Chapters 1 – 6

- Chapter 1:
  - Photogrammetry: Definition, introduction, and applications
- Chapters 2 – 4:
  - Electro-magnetic radiation
  - Optics
  - Film development and digital cameras
- Chapter 5:
  - Vertical imagery: Definitions, image scale, and relief displacement
- Chapter 6:
  - Measurement, transformation, and refinement of image coordinates

CE59700: Chapter 7

Mathematical Model

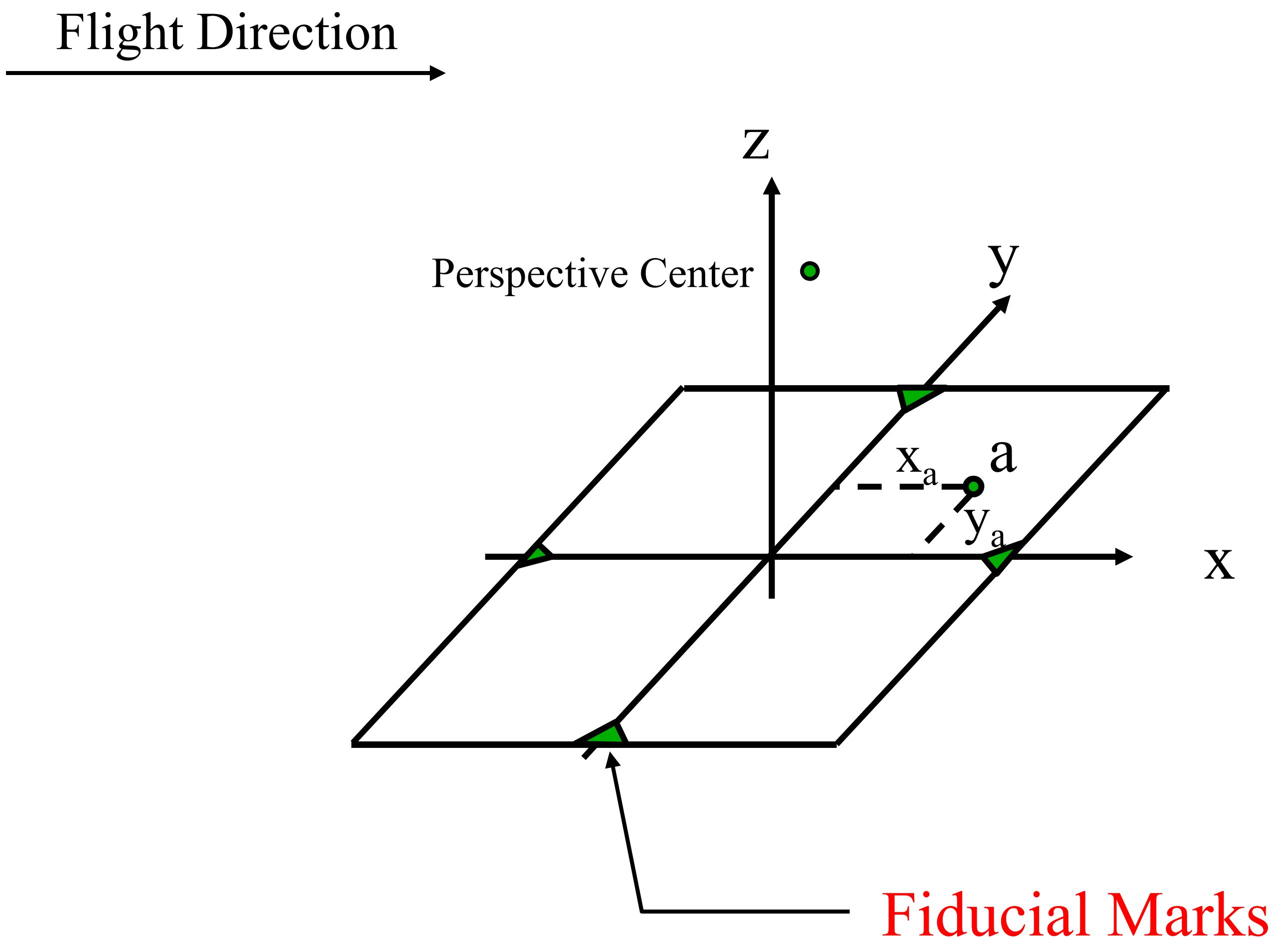
Overview

- Mathematical model: Objectives
- Mathematical model: Alternatives
- Rotation matrices (2-D and 3-D)
  - Derivation and characteristics
- Collinearity equations
  - Concept and derivation
- Bundle block adjustment
  - Concept and objectives
- Least squares adjustment in photogrammetry

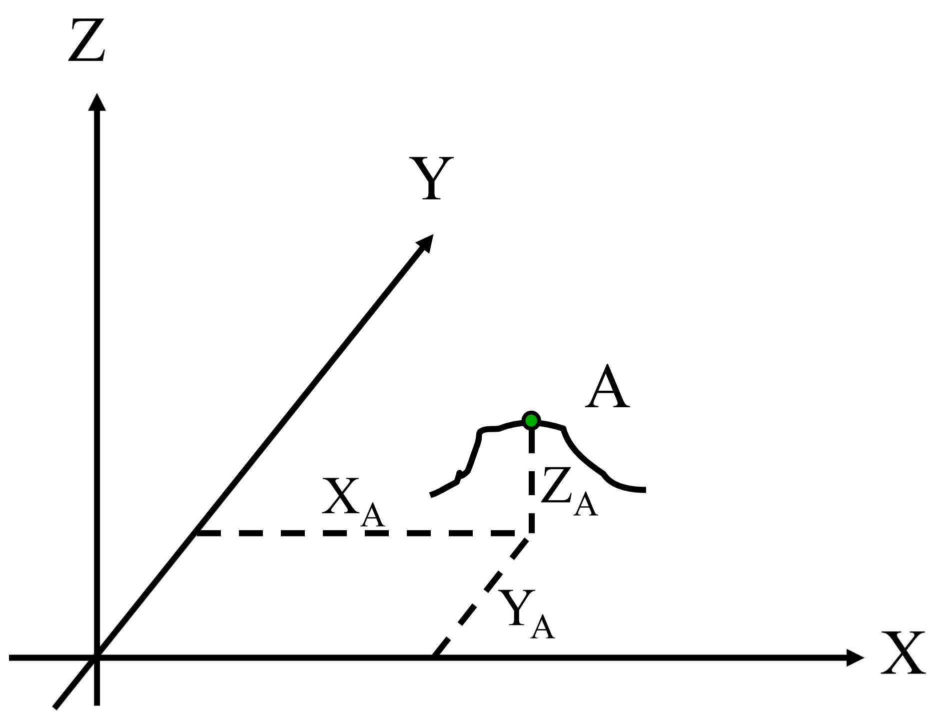
Mathematical Model

- Objective: Develop a general mathematical relationship between image and ground coordinates
- Alternatives:
  - Projective Transformation
  - Collinearity Equations
  - Direct Linear Transformation (DLT)
- The mathematical model of choice for most of the photogrammetric applications is the Collinearity Equations.

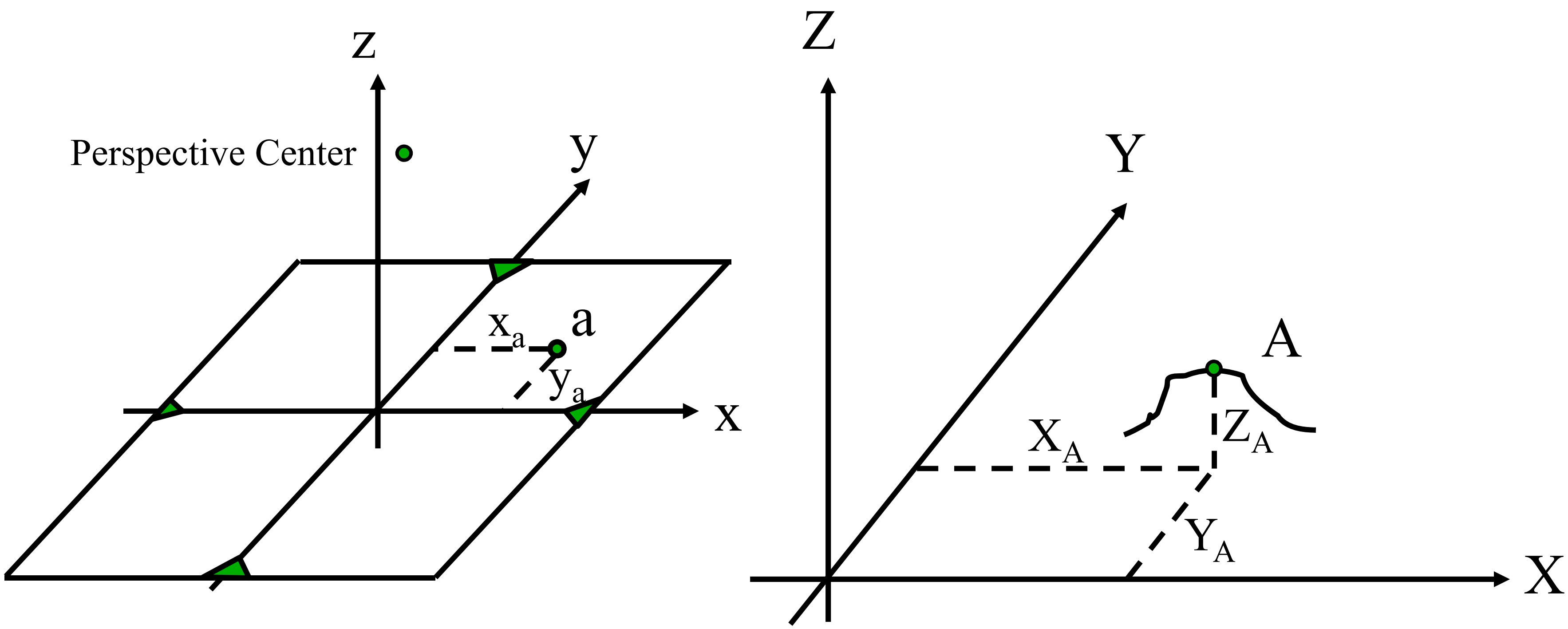
Image Coordinate System: Diapositive



Ground Coordinate System



Mathematical Model



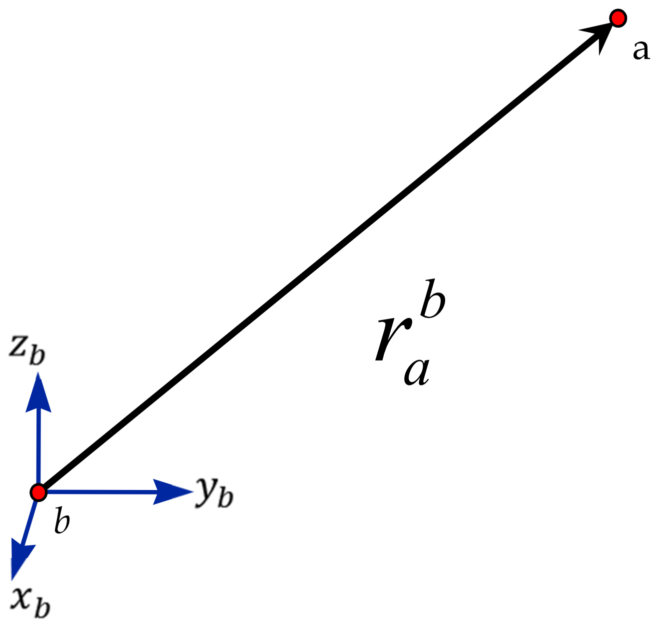
$$x_a = f_x(X_A, Y_A, Z_A, ...)$$

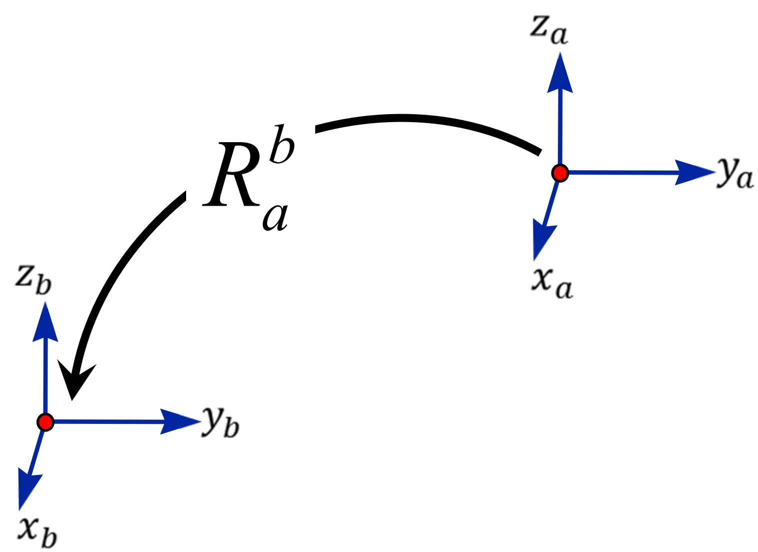
$$y_a = f_y(X_A, Y_A, Z_A, ...)$$

Mathematical Model

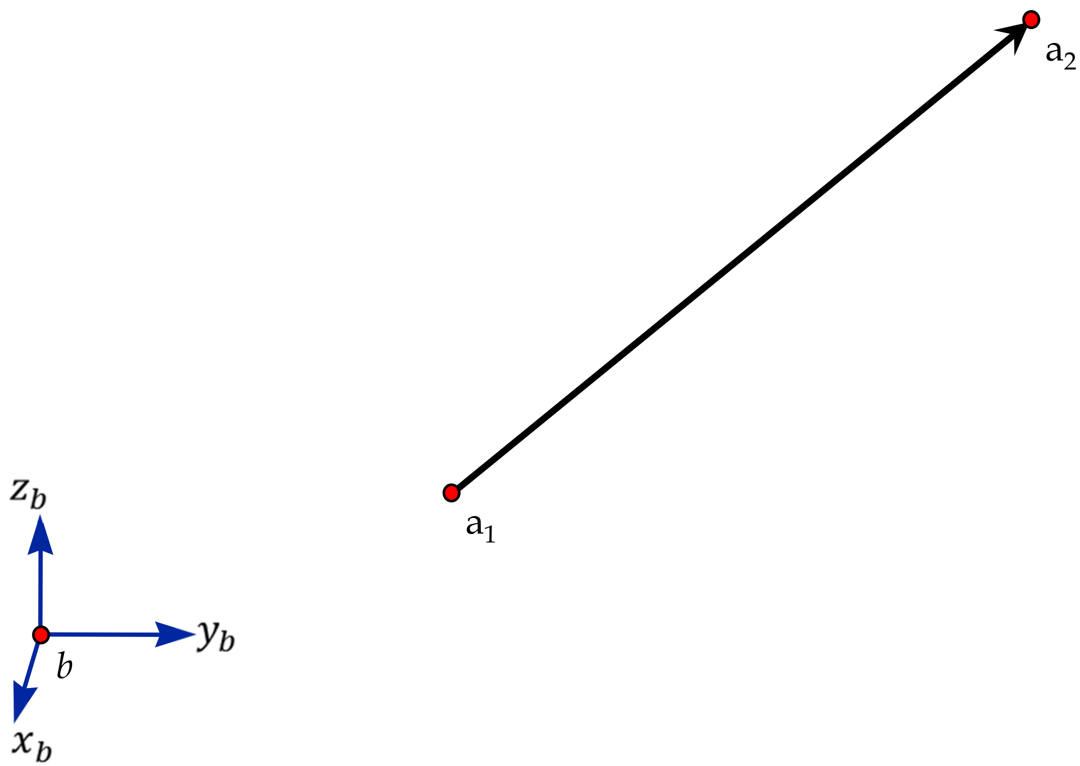
- Before looking into the mathematical model, we have to investigate the concept of rotation matrices.
  - Rotation in a plane (2D Rotation Matrices)
  - Rotation in space (3D Rotation Matrices)
  - Derivation
  - Characteristics

Notations





Notations

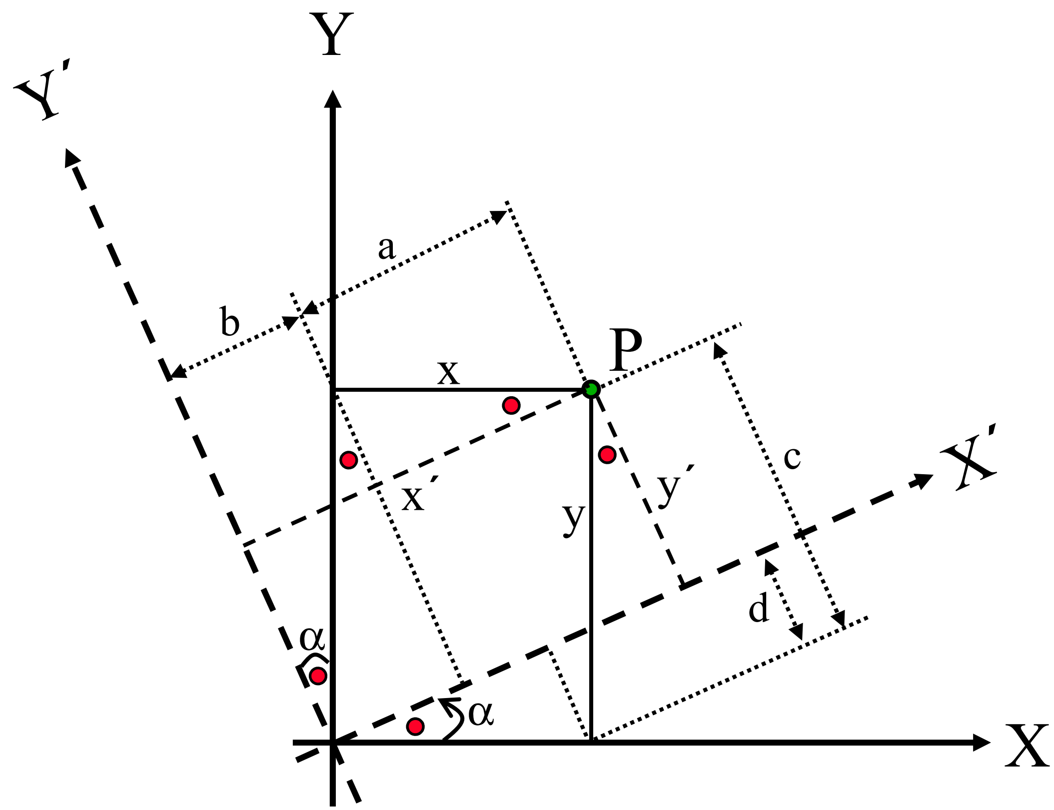


$$r^{b}_{a_1 a_2}$$

Rotation Matrices

- Rotation in a plane:
  - Given a point p(x, y) w.r.t. an xy-coordinate system
  - The system is rotated with an angle α yielding a new coordinate system x`y`.
  - We would like to express the coordinates of the point (p) w.r.t. the new system (x`, y`) as a function of:
    - The old coordinates (x, y), and
    - The rotation angle α.

Rotation in a Plane



Rotation in a Plane

$$x' = a + b = x \cos\alpha + y \sin\alpha$$

$$y' = c - d = -x \sin\alpha + y \cos\alpha$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where:

$$Rotation\ \ matrix\ (R_{xy}^{x'y'}) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Properties of the Rotation Matrix

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

where :

$r_{11} = \cos\ \alpha$

$r_{12} = \sin\ \alpha$

$r_{21} = -\ \sin\ \alpha$

$r_{22} = \cos\ \alpha$

$$r_{11}^2 + r_{21}^2 = \cos^2\alpha + \sin^2\alpha = 1$$

$$r_{12}^2 + r_{22}^2 = \sin^2\alpha + \cos^2\alpha = 1$$

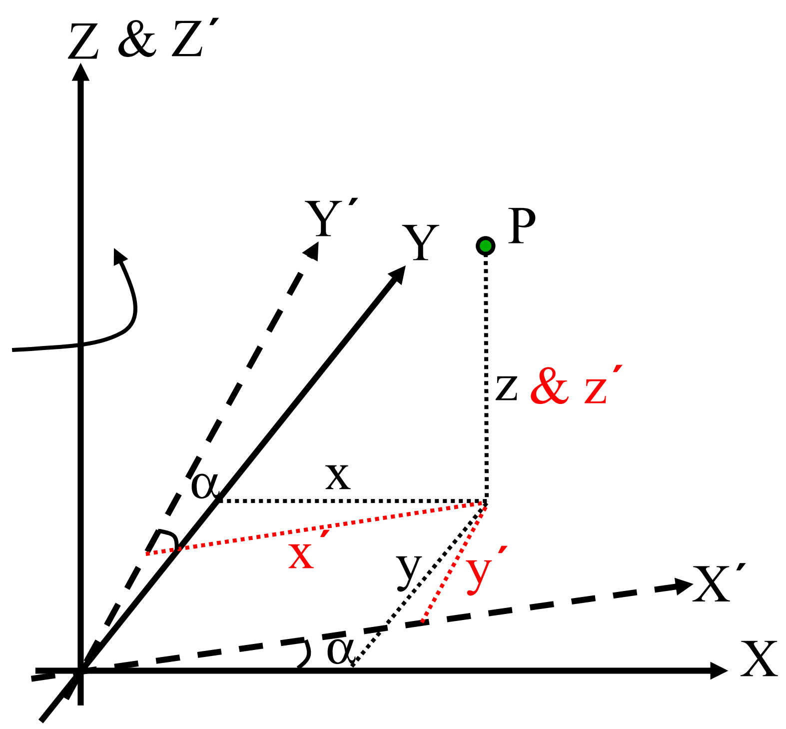
$$r_{11}\ r_{12} + r_{21}\ r_{22} = \sin\alpha\ \cos\alpha - \sin\alpha\ \cos\alpha = 0$$

$$R^T R = R R^T = I_2$$

Properties of the Rotation Matrix

- The above mentioned properties are known as the orthogonality conditions.
- The rotation matrix has four elements.
- These elements must satisfy three constraints (orthogonality conditions).
- Therefore, the rotation matrix is completely defined by one independent parameter (α).

