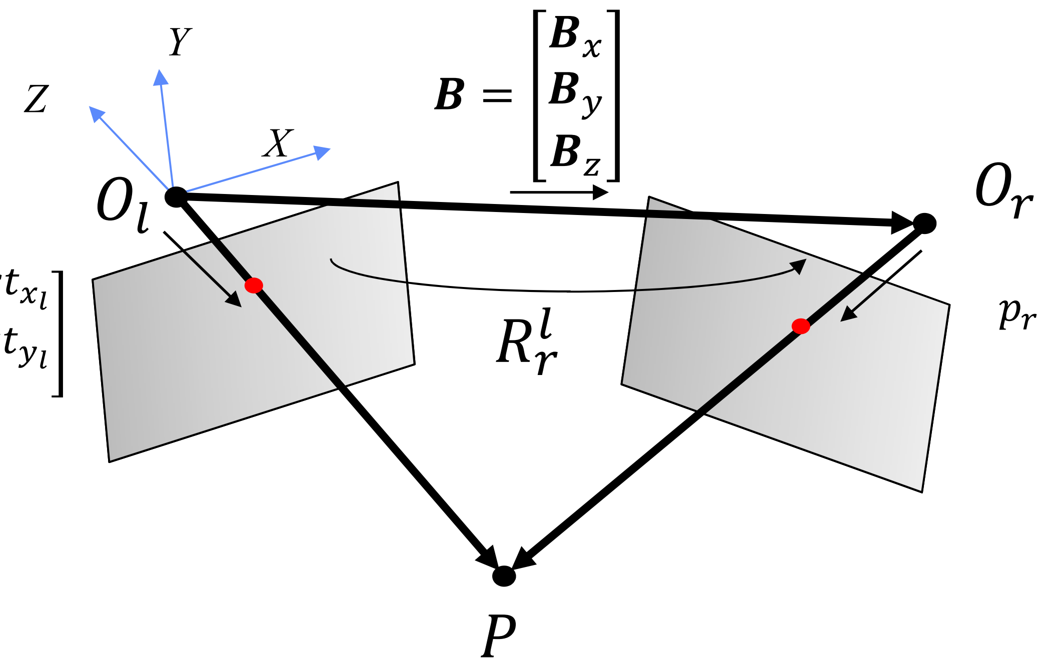
Relative Orientation

Photogrammetric Model: Coplanarity Equations

Relative Orientation

- Photogrammetric Model: Coplanarity Equations
  - The perspective centers of a stereo-pair, an object point, and the corresponding image points are coplanar.

$$p_l = \begin{bmatrix} x_l - x_p - dist_{x_l} \\ y_l - y_p - dist_{y_l} \\ -c \end{bmatrix}$$



$$p_r = \begin{bmatrix} x_r - x_p - dist_{x_r} \\ y_r - y_p - dist_{y_r} \\ -c \end{bmatrix}$$

$$p_l \cdot (\boldsymbol{B} \times \boldsymbol{R}_r^l p_r) = 0$$

$\cdot$ indicates Dot Product
$\times$ indicates Cross Product

Relative Orientation

Computer Vision Model

Relative Orientation: CV Model

For the Left Image

$$\begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = \lambda_l K_l R_m^{c_l} [I_3 \quad -X_{O_l}] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$1/_{\lambda_l} R_{c_l}^m K_l^{-1} \begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} + X_{O_l} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

For the Right Image

$$\begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} = \lambda_r K_r R_m^{c_r} [I_3 \quad -X_{O_r}] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$1/_{\lambda_r} R_{c_r}^m K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} + X_{O_r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Relative Orientation: CV Model

$$\mathbf{{}^{1}/_{\lambda_l}\, R^{m}_{c_l} K_l^{-1} \begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} + X_{O_l} = {}^{1}/_{\lambda_r}\, R^{m}_{c_r} K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} + X_{O_r}}$$

$$\mathbf{{}^{1}/_{\lambda_l}\, R^{m}_{c_l} K_l^{-1} \begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = {}^{1}/_{\lambda_r}\, R^{m}_{c_r} K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} + X_{O_r} - X_{O_l}}$$

