

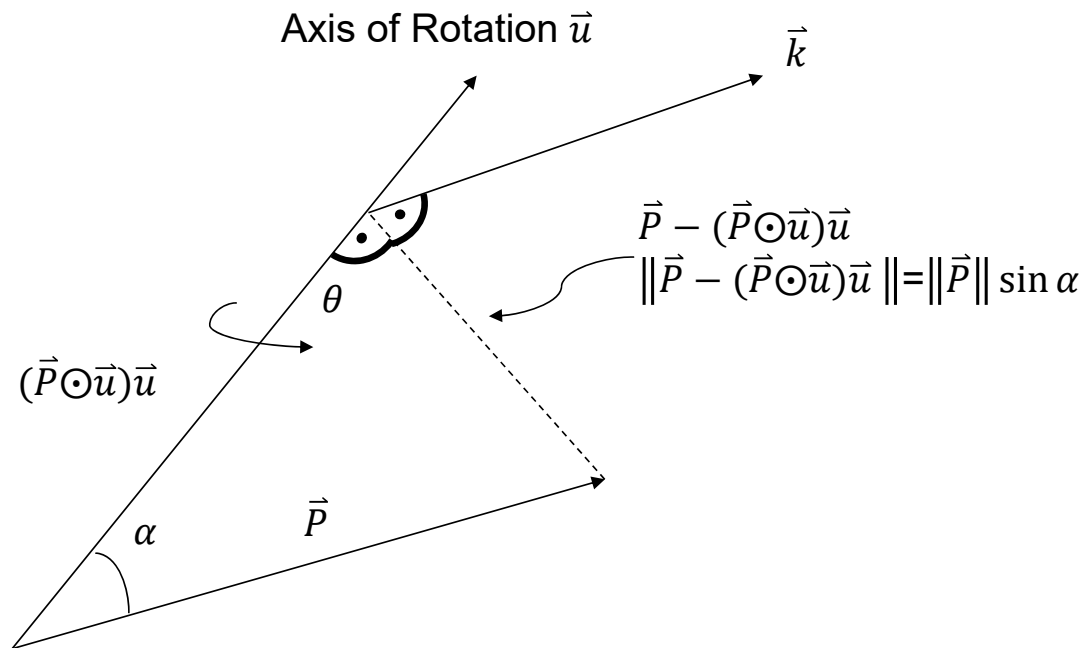


# Rodriguez Approach for Representing Rotation

Chapter 5 – A1



# Rodriguez Approach for Representing Rotation





# Rodriguez Approach for Representing Rotation

$$\vec{k} = \vec{u} \times (\vec{P} - (\vec{P} \odot \vec{u})\vec{u}) = \vec{u} \times \vec{P}$$


$$\|\vec{k}\| = \|\vec{P}\| \sin \alpha$$

Then  $\|\vec{k}\|$  &  $\|\vec{P} - (\vec{P} \odot \vec{u})\vec{u}\|$  are equivalent in size ( $\|\vec{P}\| \sin \alpha$ )

Thus

$\vec{P}'$  = rotated vector around  $\vec{u}$  with an angle  $\theta$

$$\vec{P}' = (\vec{P} \odot \vec{u})\vec{u} + \cos \theta \{\vec{P} - (\vec{P} \cdot \vec{u})\vec{u}\} + \sin \theta \vec{u} \times \vec{P} = (1 - \cos \theta)(\vec{P} \odot \vec{u})\vec{u} + \cos \theta \vec{P} + \sin \theta \hat{U}\vec{P}$$

3 × 3 matrix 

$$\hat{U} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$



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- $\vec{P}' = (1 - \cos \theta)(\vec{P} \odot \vec{u})\vec{u} + \cos \theta \vec{P} + \sin \theta \hat{U}\vec{P}$
- Now, let's focus on  $(\vec{P} \odot \vec{u})\vec{u}$
- First, let's consider  $\vec{u} \times (\vec{u} \times \vec{P}) = \hat{U}\hat{U}\vec{P} = \hat{U}^2\vec{P}$
- $\vec{u} \times (\vec{u} \times \vec{P}) = \vec{u}(\vec{u} \odot \vec{P}) - \vec{P}(\vec{u} \odot \vec{u})$
- Thus  $\vec{u} \times (\vec{u} \times \vec{P}) = \hat{U}^2\vec{P} = (\vec{u} \odot \vec{P})\vec{u} - \vec{P} = (\vec{P} \odot \vec{u})\vec{u} - \vec{P}$
- $(\vec{P} \odot \vec{u})\vec{u} = \hat{U}^2\vec{P} + \vec{P}$
- Finally:  $\vec{P}' = (1 - \cos \theta)\hat{U}^2\vec{P} + \vec{P} + \sin \theta \hat{U}\vec{P}$