Rotation in Space



Rotation in Space

$$x` = x\cos\alpha + y\sin\alpha$$

$$y` = -x\sin\alpha + y\cos\alpha$$

$$z` = z$$

$$\begin{bmatrix} x` \\ y` \\ z` \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_{xyz}^{x`y`z`} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Where :

$$R_{xyz}^{x`y`z`} = rotation\ matrix = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonality Conditions

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$r_{11}^2 + r_{21}^2 + r_{31}^2 = 1$$

$$r_{12}^2 + r_{22}^2 + r_{32}^2 = 1$$

$$r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$$

$$r_{11}\, r_{12} + r_{21}\, r_{22} + r_{31}\, r_{32} = 0$$

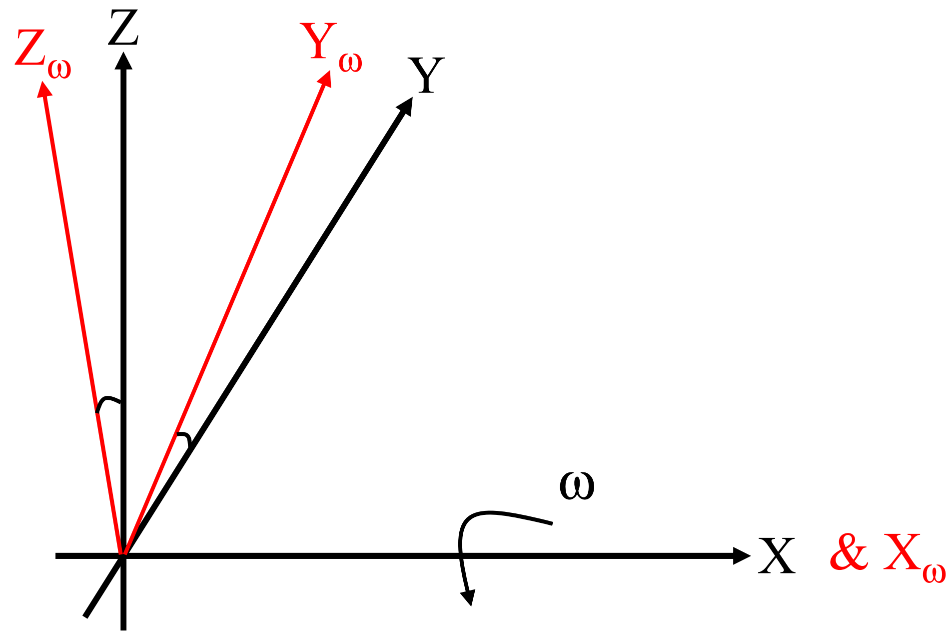
$$r_{11}\, r_{13} + r_{21}\, r_{23} + r_{31}\, r_{33} = 0$$

$$r_{12}\, r_{13} + r_{22}\, r_{23} + r_{32}\, r_{33} = 0$$

Rotation in Space

- In 3-D, the rotation matrix has nine elements.
- These nine elements must satisfy six orthogonality conditions.
- Therefore, any 3-D rotation matrix can be defined by only three independent parameters.
- In photogrammetry, we use the three rotation angles ω, ϕ, and κ.

Primary Rotation (ω)



Primary Rotation (ω)

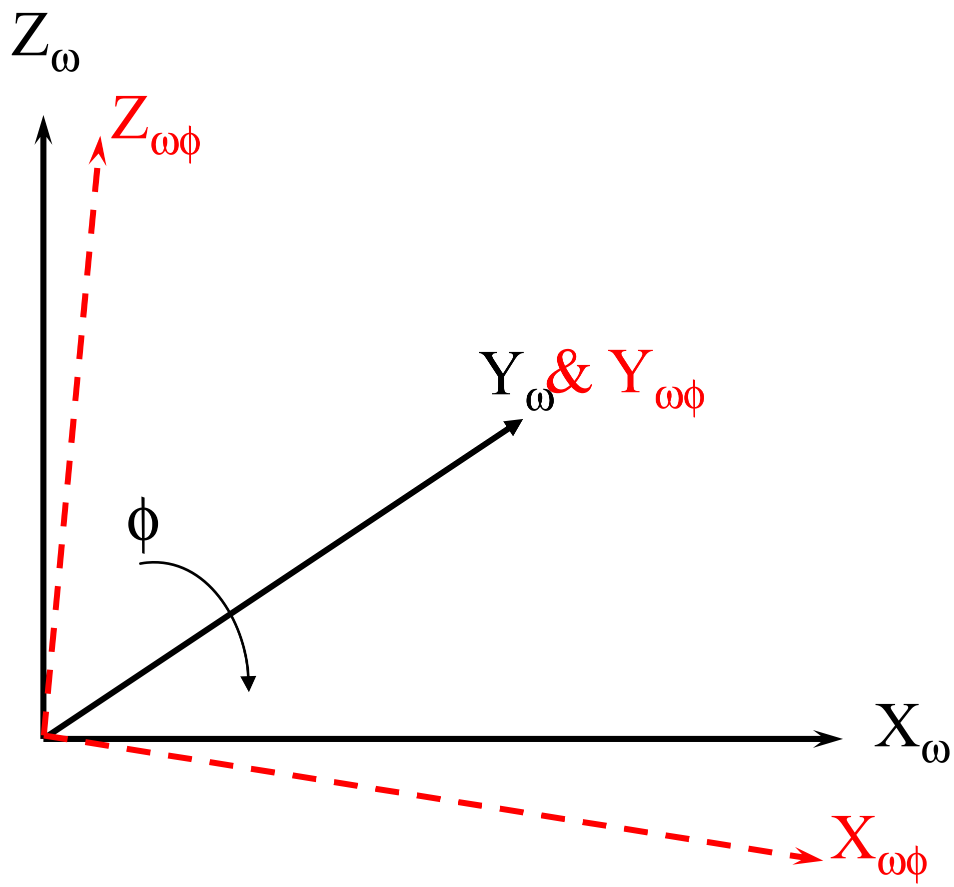
$$\begin{bmatrix} x_\omega \\ y_\omega \\ z_\omega \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x_\omega \\ y_\omega \\ z_\omega \end{bmatrix} = M_\omega \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x_\omega \\ y_\omega \\ z_\omega \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_\omega \begin{bmatrix} x_\omega \\ y_\omega \\ z_\omega \end{bmatrix}$$

Secondary Rotation (ϕ)



Secondary Rotation (ϕ)

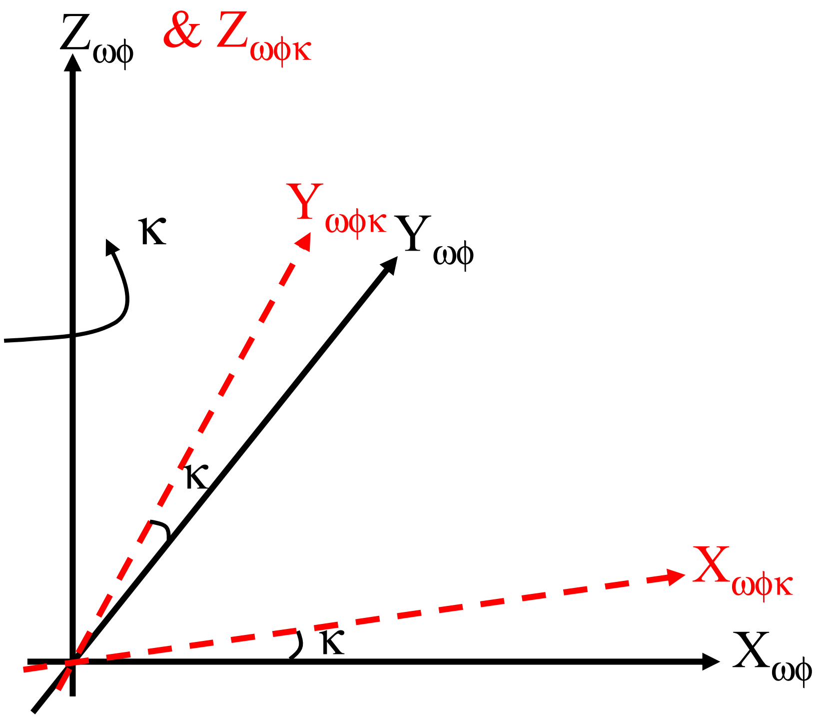
$$\begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} x_{\omega} \\ y_{\omega} \\ z_{\omega} \end{bmatrix}$$

$$\begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix} = M_{\phi} \begin{bmatrix} x_{\omega} \\ y_{\omega} \\ z_{\omega} \end{bmatrix}$$

$$\begin{bmatrix} x_{\omega} \\ y_{\omega} \\ z_{\omega} \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix}$$

$$\begin{bmatrix} x_{\omega} \\ y_{\omega} \\ z_{\omega} \end{bmatrix} = R_{\phi} \begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix}$$

Tertiary Rotation (κ)



Tertiary Rotation (κ)

$$\begin{bmatrix} x_{\omega\phi\kappa} \\ y_{\omega\phi\kappa} \\ z_{\omega\phi\kappa} \end{bmatrix} = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix}$$

$$\begin{bmatrix} x_{\omega\phi\kappa} \\ y_{\omega\phi\kappa} \\ z_{\omega\phi\kappa} \end{bmatrix} = M_{\kappa} \begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix}$$

$$\begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\omega\phi\kappa} \\ y_{\omega\phi\kappa} \\ z_{\omega\phi\kappa} \end{bmatrix}$$

$$\begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix} = R_{\kappa} \begin{bmatrix} x_{\omega\phi\kappa} \\ y_{\omega\phi\kappa} \\ z_{\omega\phi\kappa} \end{bmatrix}$$

Rotation in Space

$$\begin{bmatrix} x_{\omega\phi\kappa} \\ y_{\omega\phi\kappa} \\ z_{\omega\phi\kappa} \end{bmatrix} = M_\kappa M_\phi M_\omega \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

// to the image coordinate system // to the ground coordinate system

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_\omega R_\phi R_\kappa \begin{bmatrix} x_{\omega\phi\kappa} \\ y_{\omega\phi\kappa} \\ z_{\omega\phi\kappa} \end{bmatrix}$$

// to the ground coordinate system // to the image coordinate system

Rotation in Space

$$M_{\kappa} M_{\phi} M_{\omega} = M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

where :

$$m_{11} = \cos\phi \cos\kappa$$

$$m_{12} = \cos\omega \sin\kappa + \sin\omega \sin\phi \cos\kappa$$

$$m_{13} = \sin\omega \sin\kappa - \cos\omega \sin\phi \cos\kappa$$

$$m_{21} = -\cos\phi \sin\kappa$$

$$m_{22} = \cos\omega \cos\kappa - \sin\omega \sin\phi \sin\kappa$$

$$m_{23} = \sin\omega \cos\kappa + \cos\omega \sin\phi \sin\kappa$$

$$m_{31} = \sin\phi$$

$$m_{32} = -\sin\omega \cos\phi$$

$$m_{33} = \cos\omega \cos\phi$$

Rotation in Space

$$R_{\omega} R_{\phi} R_{\kappa} = R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

where :

$$r_{11} = \cos\phi \cos\kappa$$

$$r_{12} = -\cos\phi \sin\kappa$$

$$r_{13} = \sin\phi$$

$$r_{21} = \cos\omega \sin\kappa + \sin\omega \sin\phi \cos\kappa$$

$$r_{22} = \cos\omega \cos\kappa - \sin\omega \sin\phi \sin\kappa$$

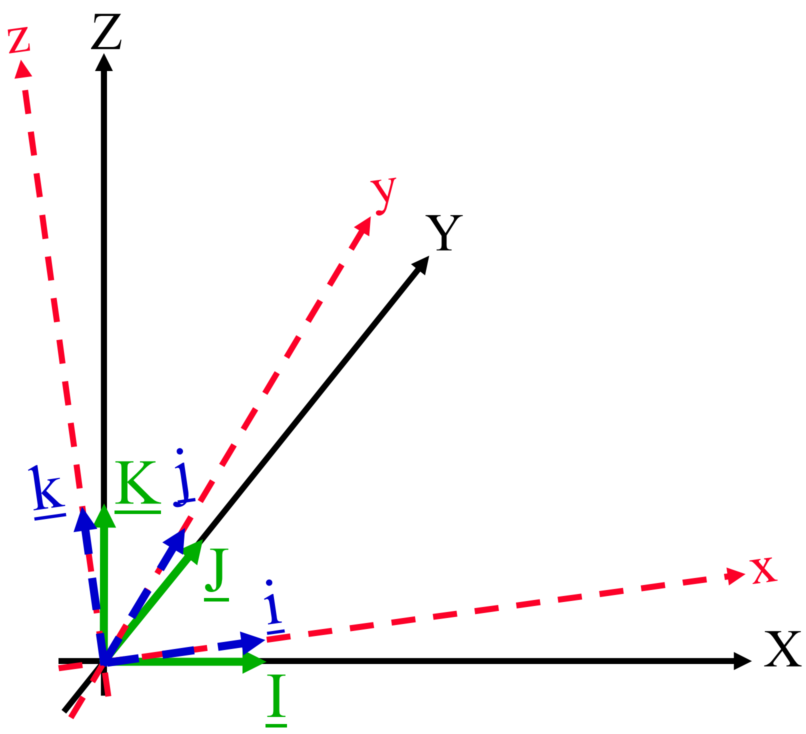
$$r_{23} = -\sin\omega \cos\phi$$

$$r_{31} = \sin\omega \sin\kappa - \cos\omega \sin\phi \cos\kappa$$

$$r_{32} = \sin\omega \cos\kappa + \cos\omega \sin\phi \sin\kappa$$

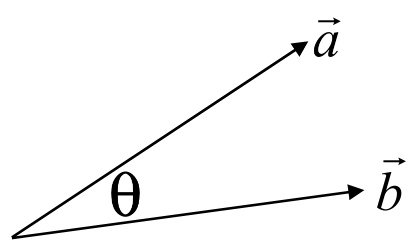
$$r_{33} = \cos\omega \cos\phi$$

Rotation Matrix: Alternative Derivation (1)



Rotation Matrix: Alternative Derivation (1)

- XYZ: Ground coordinate system
  - I, J, and K: unit vectors along the X, Y, and Z axes, respectively
- xyz: Image coordinate system
  - i, j, and k: unit vectors along the x, y, and z axes, respectively
- Remember: The component of a unit vector along another unit vector can be obtained through the dot product of these vectors.



$$\vec{a} \bullet \vec{b} = \|a\| \, \|b\| \cos\theta$$

Rotation Matrix: Alternative Derivation (1)

$$\vec{i} = \begin{bmatrix} \cos(xX) \\ \cos(xY) \\ \cos(xZ) \end{bmatrix} \quad \vec{j} = \begin{bmatrix} \cos(yX) \\ \cos(yY) \\ \cos(yZ) \end{bmatrix} \quad \vec{k} = \begin{bmatrix} \cos(zX) \\ \cos(zY) \\ \cos(zZ) \end{bmatrix}$$

- The above vectors represent the components of the unit vectors along the image coordinate system with respect to the ground coordinate system.

$$\vec{i} = \cos(xX)\,\vec{I} + \cos(xY)\,\vec{J} + \cos(xZ)\,\vec{K}$$

$$\vec{j} = \cos(yX)\,\vec{I} + \cos(yY)\,\vec{J} + \cos(yZ)\,\vec{K}$$

$$\vec{k} = \cos(zX)\,\vec{I} + \cos(zY)\,\vec{J} + \cos(zZ)\,\vec{K}$$

Rotation Matrix: Alternative Derivation (1)

- Let's consider a point whose coordinates relative to the image coordinate system are (x, y, z).

$$\vec{p} = x\vec{i} + y\vec{j} + z\vec{k}$$

w.r.t. the image coordinate system

$$\vec{P} = X\vec{I} + Y\vec{J} + Z\vec{K}$$

w.r.t. the ground coordinate system

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = x\begin{bmatrix} \cos(xX) \\ \cos(xY) \\ \cos(xZ) \end{bmatrix} + y\begin{bmatrix} \cos(yX) \\ \cos(yY) \\ \cos(yZ) \end{bmatrix} + z\begin{bmatrix} \cos(zX) \\ \cos(zY) \\ \cos(zZ) \end{bmatrix}$$

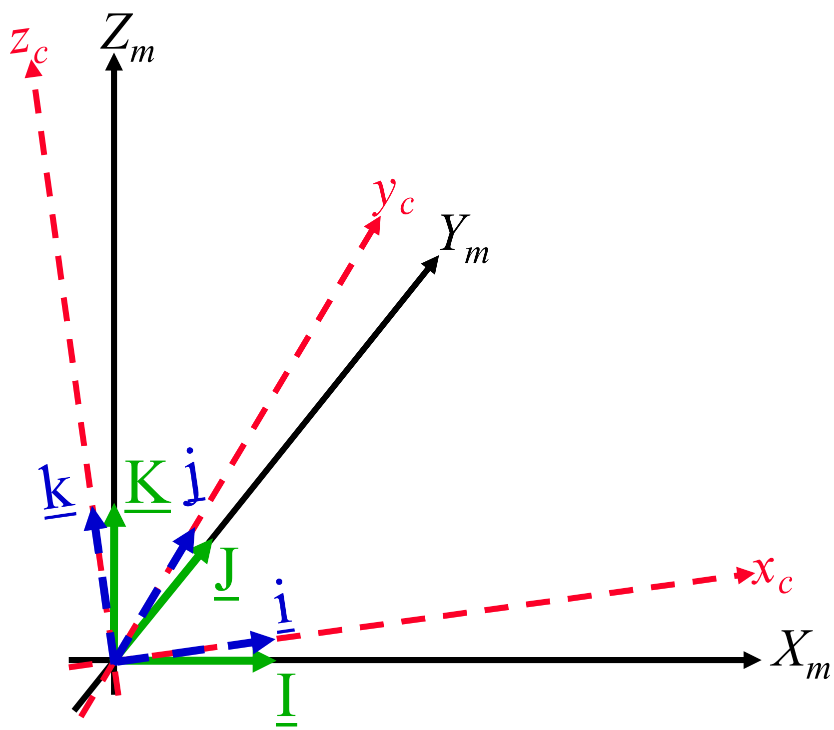
Rotation Matrix: Alternative Derivation (1)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos(xX) & \cos(yX) & \cos(zX) \\ \cos(xY) & \cos(yY) & \cos(zY) \\ \cos(xZ) & \cos(yZ) & \cos(zZ) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R_{xyz}^{XYZ} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rotation Matrix: Alternative Derivation (2)

- A rotation matrix transforms a vector from one coordinate system to another.



$$r_a^m = R_c^m \, r_a^c$$

$$R_c^m = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Rotation Matrix: Alternative Derivation (2)

- Let's consider the transformation of a unit vector along the x-axis of the camera coordinate system

$$r_a^m = R_c^m \, r_a^c$$

$$\begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} = R_c^m \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

  - The first column of the rotation matrix represents the components of a unit vector along the x-axis of the camera coordinate system w.r.t. the mapping reference frame.
  - The norm of the first column is unity.

$$r_{11}^2 + r_{21}^2 + r_{31}^2 = 1 \qquad 1$$

Rotation Matrix: Alternative Derivation (2)

- Let's consider the transformation of a unit vector along the y-axis of the camera coordinate system

$$r_a^m = R_c^m \, r_a^c$$

$$\begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} = R_c^m \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

  - The second column of the rotation matrix represents the components of a unit vector along the y-axis of the camera coordinate system w.r.t. the mapping reference frame.
  - The norm of the second column is unity.

$$r_{12}^2 + r_{22}^2 + r_{32}^2 = 1 \qquad 2$$

Rotation Matrix: Alternative Derivation (2)

- Let's consider the transformation of a unit vector along the z-axis of the camera coordinate system

$$r_a^m = R_c^m \, r_a^c$$

$$\begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} = R_c^m \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - The third column of the rotation matrix represents the components of a unit vector along the z-axis of the camera coordinate system w.r.t. the mapping reference frame.
  - The norm of the third column is unity.

$$r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$$

3

Rotation Matrix: Alternative Derivation (2)

- Since the x and y axes of the camera coordinate system are orthogonal to each other, then

$$\begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \bullet \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} = 0 \qquad r_{11} r_{12} + r_{21} r_{22} + r_{31} r_{32} = 0 \qquad 4$$

  - Since the x and z axes of the camera coordinate system are orthogonal to each other, then

$$\begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \bullet \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} = 0 \qquad r_{11} r_{13} + r_{21} r_{23} + r_{31} r_{33} = 0 \qquad 5$$

Rotation Matrix: Alternative Derivation (2)

- Since the y and z axes of the camera coordinate system are orthogonal to each other, then

$$\begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} \bullet \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} = 0 \qquad r_{12}\, r_{13} + r_{22}\, r_{23} + r_{32}\, r_{33} = 0 \quad 6$$

  - Since the nine elements of a rotation matrix must satisfy six constraints (orthogonality constraints), a 3D rotation matrix is defined by a maximum of three independent parameters/rotation angles.
  - In photogrammetry, the rotation matrix is defined by the angles (ω, ϕ, and κ).