Assuming that the mapping frame coincides with the left image coordinate system:

$$\mathbf{{}^{1}/_{\lambda_l}\, K_l^{-1} \begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = {}^{1}/_{\lambda_r}\, R^{c_l}_{c_r} K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} + B}$$

Note: $\mathbf{B \times B = \widehat{B}B = 0}$

$$\mathbf{{}^{1}/_{\lambda_l}\, \widehat{B} K_l^{-1} \begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = {}^{1}/_{\lambda_r}\, \widehat{B} R^{c_l}_{c_r} K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} + \widehat{B}B}$$

Relative Orientation: CV Model

- Cross product of a vector with itself = 0
- $\mathbf{B} \times \mathbf{B} = \begin{bmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{bmatrix} B = \hat{B}B = 0$

$$1/_{\lambda_l}\,\hat{B}K_l^{-1}\begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = 1/_{\lambda_r}\,\hat{B}R_{c_r}^{c_l}K_r^{-1}\begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} + \hat{B}B$$

$$\Downarrow$$

$$1/_{\lambda_l}\,\hat{B}K_l^{-1}\begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = 1/_{\lambda_r}\,\hat{B}R_{c_r}^{c_l}K_r^{-1}\begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix}$$

Relative Orientation: CV Model

- $\frac{1}{\lambda_l}\hat{B}K_l^{-1}\begin{bmatrix}x_l\\y_l\\1\end{bmatrix} = \frac{1}{\lambda_r}\hat{B}R_{c_r}^{c_l}K_r^{-1}\begin{bmatrix}x_r\\y_r\\1\end{bmatrix}$

- Multiply both sides dot product with the vector $K_l^{-1}\begin{bmatrix}x_l\\y_l\\1\end{bmatrix}$

$$[x_l \quad y_l \quad 1]{K_l^{-1}}^T\hat{B}K_l^{-1}\begin{bmatrix}x_l\\y_l\\1\end{bmatrix} = \frac{\lambda_l}{\lambda_r}[x_l \quad y_l \quad 1]{K_l^{-1}}^T\hat{B}R_{c_r}^{c_l}K_r^{-1}\begin{bmatrix}x_r\\y_r\\1\end{bmatrix}$$

$$\Downarrow$$

$$0 = \frac{\lambda_l}{\lambda_r}[x_l \quad y_l \quad 1]{K_l^{-1}}^T\hat{B}R_{c_r}^{c_l}K_r^{-1}\begin{bmatrix}x_r\\y_r\\1\end{bmatrix}$$

RO: CV Model – Fundamental Matrix

$$\lambda_l/\lambda_r \begin{bmatrix} x_l & y_l & 1 \end{bmatrix} {K_l^{-1}}^T \hat{B} R_{c_r}^{c_l} K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} = 0$$

$$\lambda_l/\lambda_r \neq 0$$

$$\begin{bmatrix} x_l & y_l & 1 \end{bmatrix} {K_l^{-1}}^T \hat{B} R_{c_r}^{c_l} K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} = 0$$

$$\Downarrow$$

$$\begin{bmatrix} x_l & y_l & 1 \end{bmatrix} F \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} = 0$$

F is denoted as the Fundamental Matrix

$$F = {K_l^{-1}}^T \hat{B} R_{c_r}^{c_l} K_r^{-1}$$

RO: CV Model – Essential Matrix

$$\lambda_l/\lambda_r \begin{bmatrix} x_l & y_l & 1 \end{bmatrix} {K_l^{-1}}^T \hat{B} R_{c_r}^{c_l} K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_l & y_l & 1 \end{bmatrix} {K_l^{-1}}^T \hat{B} R_{c_r}^{c_l} K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} = 0$$

$$\Downarrow$$

$$\begin{bmatrix} x_l & y_l & 1 \end{bmatrix} {K_l^{-1}}^T E K_r^{-1} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} = 0$$

E is denoted as the Essential Matrix

$$E = \hat{B} R_{c_r}^{c_l}$$

RO: CV Model – Essential Matrix

- The Essential Matrix has nine elements that should satisfy four constraints.
  - Only five degrees of freedom
- Det (E) = 0, Rank (E) = 2
- (SVD) $E = U \sum V^T$
  - $\sum = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$
  - Det (U) = Det (V) = +1
- $EE^TE - \frac{1}{2}\,\mathrm{trace}(EE^T)E = 0$