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- In other words:
- $\vec{P}' = [I_3 + \sin \theta \hat{U} + (1 - \cos \theta)\hat{U}^2]\vec{P}$
- Therefore, the rotation matrix  $R = I_3 + \sin \theta \hat{U} + (1 - \cos \theta)\hat{U}^2$
- Claim:  $R = e^{\hat{U}\theta}$
- In conclusion, Rodriguez Rotation Formula
- $\vec{P}' = [I_3 + \sin \theta \hat{U} + (1 - \cos \theta)\hat{U}^2]\vec{P}$



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$$\vec{U} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \quad \text{unit vector: } u_x^2 + u_y^2 + u_z^2 = 1$$

$$\hat{U} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

$$\hat{U}^2 = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} = \begin{bmatrix} -u_z^2 - u_y^2 & u_x u_y & u_x u_z \\ u_x u_y & -u_x^2 - u_z^2 & u_y u_z \\ u_x u_z & u_y u_z & -u_x^2 - u_y^2 \end{bmatrix}$$

$$= \begin{bmatrix} u_x^2 - 1 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 - 1 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 - 1 \end{bmatrix}$$



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$$\hat{U}^3 = \begin{bmatrix} u_x^2 - 1 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 - 1 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 - 1 \end{bmatrix} \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_x u_y u_z - u_x u_y u_z & -u_x^2 u_z + u_z + u_x^2 u_z & u_x^2 u_y - u_y - u_x^2 u_y \\ u_y^2 u_z - u_z - u_y^2 u_z & -u_x u_y u_z + u_x u_y u_z & u_x u_y^2 - u_x u_y^2 + u_x \\ u_y u_z^2 - u_y u_z^2 + u_y & -u_x u_z^2 + u_x u_z^2 - u_x & u_x u_y u_z - u_x u_y u_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & u_z & -u_y \\ -u_z & 0 & u_x \\ u_y & -u_x & 0 \end{bmatrix} = -\hat{U}$$

$$\hat{U}^4 = -\hat{U}^2 \quad \hat{U}^5 = \hat{U} \quad \hat{U}^6 = \hat{U}^2 \dots \quad \begin{matrix} \hat{U} & \hat{U}^2 & -\hat{U} & -\hat{U}^2 & \hat{U} & \hat{U}^2 & -\hat{U} & -\hat{U}^2 \\ \uparrow, & \uparrow, & \uparrow, & \uparrow, & \uparrow, & \uparrow, & \uparrow, & \uparrow \\ \hat{U}^1 & \hat{U}^2 & \hat{U}^3 & \hat{U}^4 & \hat{U}^5 & \hat{U}^6 & \hat{U}^7 & \hat{U}^8 \end{matrix}$$



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Now, let's evaluate  $e^{\hat{U}\theta}$

$$\begin{aligned}e^{\hat{U}\theta} &= I_3 + \frac{\theta\hat{U}}{1!} + \frac{\theta^2\hat{U}^2}{2!} + \frac{\theta^3\hat{U}^3}{3!} + \frac{\theta^4\hat{U}^4}{4!} + \frac{\theta^5\hat{U}^5}{5!} + \frac{\theta^6\hat{U}^6}{6!} + \dots \\&= I_3 + \theta\hat{U} + \frac{\theta^2}{2!}\hat{U}^2 - \frac{\theta^3}{3!}\hat{U} - \frac{\theta^4}{4!}\hat{U}^2 + \frac{\theta^5}{5!}\hat{U} + \frac{\theta^6}{6!}\hat{U}^2 - \frac{\theta^7}{7!}\hat{U} + \dots \\&= I_3 + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)\hat{U} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \frac{\theta^8}{8!} + \dots\right)\hat{U}^2 \\&= I_3 + \sin\theta\hat{U} + (1 - \cos\theta)\hat{U}^2 \quad \text{we need to prove that}\end{aligned}$$





# Rodriguez Approach for Representing Rotation

According to Taylor Series expansion

$$\sin \theta = \sin(0) + \cos(0)\theta - \sin(0)\frac{\theta^2}{2!} - \cos(0)\frac{\theta^3}{3!} + \sin(0)\frac{\theta^4}{4!} + \cos(0)\frac{\theta^5}{5!}$$

$$- \sin(0)\frac{\theta^6}{6!} - \cos(0)\frac{\theta^7}{7!} + \dots$$

$$= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}$$

$$\cos \theta = \cos(0) - \sin(0)\theta - \cos(0)\frac{\theta^2}{2!} + \sin(0)\frac{\theta^3}{3!} + \cos(0)\frac{\theta^4}{4!} - \sin(0)\frac{\theta^5}{5!}$$

$$- \cos(0)\frac{\theta^6}{6!} + \sin(0)\frac{\theta^7}{7!} + \cos(0)\frac{\theta^8}{8!} \dots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!}$$



# Rodriguez Approach for Representing Rotation

So, finally

$$R = I_3 + \sin \theta \hat{U} + (1 - \cos \theta) \hat{U}^2 = e^{\hat{U}\theta}$$

Rodriguez Rotation Formula

$$\vec{P}' = [I_3 + \sin \theta \hat{U} + (1 - \cos \theta) \hat{U}^2] \vec{P}$$

where  $\hat{U} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$  &  $\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$  is unit direction vector