Rotation Matrix: Orthogonality Conditions

$$R = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{bmatrix}$$

- $\vec{r}_1$: The components of a unit vector along the x-axis of the camera coordinate system w.r.t. the ground coordinate system
- $\vec{r}_2$: The components of a unit vector along the y-axis of the camera coordinate system w.r.t. the ground coordinate system
- $\vec{r}_3$: The components of a unit vector along the z-axis of the camera coordinate system w.r.t. the ground coordinate system

Rotation Matrix: Orthogonality Conditions

$$\left\|\vec{r}_1\right\| = 1 \quad \left\|\vec{r}_2\right\| = 1 \quad \left\|\vec{r}_3\right\| = 1$$

$$\vec{r}_1 \bullet \vec{r}_2 = 0.0$$

$$\vec{r}_1 \bullet \vec{r}_3 = 0.0$$

$$\vec{r}_2 \bullet \vec{r}_3 = 0.0$$

- These are the orthogonality conditions for rotation matrices.

$$R^{-1} = R^T$$

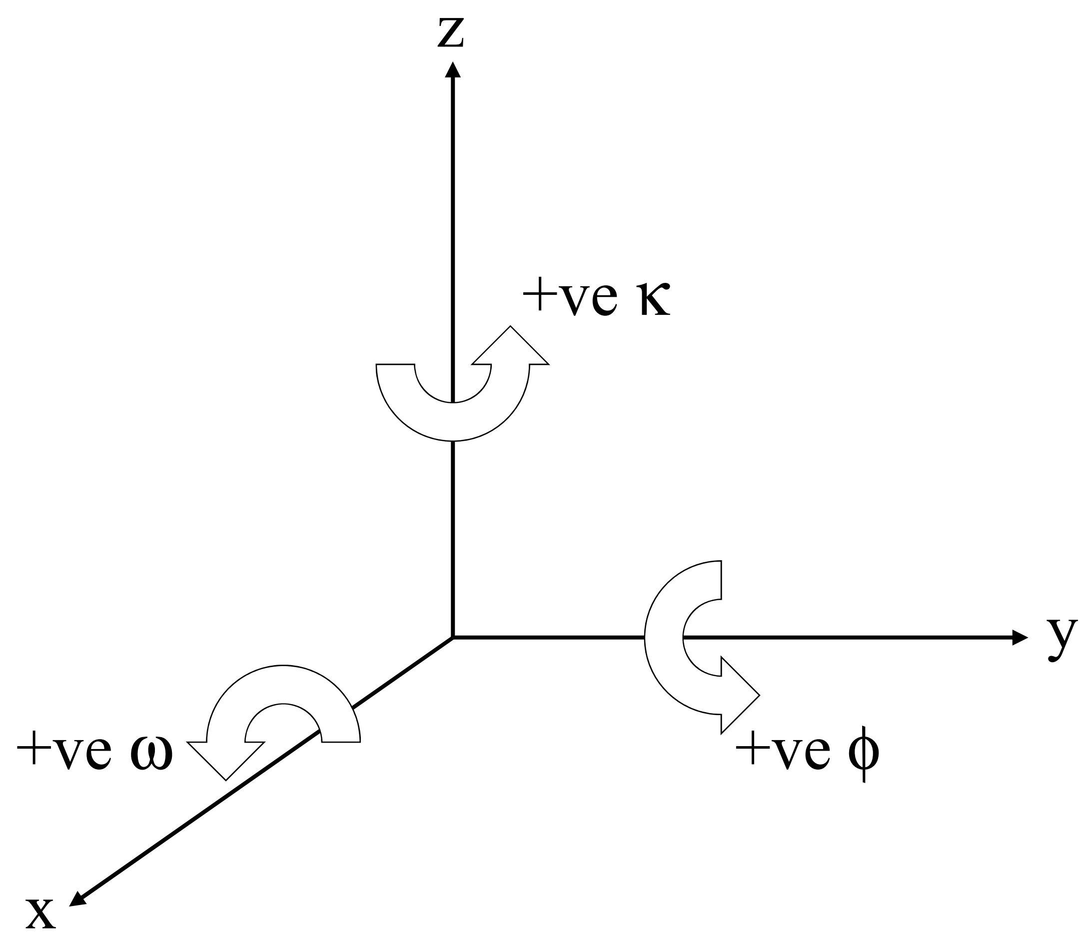
$$R^T R = R R^T = I_3$$

Rotation in Space: Final Remarks

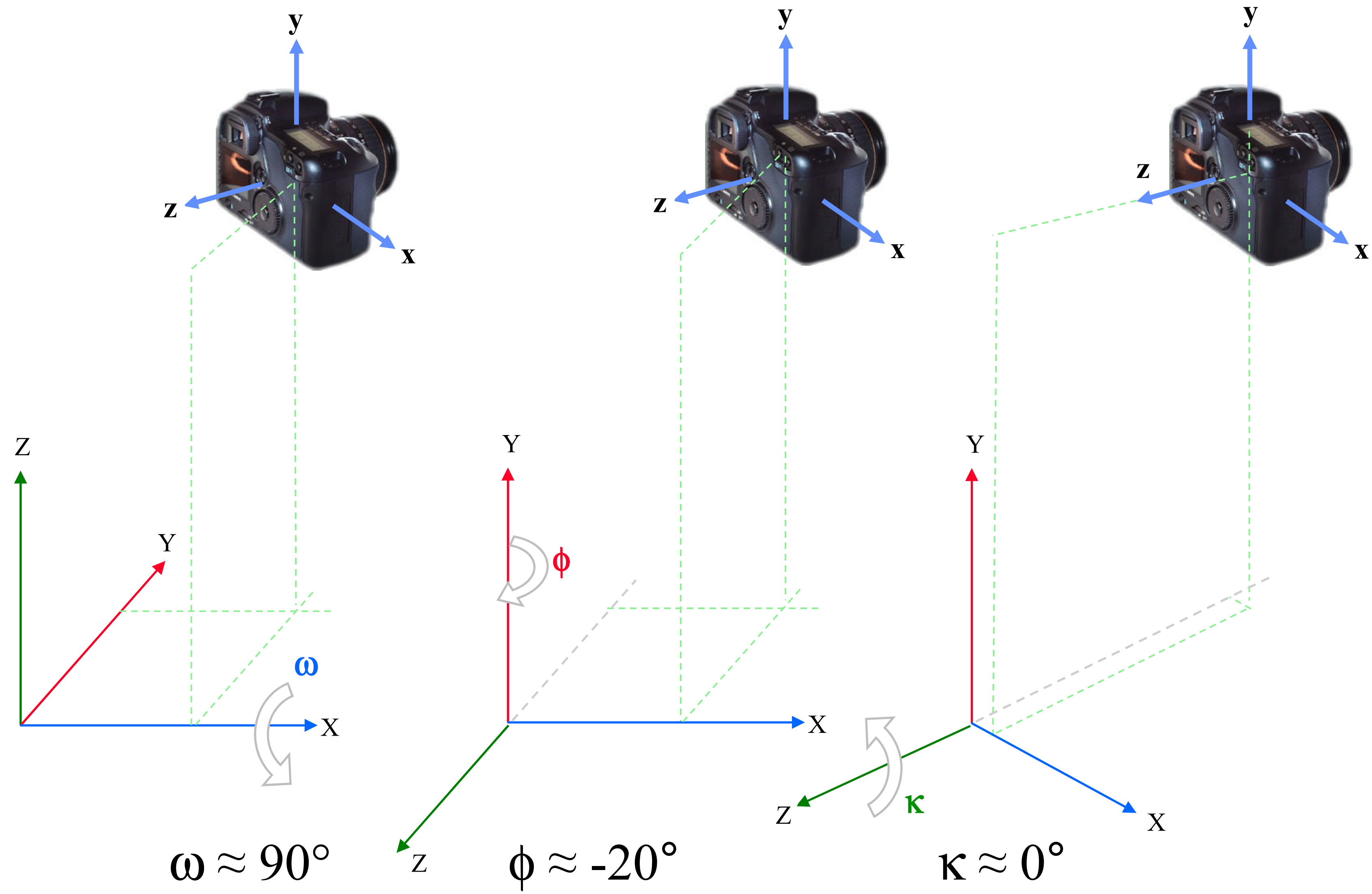
- The elements of the rotation matrix depend on the sequence of rotations.
  - i.e., $R_\kappa \, R_\phi \, R_\omega \neq R_\kappa \, R_\omega \, R_\phi$.
- A positive rotation angle is defined as the one that is counter clock wise when looking at the system with the positive direction of the axis of rotation is pointing towards us.
- The angles (ω, ϕ, κ) are the rotation angles that need to be applied to the ground coordinate system until it becomes parallel to the image coordinate system.

Positive Rotation Angles

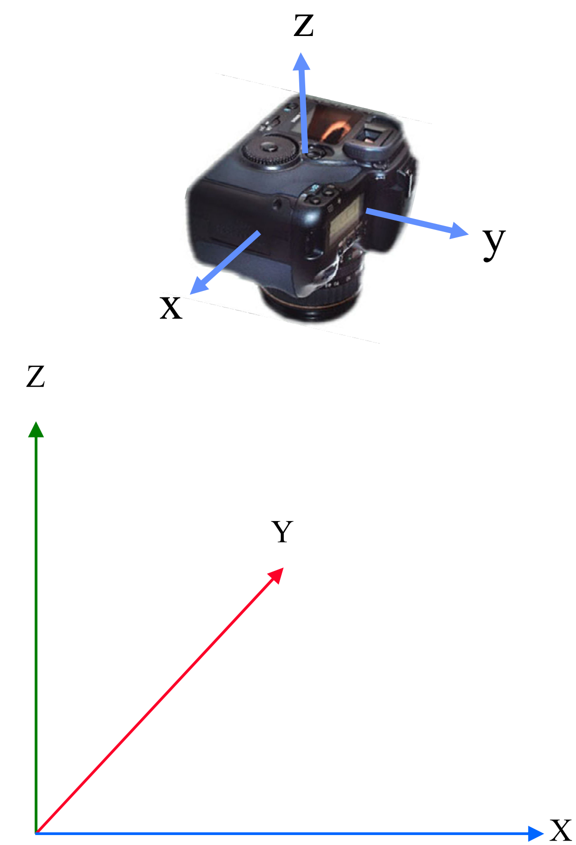
(Right Handed System)



Rotation Angles (ω, ϕ, κ)

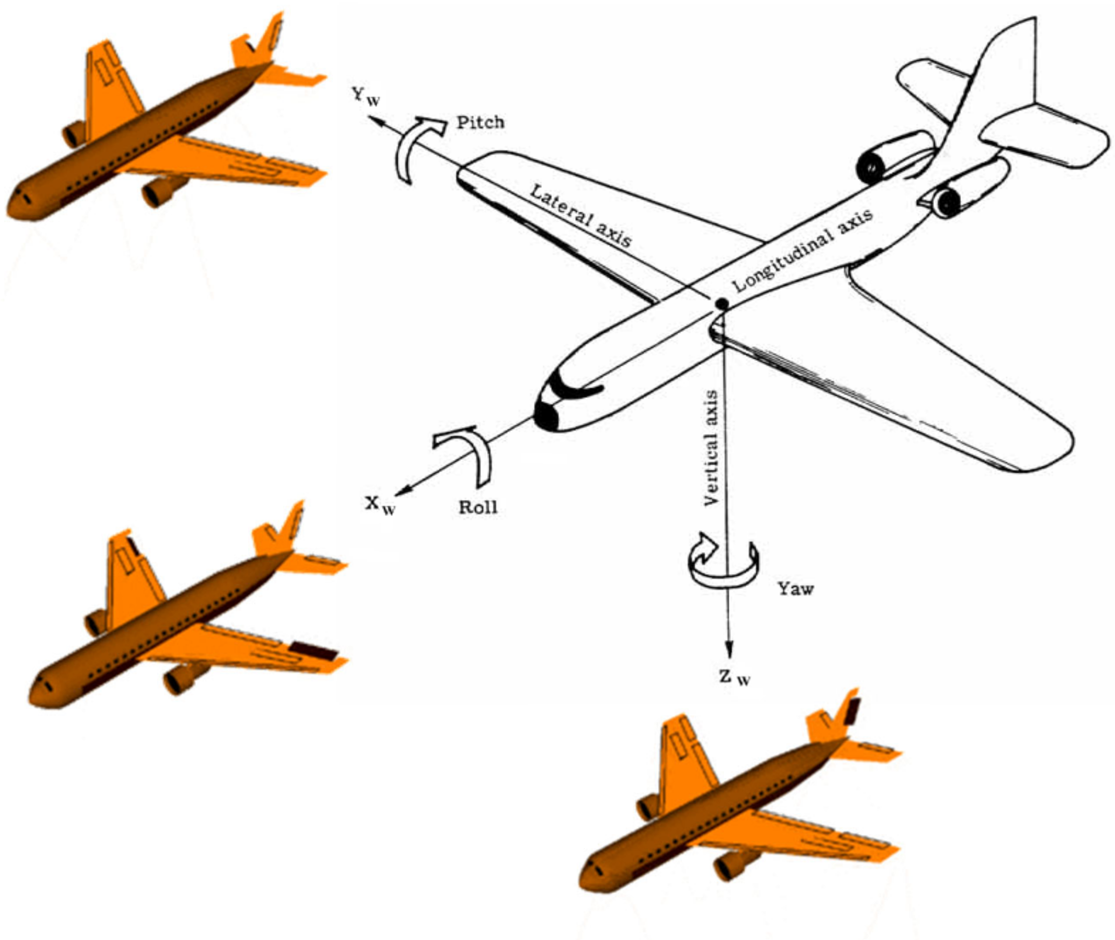


Rotation Angles (ω, ϕ, κ)



$\omega \approx ?$, $\phi \approx ?$, and $\kappa \approx ?$

Rotation Angles (Azimuth, Pitch, Roll)



Azimuth ≡ Yaw

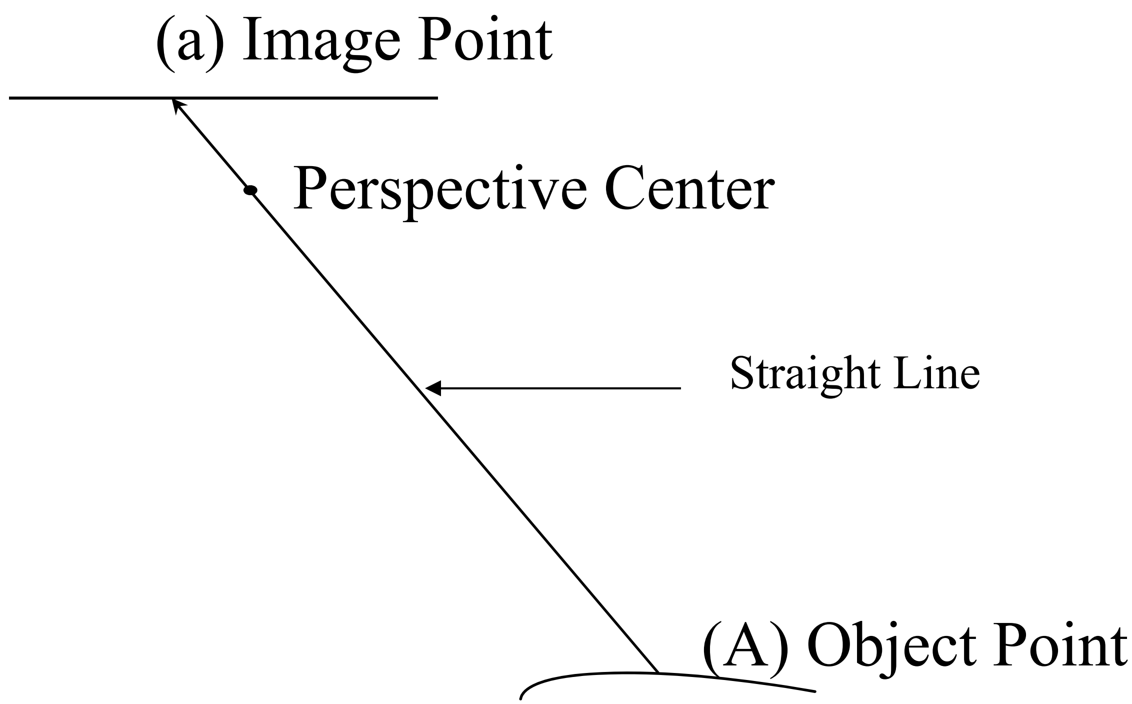
http://mtp.mjmahoney.net/www/notes/pointing/pointing.html

http://en.wikipedia.org/wiki/Aircraft_principal_axes

Collinearity Equations

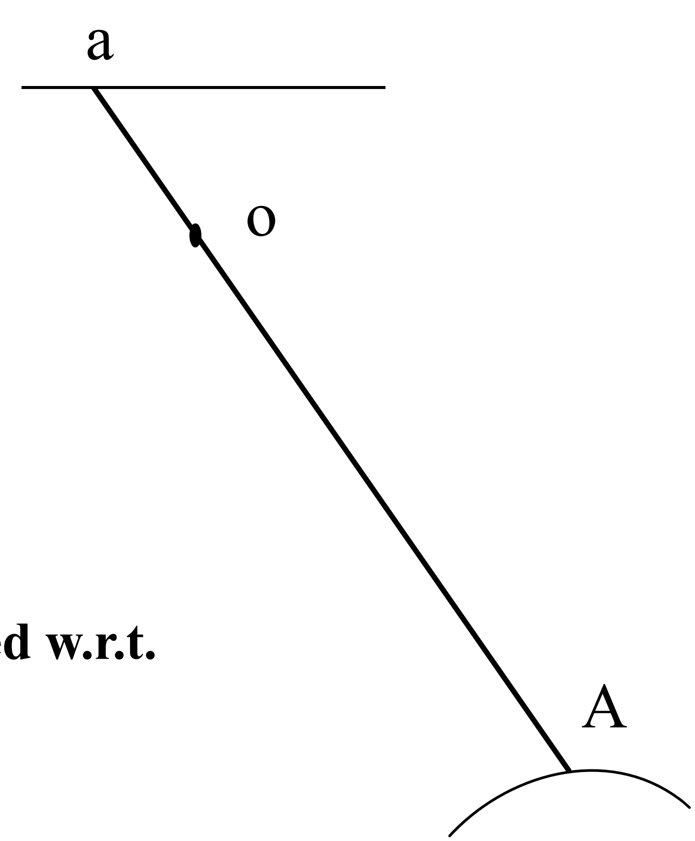
- Objective:
  - Mathematically represent the general relationship between corresponding image and ground coordinates
- Concept:
  - Image Point, Object Point, and the Perspective Center are collinear.

