z - Z_O) \\ r_{13}(X - X_O) + r_{23}(Y - Y_O) + r_{33}(Z - Z_O) \end{bmatrix}$$

- Since the scale factor is always +ve

$$r_{13}(X - X_O) + r_{23}(Y - Y_O) + r_{33}(Z - Z_O) \Rightarrow -ve$$

- Assuming that the origin (0, 0, 0) is visible in the imagery

$$-r_{13} X_O - r_{23} Y_O - r_{33} Z_O \Rightarrow -ve$$

DLT → IOP & EOP

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}$$

- By choosing $L_{12} = 1$.

$$L_{12} = -\lambda\ (r_{13}\, X_O + r_{23}\, Y_O + r_{33}\, Z_O)$$

$$1 = \lambda\ (-r_{13}\, X_O - r_{23}\, Y_O - r_{33}\, Z_O)$$

$$\lambda = 1 / (-r_{13}\, X_O - r_{23}\, Y_O - r_{33}\, Z_O)$$

λ is Negative

$$\lambda = -\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}$$

DLT → IOP & EOP

$$L_9 = \lambda \; r_{13} = \lambda \; \sin\phi$$

$$\sin\phi = L_9 \Big/ -\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}$$

- No sign Ambiguity

DLT → IOP & EOP

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

$$L_{10} = \lambda\, r_{23} = -\lambda \sin\omega \cos\phi$$

$$L_{11} = \lambda\, r_{33} = \lambda \cos\omega \cos\phi$$

$$\tan\omega = {-L_{10}} / {L_{11}}$$

No Sign Ambiguity

DLT → IOP & EOP

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \lambda \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

- Retrieve κ

$$\cos\kappa = r_{11} / \cos\phi$$

- Note: There is an ambiguity in κ determination ($\pm\kappa$ cannot be distinguished).

DLT $\rightarrow$ IOP & EOP

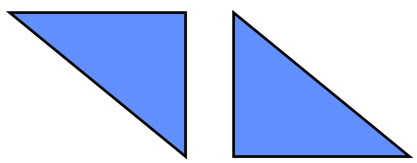
Approach # 2: Matrix Factorization

DLT → IOP (Factorization # 1)

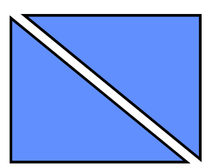
- Conceptual basis: Direct derivation of the calibration matrix

$$D\,D^T = (\lambda\,K\,R^T)(\lambda\,K\,R^T)^T = \lambda^2\,KK^T = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$D\,D^T = \lambda^2 \begin{bmatrix} -c_x & -\alpha\,c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & 0 & 0 \\ -\alpha\,c_x & -c_y & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

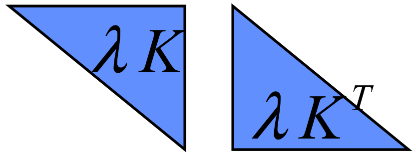


- Cholesky Decomposition of $DD^T \rightarrow \lambda K$ (Calibration Matrix)? Wrong



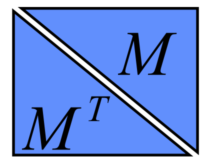
DLT → IOP (Factorization # 2)

$$N = D\,D^T$$



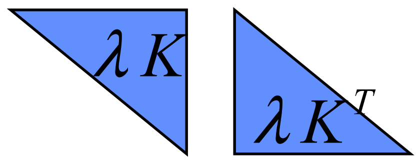
$$CHO(N^{-1}) = M$$

$$M^T\,M = N^{-1}$$



$$N^{-1} = M^T\,M$$

$$N = M^{-1}\,M^{T^{-1}} = \lambda^2\;K\;K^T$$



$$\lambda\,K = [CHO(\{DD^T\}^{-1})]^{-1}$$

DLT → Rotation Angles

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \lambda \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

- Using the rotation matrix R, one can derive the individual rotation angles ω, φ and κ.

Analysis

Perspective Center

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix}$$

$$L_1 X_o + L_2 Y_o + L_3 Z_o = -L_4$$

$$L_5 X_o + L_6 Y_o + L_7 Z_o = -L_8$$

$$L_9 X_o + L_{10} Y_o + L_{11} Z_o = -L_{12}$$

- (X_O, Y_O , Z_O) is the intersection point of three different planes whose surface normals are (L_1, L_2, L_3), (L_5, L_6, L_7) and (L_9, L_{10}, L_{11}), respectively.

Perspective Center

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- Assuming:
  - $x_p \approx 0.0$ and $y_p \approx 0.0$
  - $-\alpha c_x \approx 0.0$

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x r_{11} & -c_x r_{21} & -c_x r_{31} \\ -c_y r_{12} & -c_y r_{22} & -c_y r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- The three surfaces are orthogonal to each other.
  - This would lead to better intersection.

Perspective Center

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- Assuming:
  - $x_p \neq 0.0$ and $y_p \neq 0.0$
  - $-\alpha c_x \approx 0.0$

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x r_{11} + x_p r_{13} & -c_x r_{21} + x_p r_{23} & -c_x r_{31} + x_p r_{33} \\ -c_y r_{12} + y_p r_{13} & -c_y r_{22} + y_p r_{23} & -c_y r_{32} + y_p r_{33} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- As $\boldsymbol{x_p}$ and $\boldsymbol{y_p}$ increase, the surface normals become almost parallel.
  - This would lead to weak intersection.

Rotation Angles

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \lambda \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

- Assuming:
  - $x_p \approx 0.0$ and $y_p \approx 0.0$
  - $-\alpha c_x \approx 0.0$

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x r_{11} & -c_x r_{21} & -c_x r_{31} \\ -c_y r_{12} & -c_y r_{22} & -c_y r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- The rows of D are not correlated:
  - They are orthogonal to each other.
- L^{-1} is well defined.

Rotation Angles

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \lambda \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

- Assuming:
  - $x_p \neq 0.0$ and $y_p \neq 0.0$
  - $-\alpha c_x \approx 0.0$

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x r_{11} + x_p r_{13} & -c_x r_{21} + x_p r_{23} & -c_x r_{31} + x_p r_{33} \\ -c_y r_{12} + y_p r_{13} & -c_y r_{22} + y_p r_{23} & -c_y r_{32} + y_p r_{33} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- The rows of D tend to be highly correlated.
- L^{-1} is not well defined.