Collinearity Equations



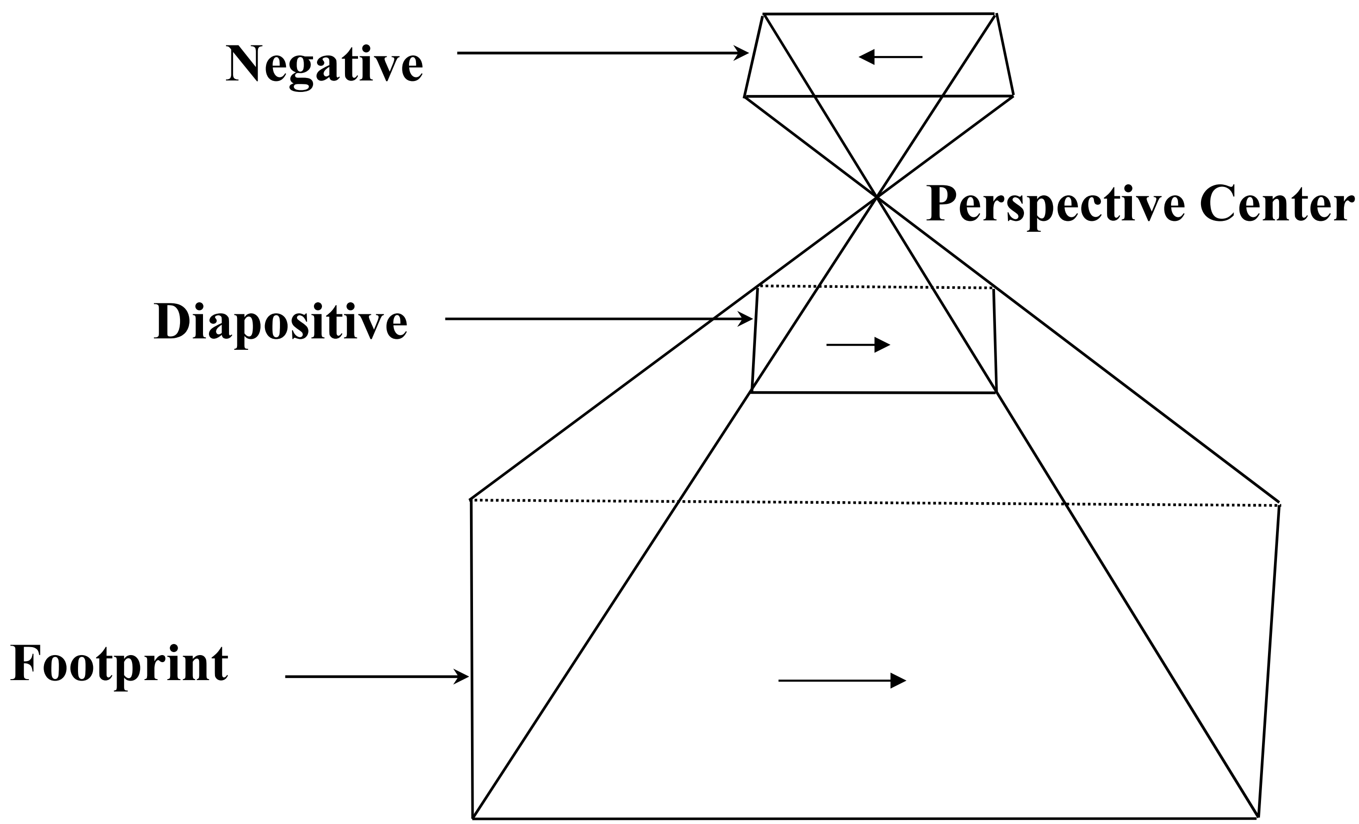
Collinearity Equations

$$\vec{oa} = \lambda \ \vec{oA}$$



These vectors should be defined w.r.t. the same coordinate system.

Frame Camera

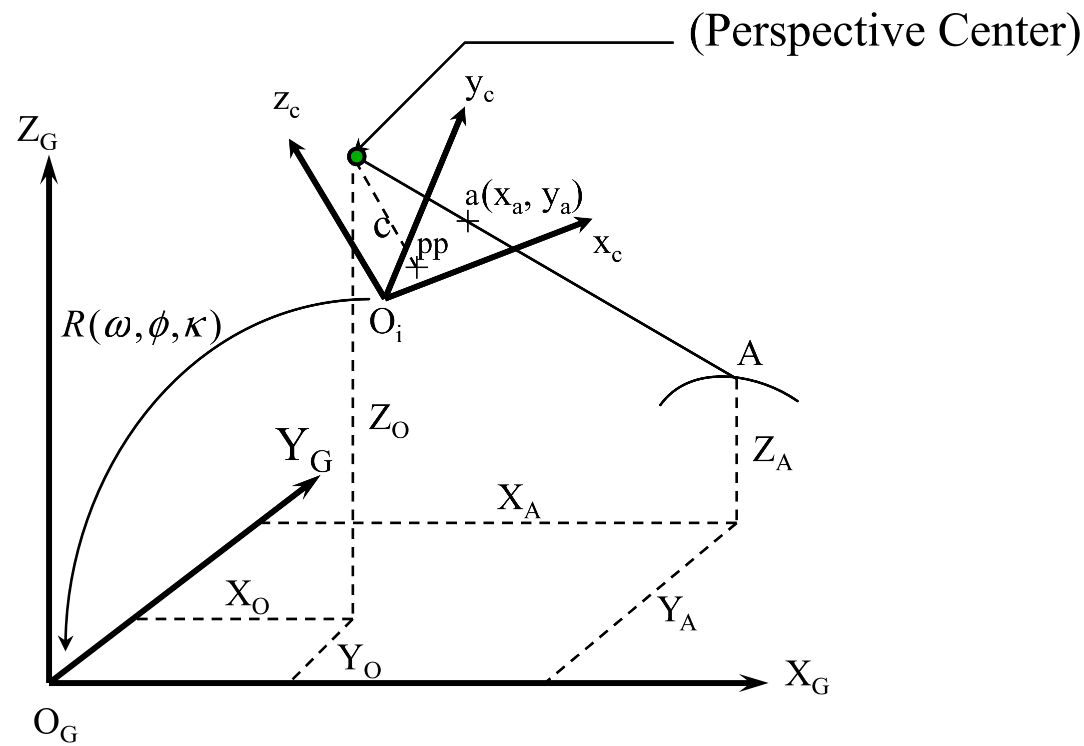


Negative Versus Diapositive Films

Negative Film

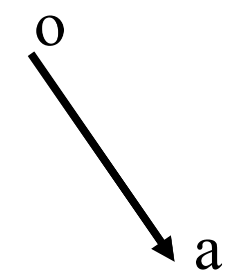
Diapositive

Collinearity Equations



Collinearity Equations

The vector connecting the perspective center to the image point



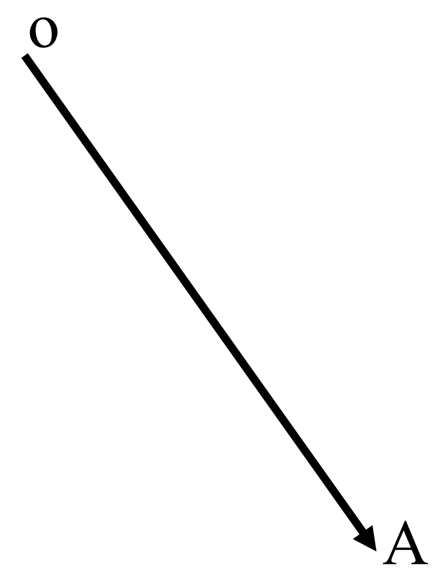
$$\vec{v}_i = r_{oa}^{c} = \begin{bmatrix} x_a - dist_x \\ y_a - dist_y \\ 0 \end{bmatrix} - \begin{bmatrix} x_p \\ y_p \\ c \end{bmatrix} = \begin{bmatrix} x_a - x_p - dist_x \\ y_a - y_p - dist_y \\ -c \end{bmatrix}$$

w.r.t. the camera/image coordinate system

Collinearity Equations

The vector connecting the perspective center to the object point

$$\vec{V}_o = r_{oA}^m = \begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} - \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} = \begin{bmatrix} X_A - X_o \\ Y_A - Y_o \\ Z_A - Z_o \end{bmatrix}$$



w.r.t. the ground coordinate system

Collinearity Equations

$$\overrightarrow{oa} = \lambda \overrightarrow{oA}$$

$$\vec{v}_i = r_{oa}^c = \lambda \;\; M(\omega, \varphi, \kappa) \;\; \vec{V}_O = \lambda \;\; R_m^c \, r_{oA}^m$$

$$\begin{bmatrix} x_a - x_p - dist_x \\ y_a - y_p - dist_y \\ -c \end{bmatrix} = \lambda \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} X_A - X_o \\ Y_A - Y_o \\ Z_A - Z_o \end{bmatrix}$$

Where: λ is a scale factor

- Questions:
  - Can you come up with an average estimate of λ?
  - Is λ constant for a given image? Why?

Collinearity Equations

$$M = R_m^c$$

$$x_a = x_p - c\frac{m_{11}(X_A - X_o) + m_{12}(Y_A - Y_o) + m_{13}(Z_A - Z_o)}{m_{31}(X_A - X_o) + m_{32}(Y_A - Y_o) + m_{33}(Z_A - Z_o)} + dist_x$$

$$y_a = y_p - c\frac{m_{21}(X_A - X_o) + m_{22}(Y_A - Y_o) + m_{23}(Z_A - Z_o)}{m_{31}(X_A - X_o) + m_{32}(Y_A - Y_o) + m_{33}(Z_A - Z_o)} + dist_y$$

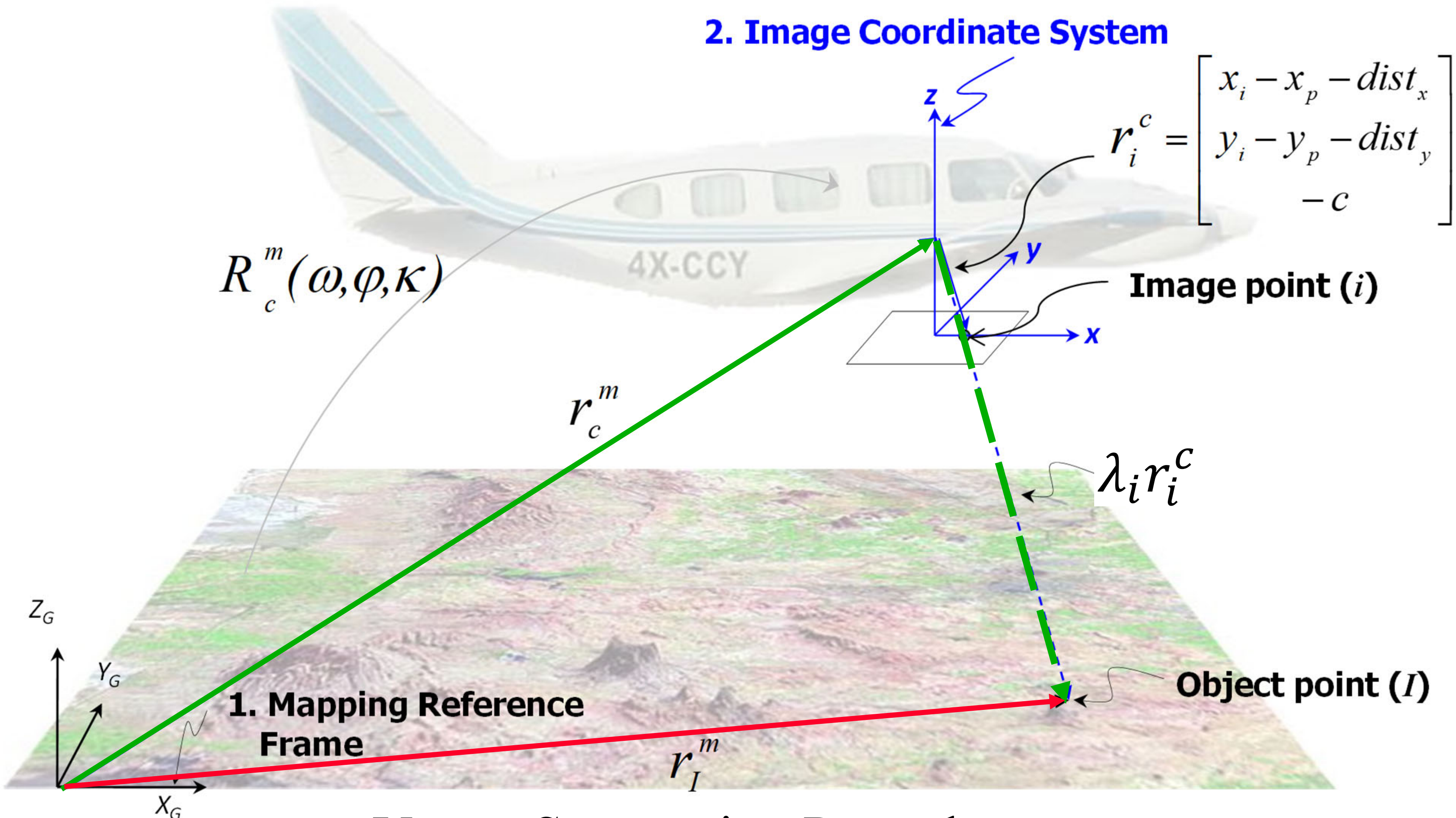
$$R = R_c^m$$

$$x_a = x_p - c\frac{r_{11}(X_A - X_o) + r_{21}(Y_A - Y_o) + r_{31}(Z_A - Z_o)}{r_{13}(X_A - X_o) + r_{23}(Y_A - Y_o) + r_{33}(Z_A - Z_o)} + dist_x$$

$$y_a = y_p - c\frac{r_{12}(X_A - X_o) + r_{22}(Y_A - Y_o) + r_{32}(Z_A - Z_o)}{r_{13}(X_A - X_o) + r_{23}(Y_A - Y_o) + r_{33}(Z_A - Z_o)} + dist_y$$

Collinearity Equations

$$r_I^m = r_c^m + \lambda_i \, R_c^m(\omega, \phi, \kappa) \; r_i^c$$



Vector Summation Procedure

Collinearity Equations

$$r_I^m = r_c^m + \lambda_i \, R_c^m(\omega, \phi, \kappa) \, r_i^c$$

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} + \lambda_i \, R_c^m(\omega, \phi, \kappa) \begin{bmatrix} x_i - x_p - dist_{x_i} \\ y_i - y_p - dist_{y_i} \\ -c \end{bmatrix}$$

$$\begin{bmatrix} x_i - x_p - dist_{x_i} \\ y_i - y_p - dist_{y_i} \\ -c \end{bmatrix} = 1/\lambda_i \, R_m^c(\omega, \phi, \kappa) \left[\vec{X}_G - \vec{X}_o\right] = 1/\lambda_i \begin{bmatrix} N_x \\ N_y \\ D \end{bmatrix}$$

$$x_i = x_p - c \, N_x / D + dist_{x_i}$$

$$y_i = y_p - c \, N_y / D + dist_{y_i}$$

Vector Summation Procedure

Collinearity Equations

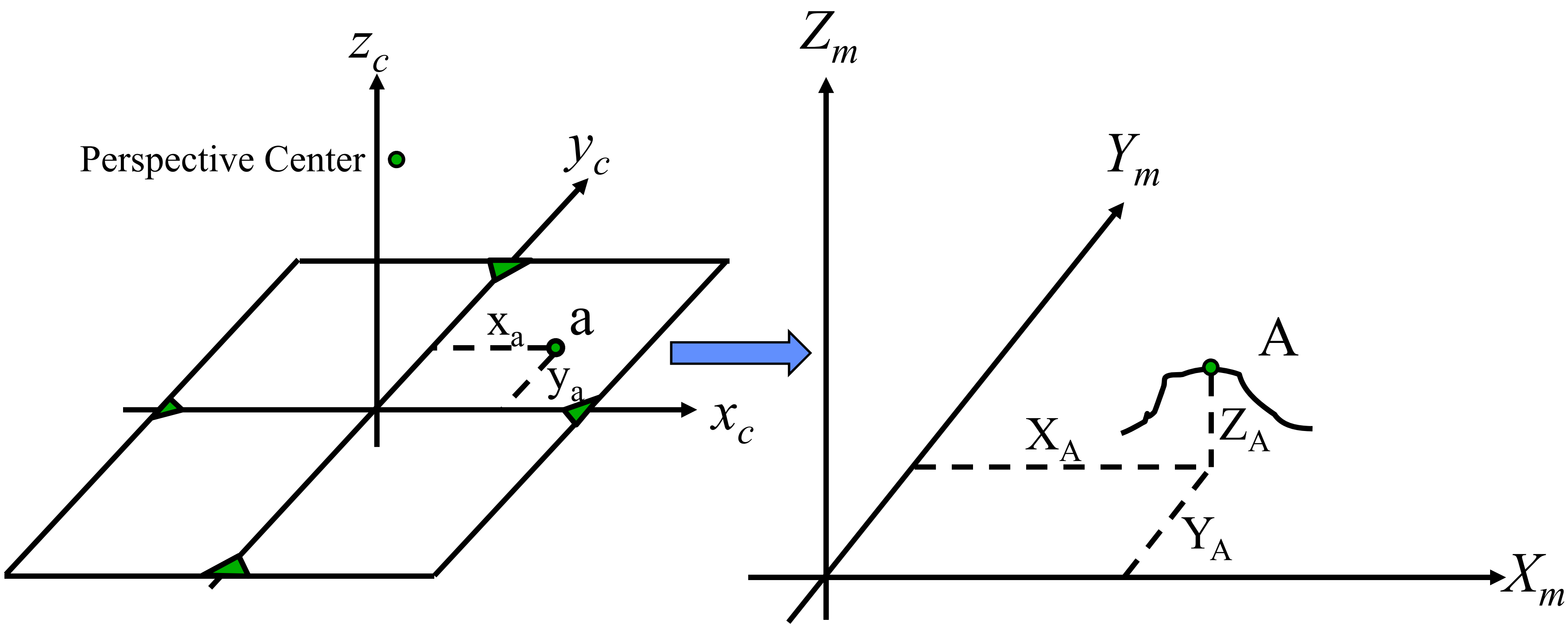
$$R = R_c^m$$

$$x_a = x_p - c\frac{r_{11}(X_A - X_o) + r_{21}(Y_A - Y_o) + r_{31}(Z_A - Z_o)}{r_{13}(X_A - X_o) + r_{23}(Y_A - Y_o) + r_{33}(Z_A - Z_o)} + dist_x$$

$$y_a = y_p - c\frac{r_{12}(X_A - X_o) + r_{22}(Y_A - Y_o) + r_{32}(Z_A - Z_o)}{r_{13}(X_A - X_o) + r_{23}(Y_A - Y_o) + r_{33}(Z_A - Z_o)} + dist_y$$

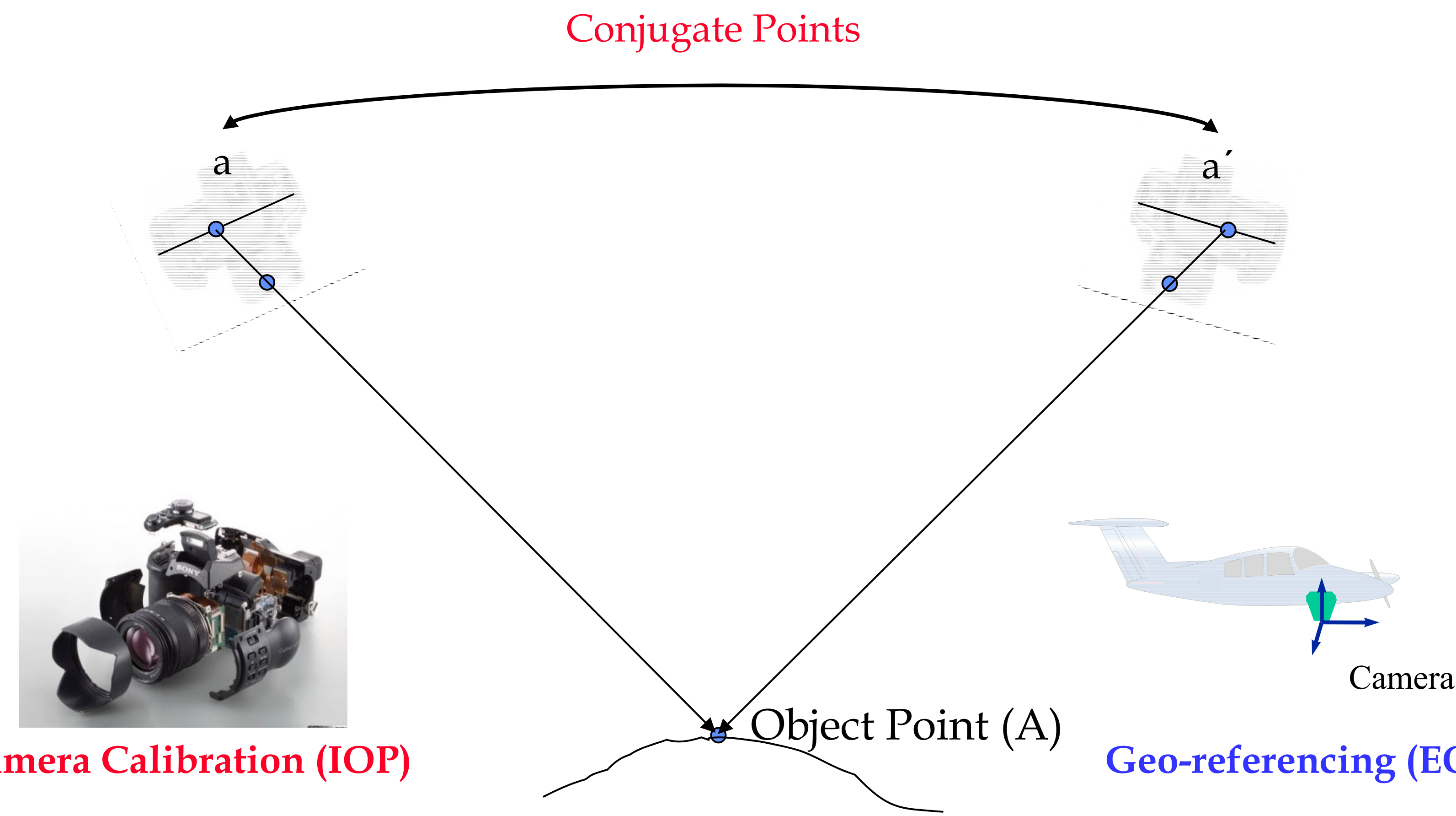
- Involved parameters:
  - Image coordinates (x_a, y_a)
  - Ground coordinates (A_A, Y_A, Z_A)
  - Exterior Orientation Parameters – EOPs (X_O, Y_O, Z_O, ω, ϕ, κ)
  - Interior Orientation Parameters – IOPs (x_p, y_p, c, and the coefficients describing $dist_x$ and $dist_y$)

Collinearity Equations



$$x_a = f_x(X_A, Y_A, Z_A, IOP, EOP)$$
$$y_a = f_y(X_A, Y_A, Z_A, IOP, EOP)$$

Photogrammetric Point Positioning



Camera Calibration (IOP)

Geo-referencing (EOP)

- **The interior orientation parameters (IOP) of the involved cameras have to be known.**
- **The position and the orientation (EOP) of the camera stations have to be known.**

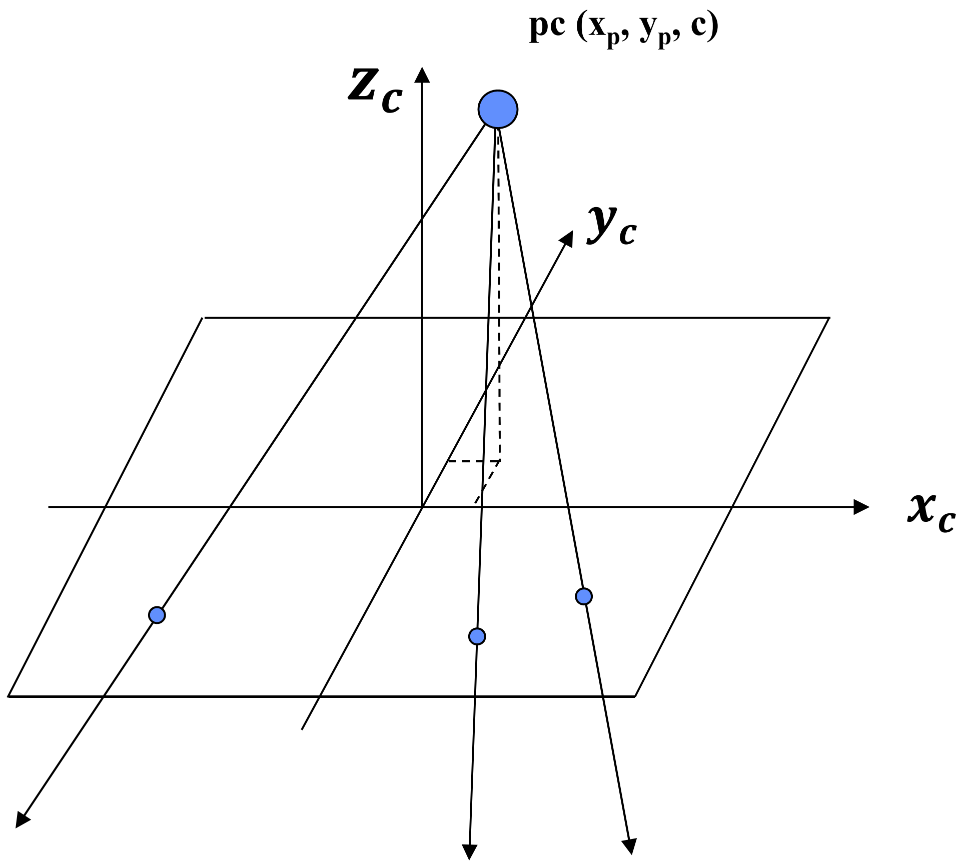
Interior Orientation

- Interior Orientation Parameters (IOP) describe the internal characteristics of the implemented camera.
  - IOP include the principal distance, principal point coordinates, and distortion parameters.
  - IOP are determined using a calibration procedure.

http://www.dpreview.com/reviews/sonydscf828/3

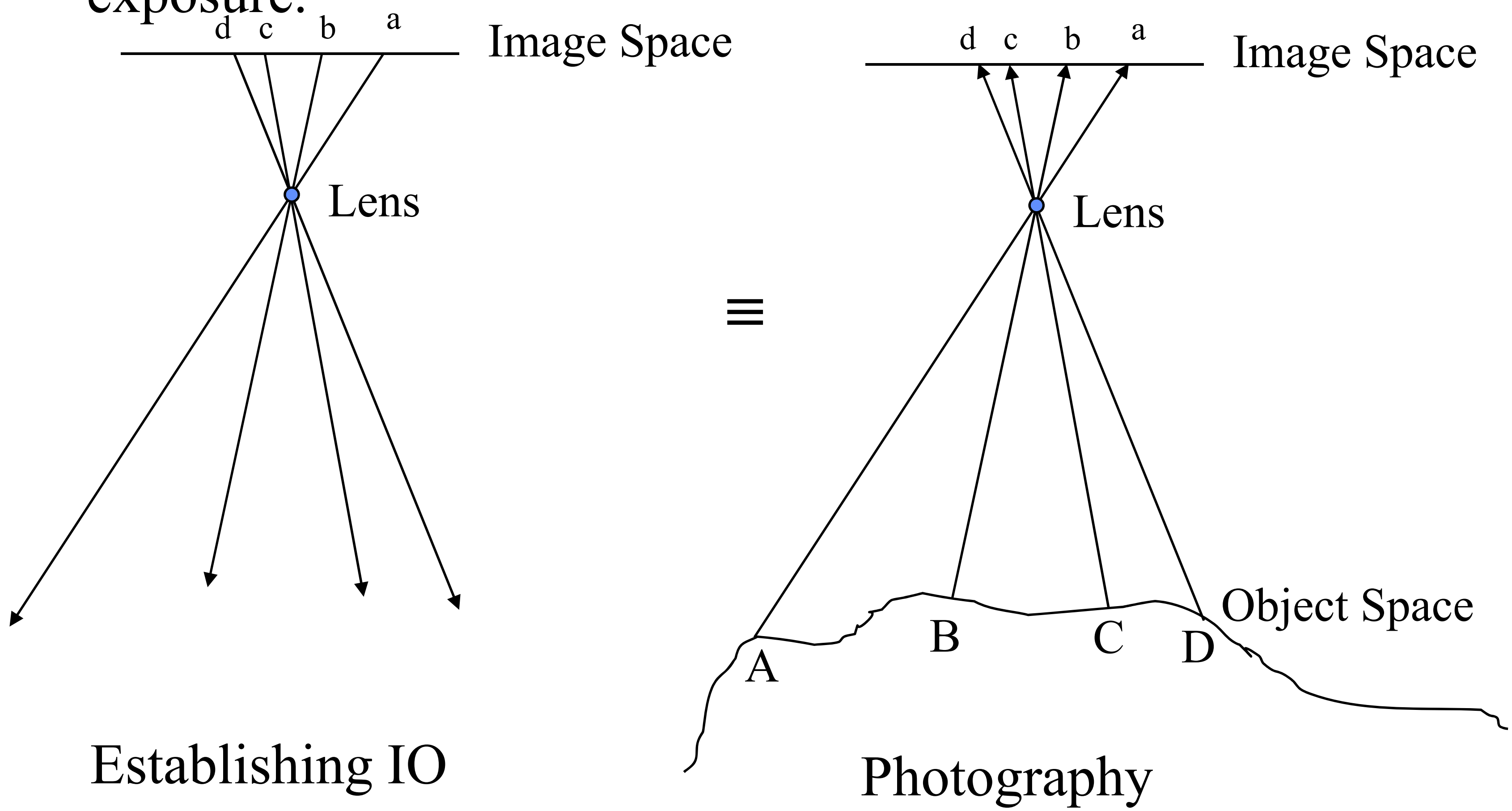
Interior Orientation

- IOP together with the image coordinates of selected features define a bundle of light rays (image bundle).



Interior Orientation

- **Target function of the Interior Orientation:**
  - The defined bundle by the IOP should be as similar as possible to the incident bundle onto the camera at the moment of exposure.

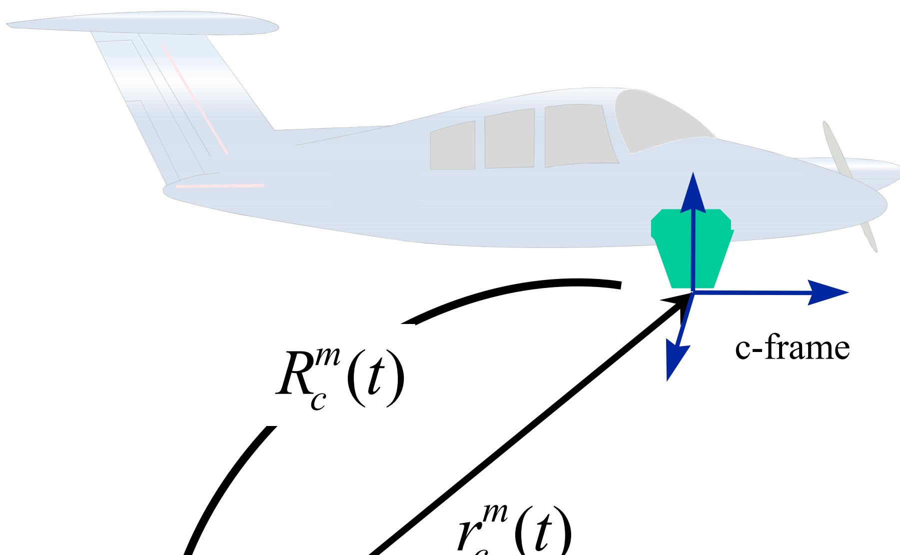


Exterior Orientation

- Exterior Orientation Parameters (EOP) – geo-referencing parameters – define the position and attitude of the image bundle relative to the ground coordinate system.
  - The position of the bundle is defined by (X_o, Y_o, Z_o).
  - The attitude of the bundle (camera/image coordinate system) relative to the ground coordinate system is defined by the rotation angles (ω, ϕ, κ).
- EOP can be either:
  - Indirectly estimated using Ground Control Points (GCPs), or
  - Directly derived using GNSS/INS unit onboard the imaging platform.

Exterior Orientation

- The Exterior Orientation Parameters (EOP) define the camera position, $r_c^m(t)$, and orientation, $R_c^m(t)$, relative to ground coordinate system at the moment of exposure.

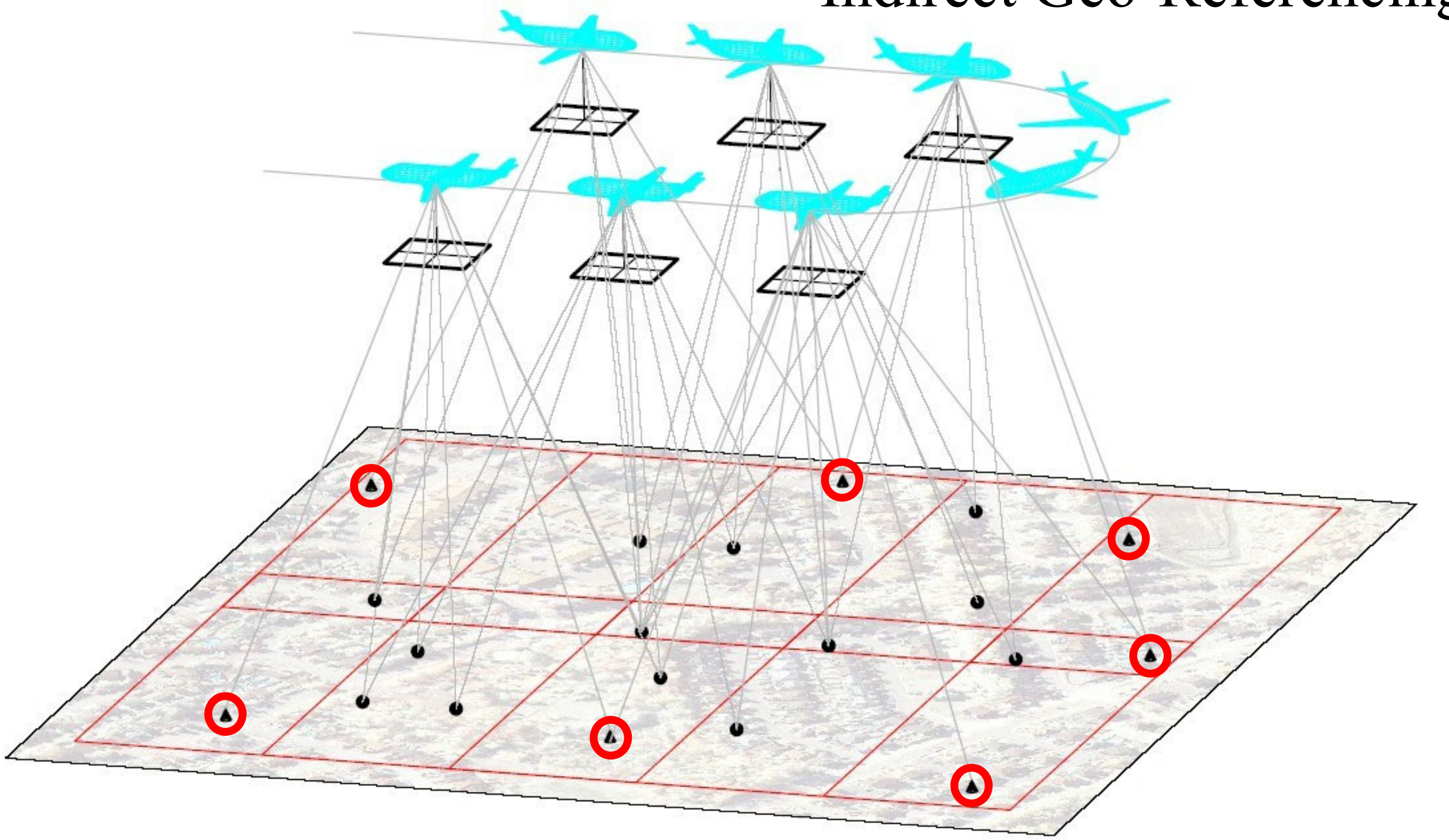


EOP can be either:

- Indirectly estimated using Ground Control Points (GCPs), or
- Directly derived using GNSS/INS unit onboard the imaging platform.

Exterior Orientation

Indirect Geo-Referencing



Ground Control Points
Tie Points

Exterior Orientation

Indirect Geo-Referencing

Signalized Targets

Exterior Orientation

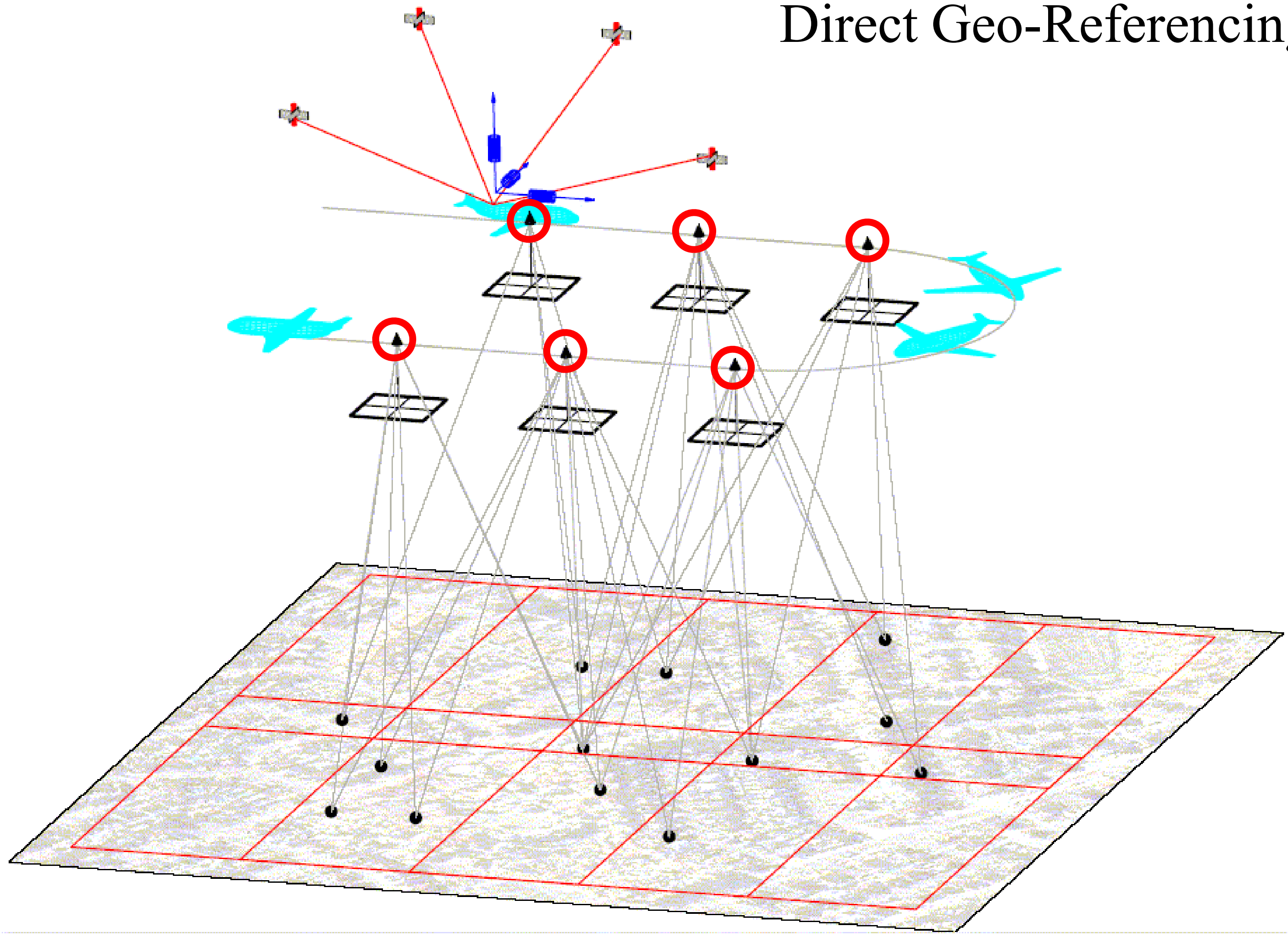
http://videoindustrial.files.wordpress.com/2011/09/gps.jpg

Indirect Geo-Referencing

Natural Targets

Exterior Orientation

Direct Geo-Referencing



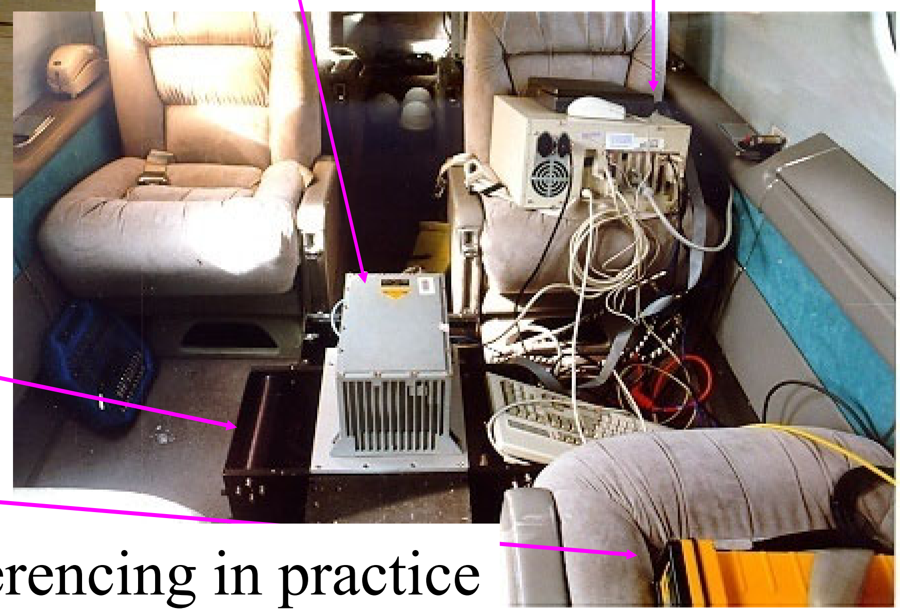
Exterior Orientation

Direct Geo-Referencing

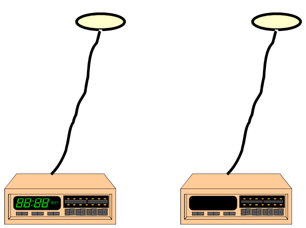
GNSS Antenna

INS

PC



Two Base Stations



Camera

GNSS Receiver

Direct geo-referencing in practice

Exterior Orientation

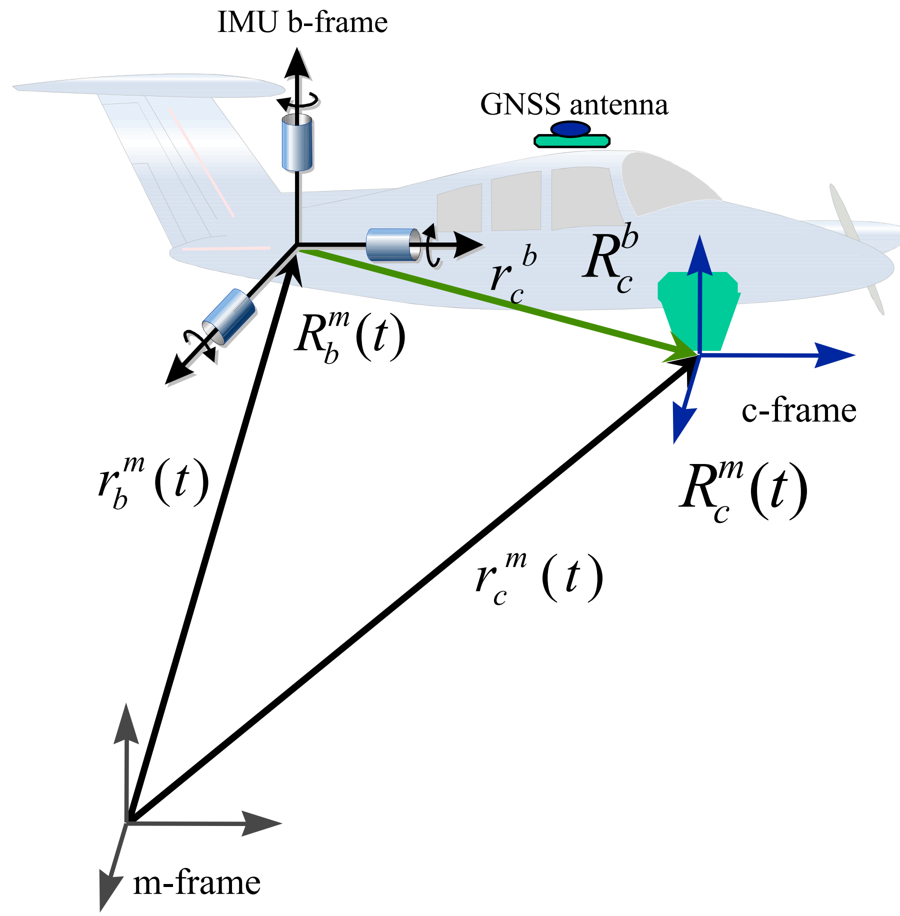
$$r_c^m(t) = r_b^m(t) + R_b^m(t)\, r_c^b$$

- $r_c^m(t)$ → Camera position
- $r_b^m(t)$ → GNSS/INS position
- $R_b^m(t)$ → GNSS/INS attitude
- r_c^b → Calibration

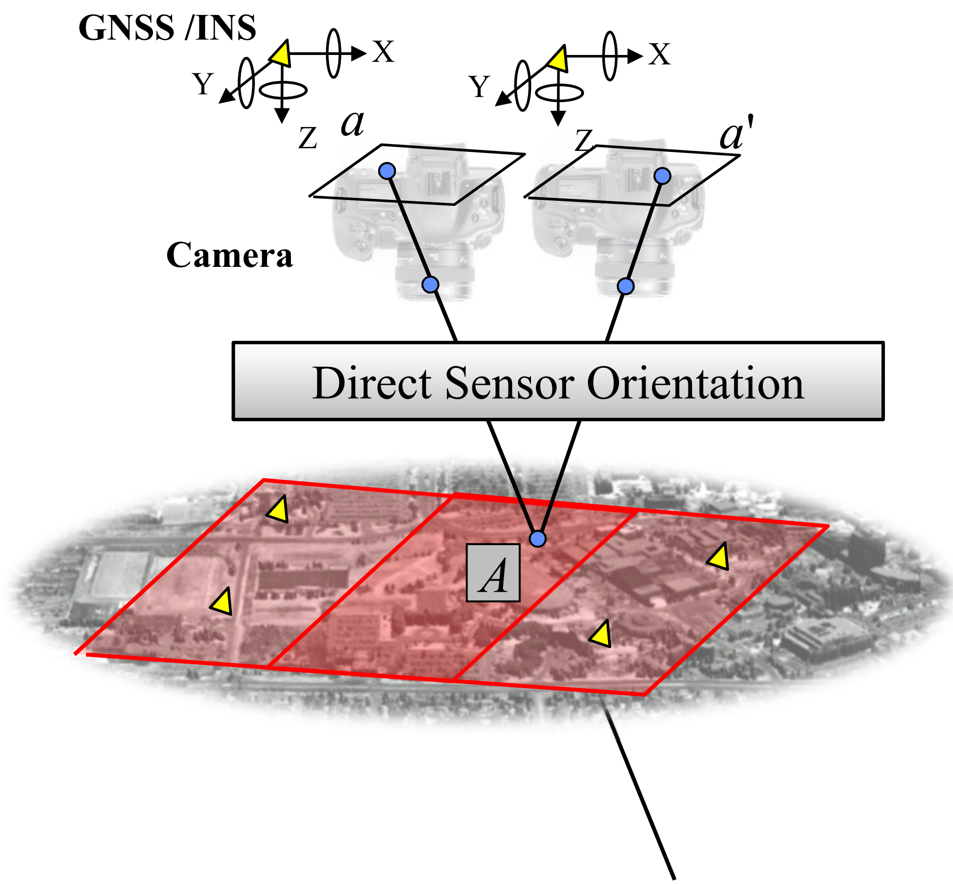
$$R_c^m(t) = R_b^m(t)\; R_c^b$$

- $R_c^m(t)$ → Camera attitude
- $R_b^m(t)$ → GNSS/INS attitude
- R_c^b → Calibration

Direct Geo-Referencing

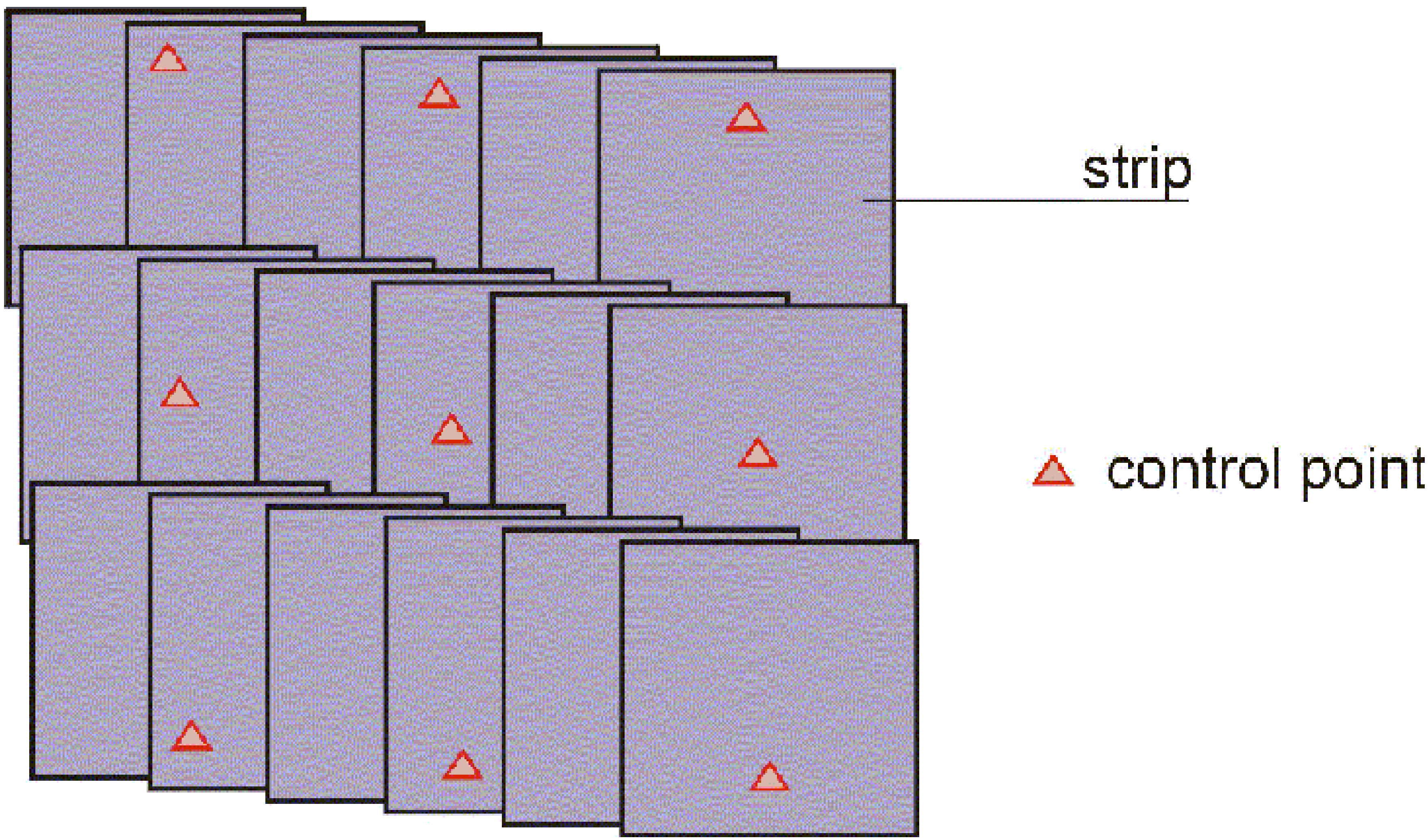


Photogrammetric Point Positioning

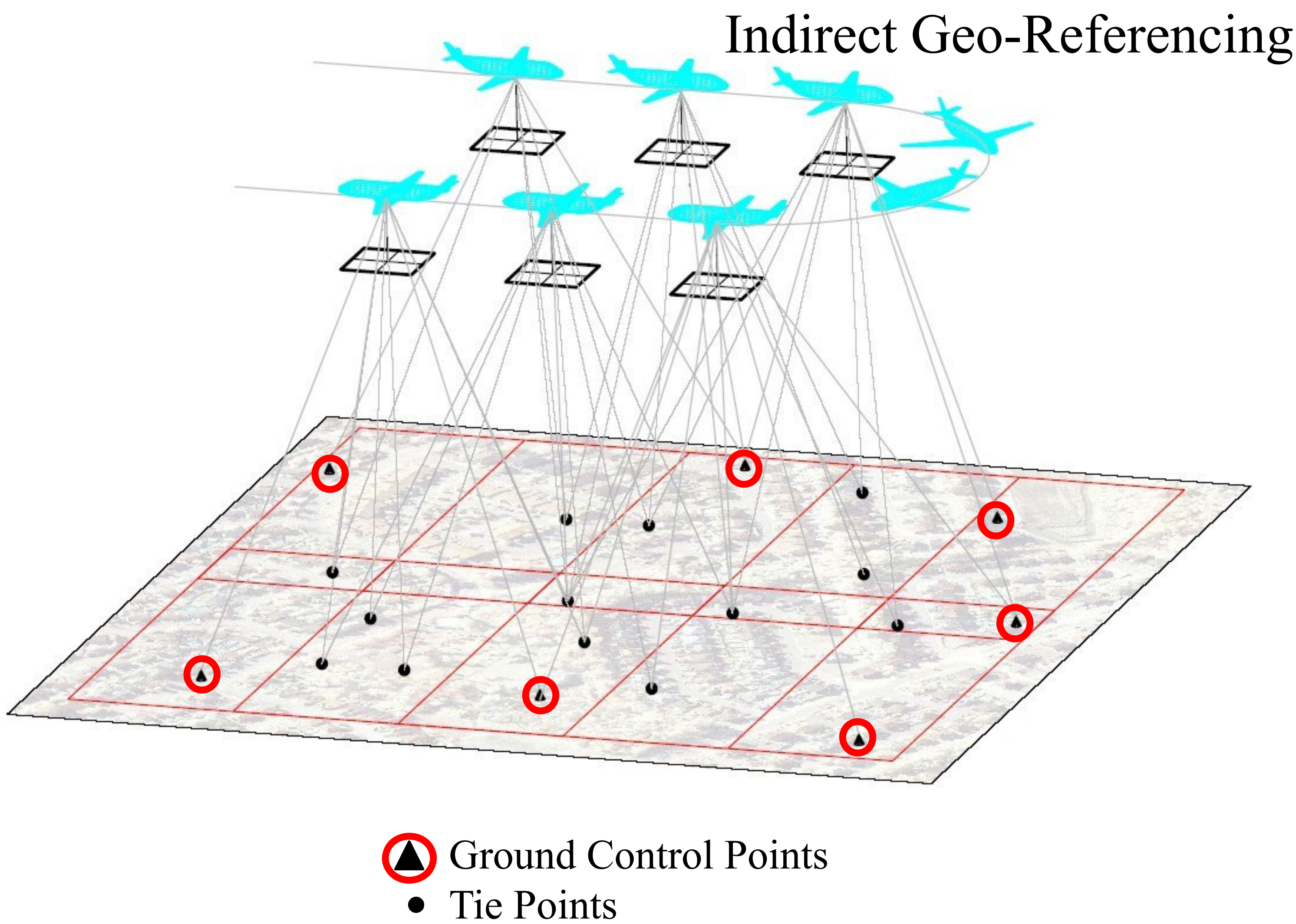


Bundle Block Adjustment

Bundle Block Adjustment

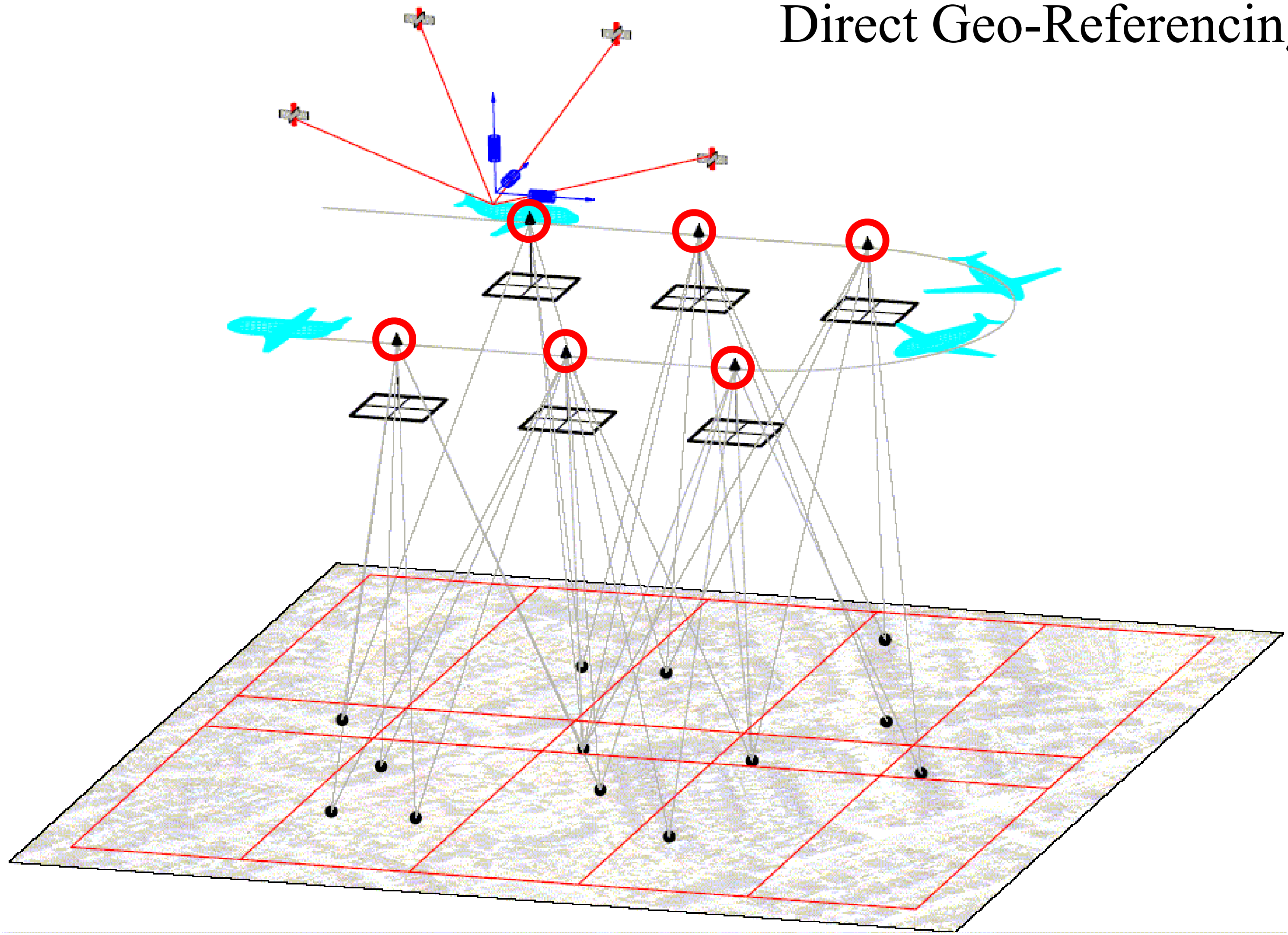


Bundle Block Adjustment



Bundle Block Adjustment

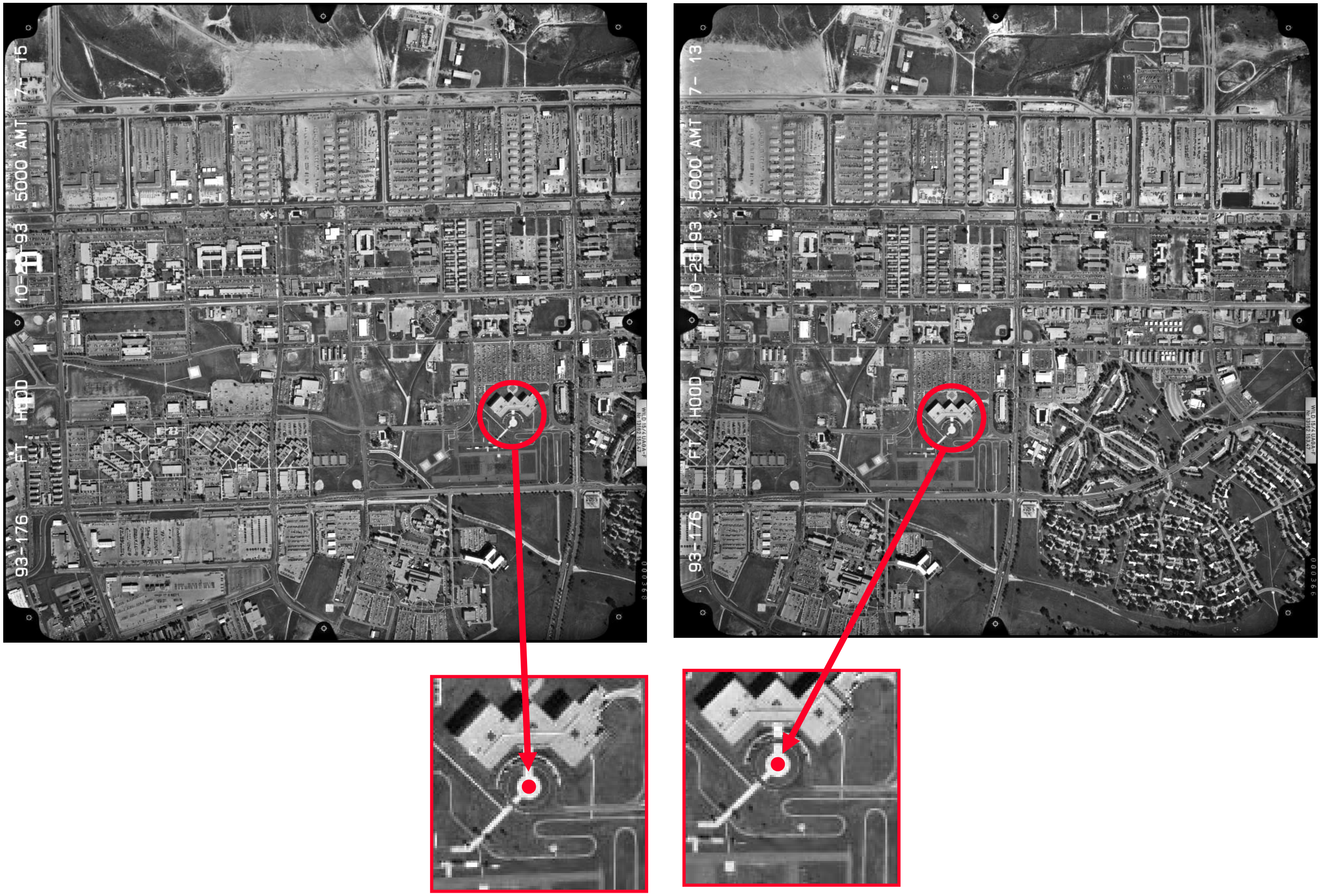
Direct Geo-Referencing



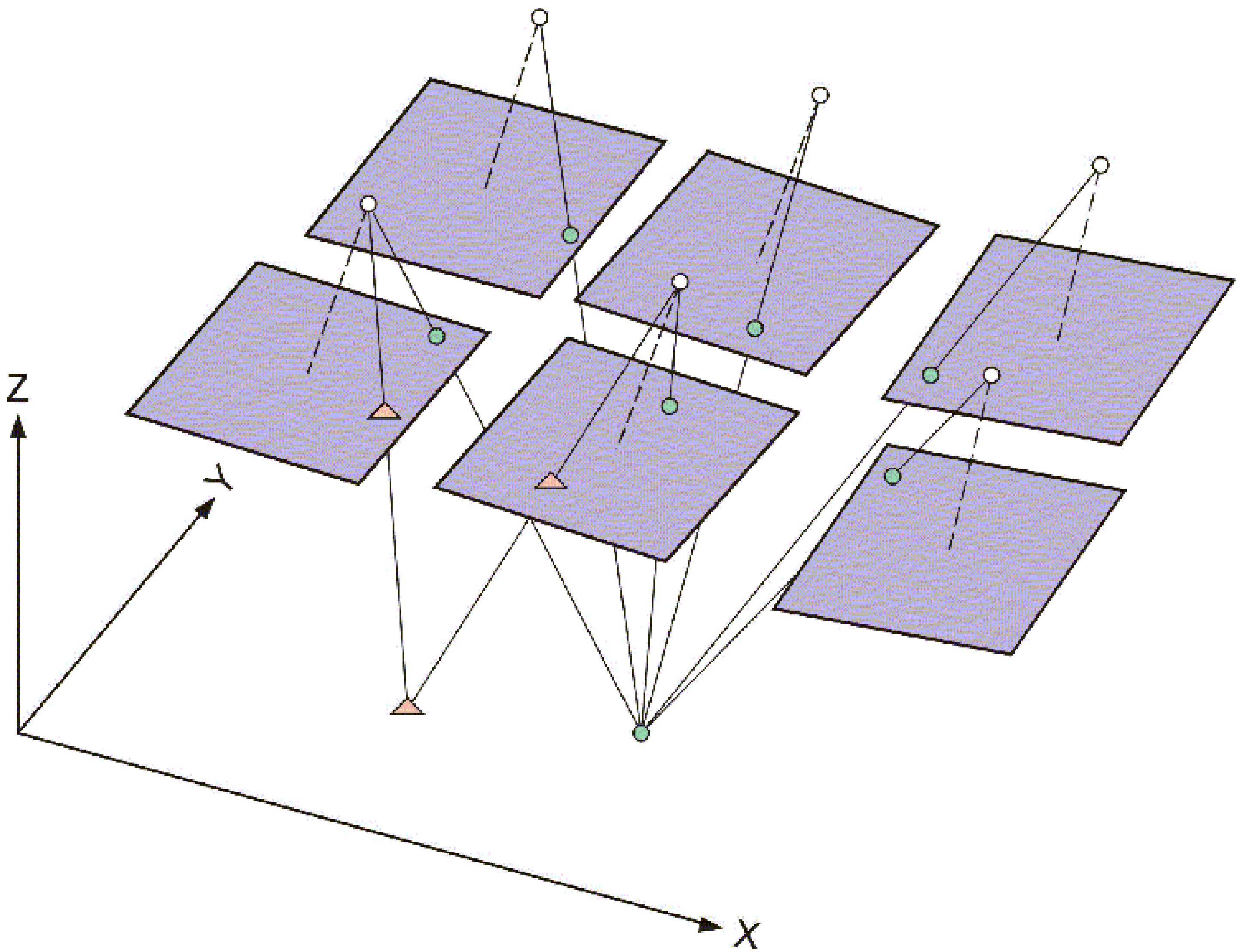
Bundle Adjustment: Concept

- During the bundle block adjustment procedure, the bundles within the block are rotated and shifted in space until:
  - Conjugate light rays (corresponding to tie points) intersect as well as possible.
  - Light rays that correspond to ground control points (GCPs) pass as close as possible to their ground locations.

Tie Points



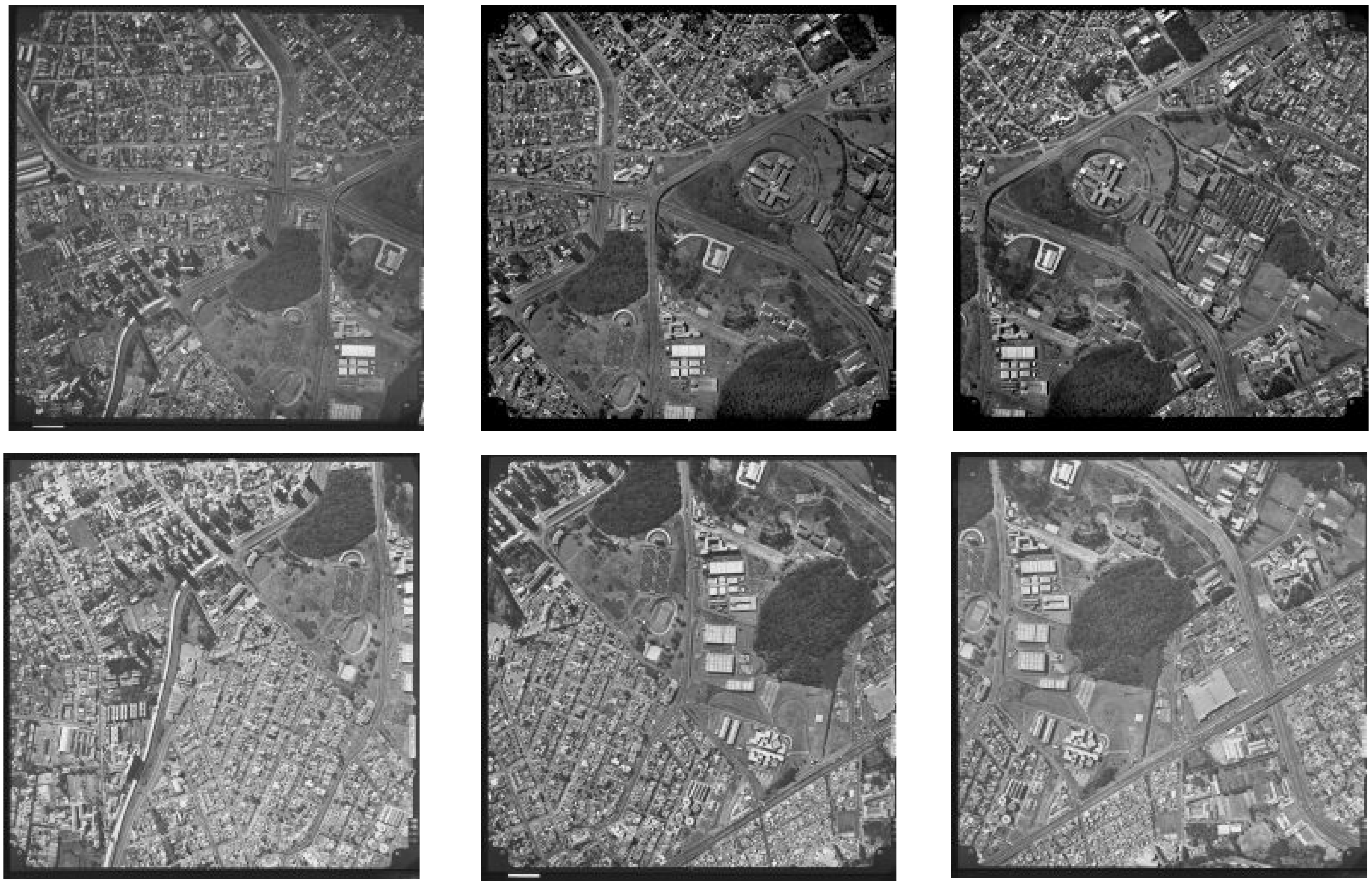
Bundle Adjustment: Concept



Analog Camera (WILD RC10)

http://arsf.nerc.ac.uk/images/rc-10.jpg

Aerial Block (WILD RC10)



Digital Camera: SONY DSC 717

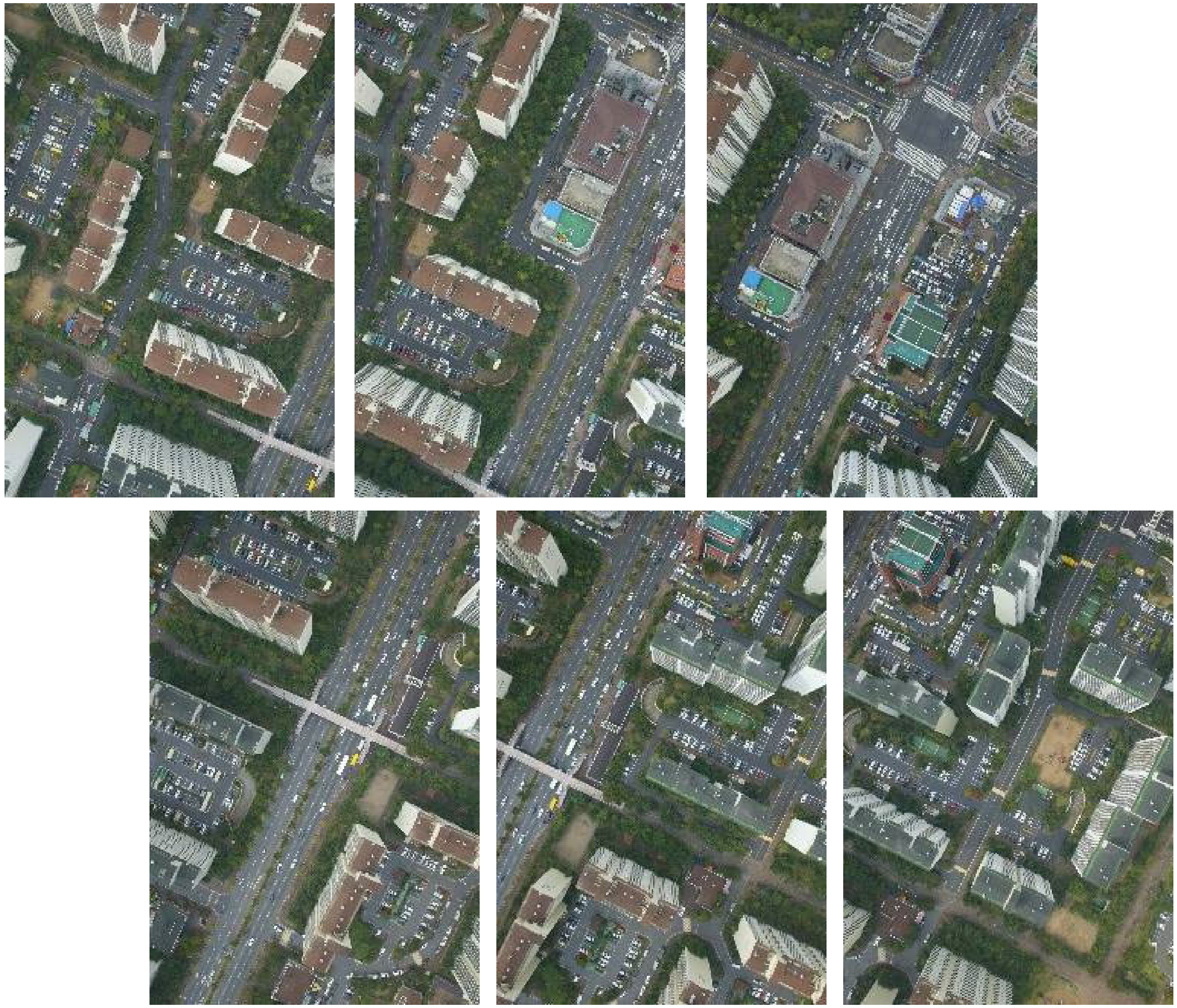
http://www.dpreview.com/reviews/sonydscf717

SONY DSC F717 Aerial Block

Digital Camera: EOS-1D

http://www.canon.ca/inetCA/

Canon EOS-1D Aerial Block



Least Squares Adjustment in Photogrammetry

Least Squares Adjustment in Photogrammetry

- Prior to the adjustment, we need to identify:
  - Unknown parameters,
  - Observable quantities, and
  - The mathematical relationship between the unknown parameters and the observable quantities.
- Linearize the mathematical relationship (if it is not linear)
- Apply least squares adjustment formulas

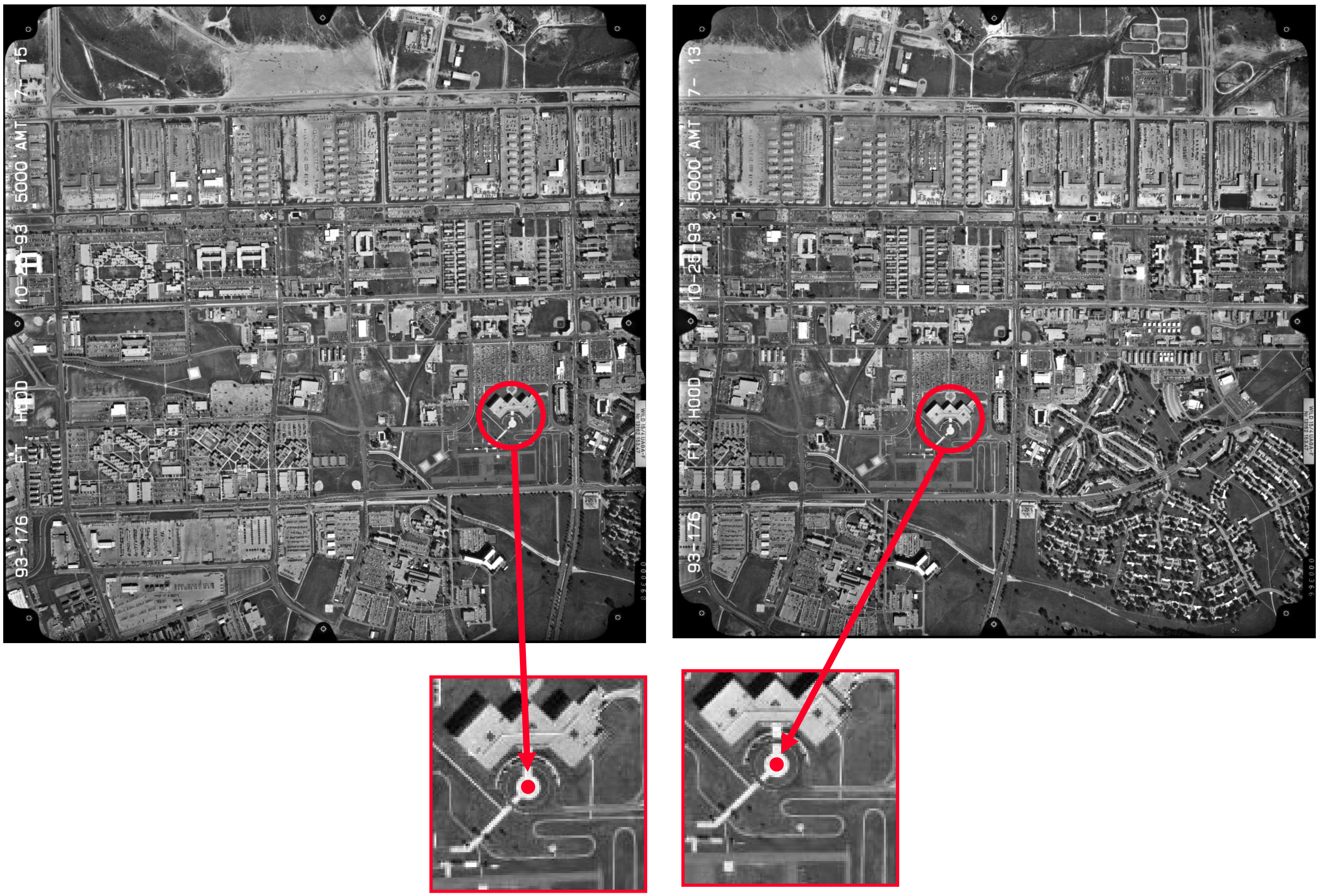
Unknown Parameters

- Unknown parameters might include:
  - Ground coordinates of tie points (points that appear in more than one image)
  - Exterior orientation parameters (EOP) of the involved imagery
  - Interior orientation parameters (IOP) of the involved cameras (for camera calibration purposes)

Observable Quantities

- Observable quantities <u>might</u> include:
  - Image coordinates of tie as well as control points
  - The ground coordinates of control points (GCPs)
  - Interior orientation parameters (IOP) of the involved cameras (Camera Calibration Certificate)
  - Exterior orientation parameters (EOP) of the involved imagery (from a GNSS/INS unit onboard)

Tie Points


93-176 FT. HOOD 10-20-93 5000' AMT 7- 15
93-176 FT. HOOD 10-20-93 5000' AMT 7- 13

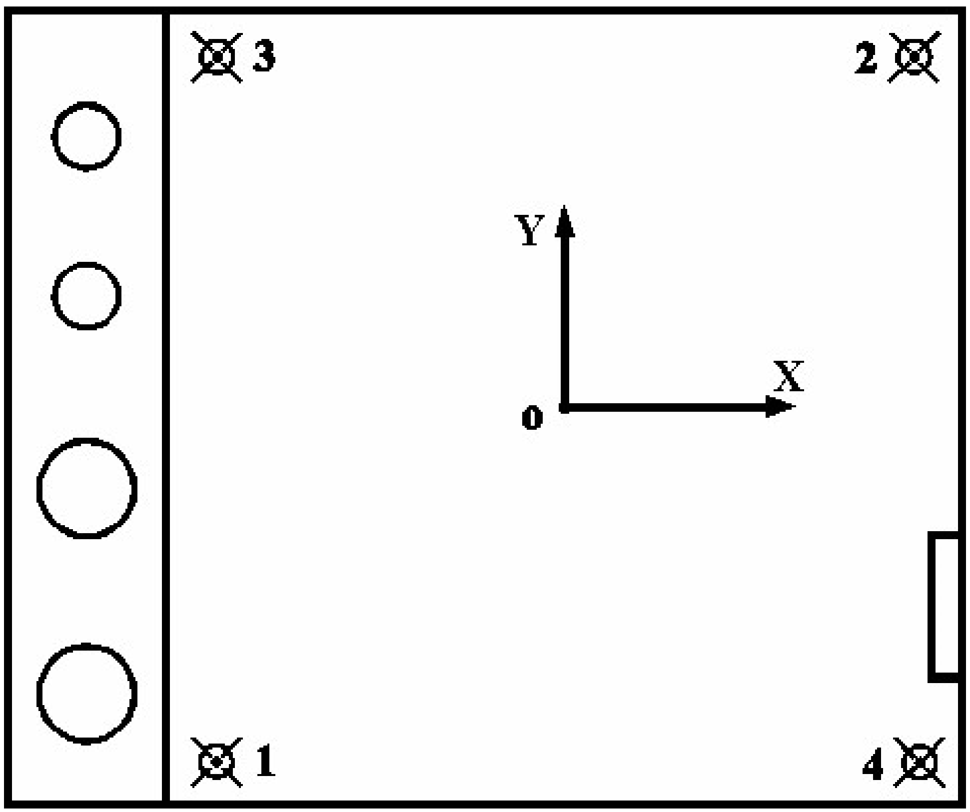
Ground Control Points: Collection

http://videoindustrial.files.wordpress.com/2011/09/gps.jpg

IOP: Camera Calibration Certificate

- Wild Heerbrugg Instruments Inc.
- Camera type: Wild RC10
  - Identification number: 2061
- Lens: Wild 15 UAG I
  - Identification Number: 6029
- Calibrated Focal Length: C = 153.167 mm
- Principal point coordinates in the Fiducial system:
  - x_p = 0.001 mm
  - y_p = -0.053 mm

IOP: Camera Calibration Certificate



IOP: Camera Calibration Certificate

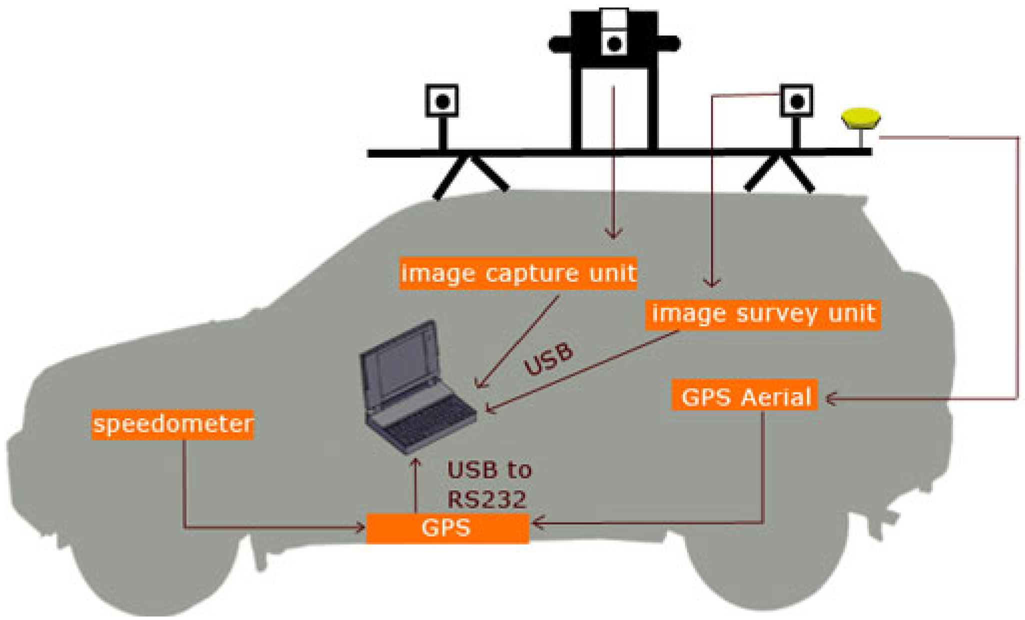
- Fiducial Marks:

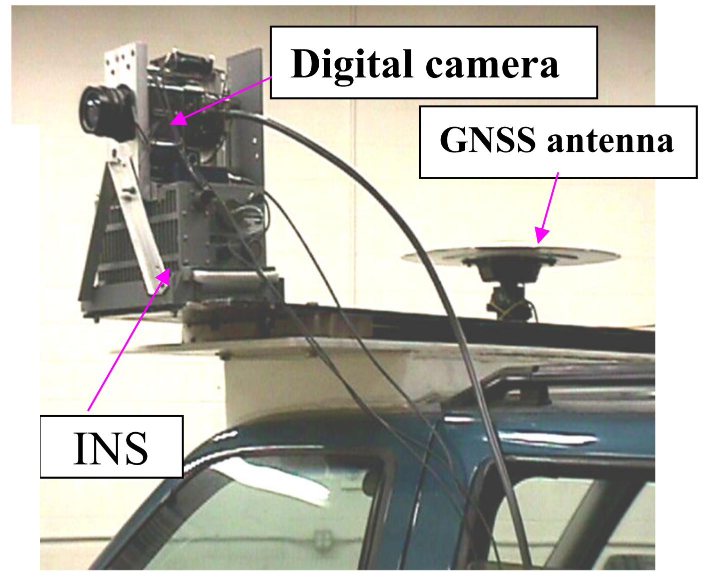
ID	x-Coordinate mm	y-Coordinate mm
01	-105.999	-105.978
02	105.996	106.022
03	-106.018	106.021
04	105.988	-105.978

IOP: Camera Calibration Certificate

- Radial Lens Distortion Coefficients:
  - $K_1 = 2.99778547E\text{-}08\ mm^{-2}$
  - $K_2 = -3.15091119E\text{-}12\ mm^{-4}$
  - $K_3 = 6.05776623E\text{-}17\ mm^{-6}$
- De-centering Lens Coefficients :
  - $P_1 = 2.76490955E\text{-}07\ mm^{-1}$
  - $P_2 = -1.06518601E\text{-}06\ mm^{-1}$

EOP: GNSS-INS (Land Based Systems)





http://www.easypano.com/images/city8/street-view-car-structure.jpg

EOP: GNSS-INS (Airborne System)

Mathematical Model

$$x_a = x_p - c\ \frac{r_{11}\,(X_A - X_O) + r_{21}\,(Y_A - Y_O) + r_{31}\,(Z_A - Z_O)}{r_{13}\,(X_A - X_O) + r_{23}\,(Y_A - Y_O) + r_{33}\,(Z_A - Z_O)} + dist_x + e_x$$

$$y_a = y_p - c\ \frac{r_{12}\,(X_A - X_O) + r_{22}\,(Y_A - Y_O) + r_{32}\,(Z_A - Z_O)}{r_{13}\,(X_A - X_O) + r_{23}\,(Y_A - Y_O) + r_{33}\,(Z_A - Z_O)} + dist_y + e_y$$

$$\begin{bmatrix} e_x \\ e_y \end{bmatrix} \sim (0, \sigma_o^2\ P^{-1})$$

Mathematical Model

- $dist_x = \Delta x_{\text{Radial Lens Distortion}} + \Delta x_{\text{De-centering Lens Distortion}}$
  $+ \Delta x_{\text{Atmospheric Refraction}} + \Delta x_{\text{Affine Deformation}}$
  $+$ etc….

- $dist_y = \Delta y_{\text{Radial Lens Distortion}} + \Delta y_{\text{De-centering Lens Distortion}}$
  $+ \Delta y_{\text{Atmospheric Refraction}} + \Delta y_{\text{Affine Deformations}}$
  $+$ etc….

Distortion Parameters

$$\Delta x_{\text{Radial Lens Distortion}} = \bar{x}\ (k_1 r^2 + k_2 r^4 + k_3 r^6 + \ldots.)$$

$$\Delta y_{\text{Radial Lens Distortion}} = \bar{y}\ (k_1 r^2 + k_2 r^4 + k_3 r^6 + \ldots.)$$

$$\Delta x_{\text{De-centering Lens Distortion}} = (1 + p_3\, r^2)\{p_1\, (r^2 + 2\bar{x}^2) + 2 p_2\, \bar{x}\, \bar{y}\}$$

$$\Delta y_{\text{De-centering Lens Distortion}} = (1 + p_3\, r^2)\{2 p_1\, \bar{x}\, \bar{y} + p_2\, (r^2 + 2\bar{y}^2)\}$$

where: $r = \{(x - x_p)^2 + (y - y_p)^2\}^{0.5}$

$$\bar{x} = x - x_p$$

$$\bar{y} = y - y_p$$

Least Squares Adjustment

$y = A\,x + e \qquad\qquad e \sim (0, \sigma_o^2 P^{-1})$

$y \qquad n \times 1$ ***observation vector***

$A \qquad n \times m$ ***design matrix***

$x \qquad m \times 1$ ***vector of unknowns***

$e \qquad n \times 1$ ***noise contaminating the observation vector***

$\sigma_o^2 P^{-1} \qquad n \times n$ ***variance covariance matrix of the noise vector***

Least Squares Adjustment

$$\hat{x} = (A^T P A)^{-1} A^T P y$$

$$D\{\hat{x}\} = \sigma_o^2 \, (A^T P A)^{-1}$$

$$\tilde{e} = y - A\hat{x}$$

$$\hat{\sigma}_o^2 = (\tilde{e}^T P \tilde{e}) \,/\, (n - m)$$

Non Linear System

$Y = a(X) + e$

$a(X)$ is non – linear function

We use Taylor Series Expansion

$$Y \approx a(X_o) + \left.\frac{\partial a}{\partial X}\right|_{X_o} (X - X_o) + e \qquad \text{(We ignore higher order terms)}$$

Where :

X_o are approximate values for the unknown parameters

$$Y - a(X_o) = \left.\frac{\partial a}{\partial X}\right|_{X_o} (X - X_o) + e$$

$y = A\,x + e$

Where:

$y = Y - a(X_o)$

$$A = \left.\frac{\partial a}{\partial X}\right|_{X_o}$$

- Iterative solution for the unknown parameters
- When should we stop the iterations?

Examples

- Linear System:
  - 2-D Similarity Transformation
    - Given:
      - The coordinates of a set of points in two different coordinate systems, which can be related to each other through 2-D similarity transformation
    - Unknown parameters:
      - The 2-D similarity transformation parameters (x_t, y_t, a, b)
- Non-Linear System:
  - Computing the radius of a circle
    - By observing the area of a circle, we would like to estimate its radius.

2-D Similarity Transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} e_x \\ e_y \end{bmatrix}$$

$$\begin{bmatrix} e_x \\ e_y \end{bmatrix} \sim (0, \sigma_o^2 P^{-1})$$

2-D Similarity Transformation

- Assuming that we observed three points

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x'_1 & -y'_1 \\ 0 & 1 & y'_1 & x'_1 \\ 1 & 0 & x'_2 & -y'_2 \\ 0 & 1 & y'_2 & x'_2 \\ 1 & 0 & x'_3 & -y'_3 \\ 0 & 1 & y'_3 & x'_3 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ a \\ b \end{bmatrix} + \begin{bmatrix} e_{x_1} \\ e_{y_1} \\ e_{x_2} \\ e_{y_2} \\ e_{x_3} \\ e_{y_3} \end{bmatrix}$$

Estimating the Radius of a Circle

$$Area_{obs.} = \pi\ R^2 + e_{Area_{obs.}}$$

$$e_{Area_{obs.}} \sim (0, \sigma_o^2\ P^{-1})$$

$$Area_{obs.} = \pi\ R_o^2 + 2\ \pi\ R_o\ (R - R_o) + ... + e_{Area_{obs.}}$$

$$\underbrace{Area_{obs.} - \pi\ R_o^2}_{y} = \underbrace{2\ \pi\ R_o}_{A}\ \underbrace{\Delta R}_{x} + e_{Area_{obs.}}$$

Estimating the Radius of a Circle

- Assuming that we measured the area three times

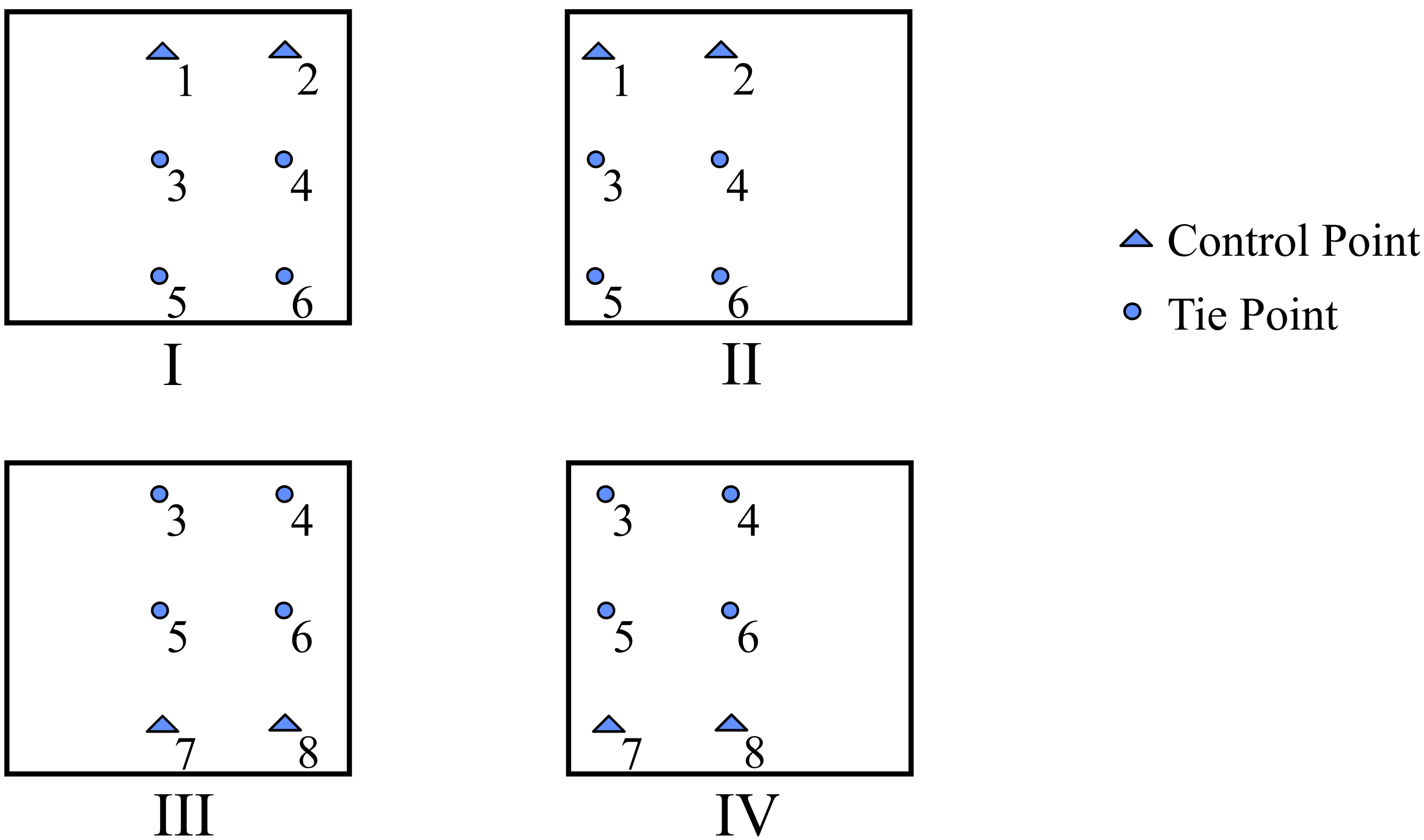
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2\,\pi\,R_o \\ 2\,\pi\,R_o \\ 2\,\pi\,R_o \end{bmatrix} \left[\Delta R\right] + \begin{bmatrix} e_{Area_1} \\ e_{Area_2} \\ e_{Area_3} \end{bmatrix}$$

Least Squares Adjustment

$$y = Ax + e \qquad\qquad Y = a(X) + e \Rightarrow y = Ax + e$$

Linear System	Non-Linear System → Linearized System
y is the observations vector.	**y is the vector of differences between the measured and computed image coordinates using the approximate values for the unknown parameters.**
A is the design matrix.	**A is the design matrix composed of the partial derivatives.**
x is the vector of unknown parameters.	**x is the vector of corrections to the approximate values of the unknown parameters.**
e is the error vector.	**e is the error vector.**

Example (4 Images in Two Strips)



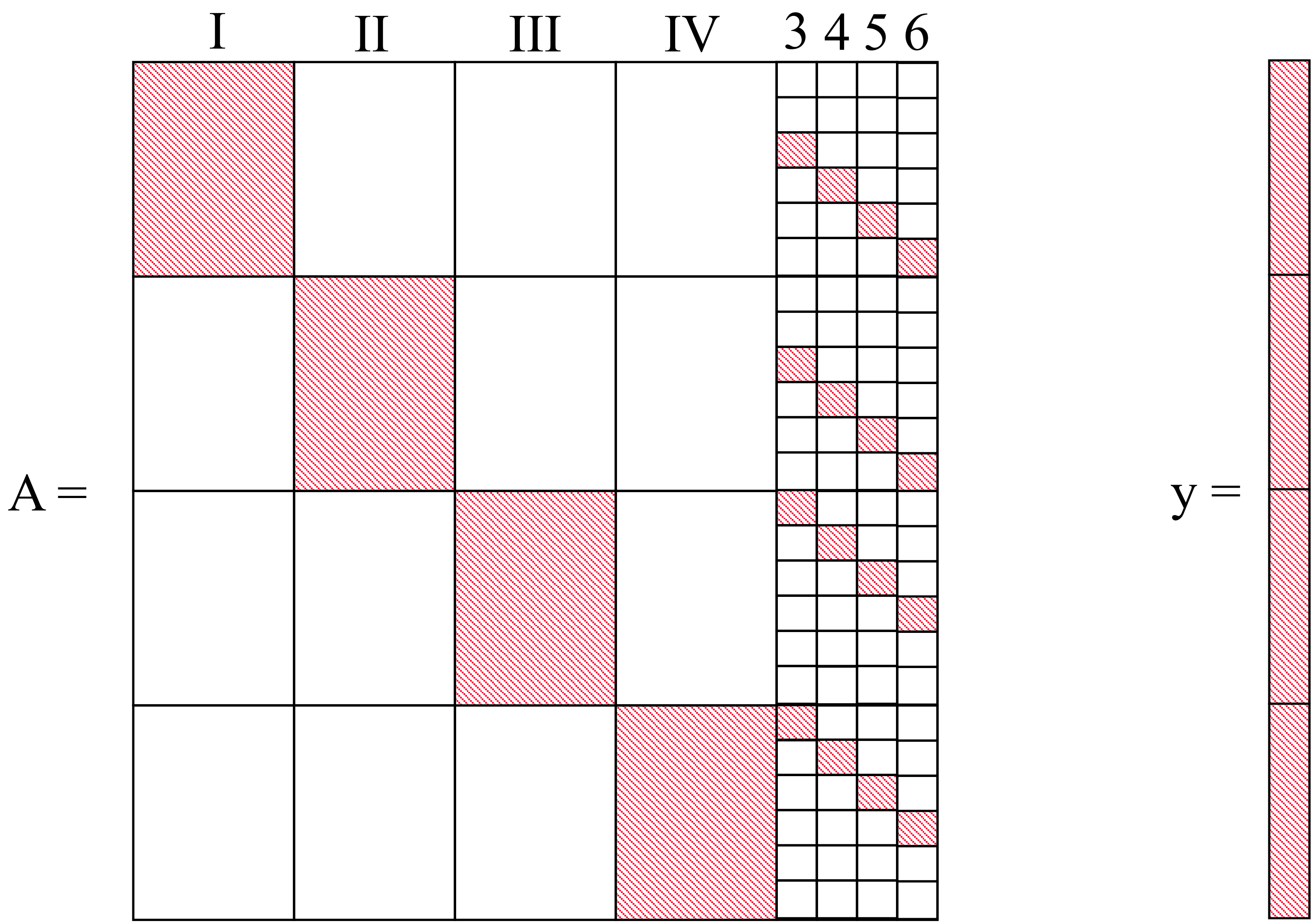
Balance Between Observations & Unknowns

- Number of equations:
  - 4 x 6 x 2 = 48 equations (collinearity equations)
- Number of unknowns:
  - 4 x 6 + 3 x 4 = 36 unknowns
- Redundancy:
  - 12
- Assumptions:
  - IOP are known and are errorless.
  - The ground coordinates of the control points are errorless.

Structure of the Design Matrix (BA)

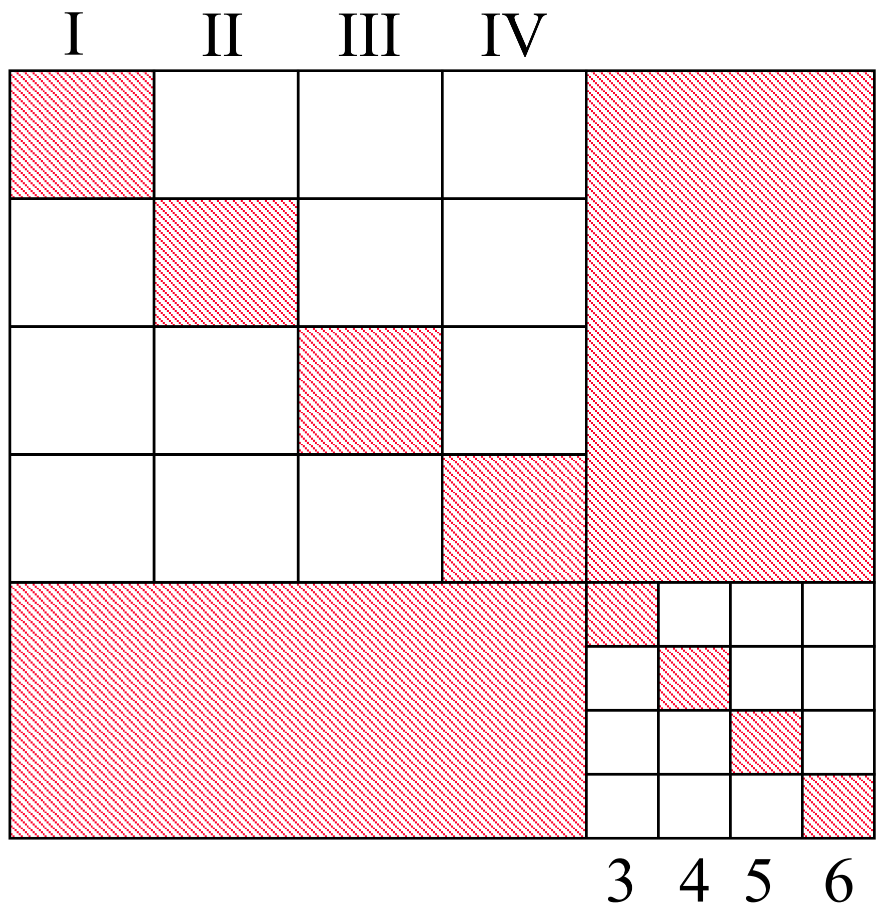
- $Y = a(X) + e \qquad e \sim (0, \sigma^2 P^{-1})$
- Using approximate values for the unknown parameters (X_o) and partial derivatives, the above equations can be linearized leading to the following equations:
- $y_{48x1} = A_{48x36}\, x_{36x1} + e_{48x1} \qquad e \sim (0, \sigma^2 P^{-1})$

Structure of the Design Matrix



Structure of the Normal Matrix

$$[N] =$$



Sample Data

- 2 cameras
- 4 images
- 16 points

- All the points appear in all the images.
- Two images were captured by each camera.

Structure of the Normal Matrix: Example