Least Squares Adjustment with Rank-Deficient Weight Matrix

1. Model

$$\vec{y} = A\delta\vec{x} + \vec{d} + \vec{e} \quad \vec{e} \sim (0, \Sigma^{`}) \quad where\ \Sigma^{`} = \sigma_o^2 P^{`+}\ and\ P^{`}\vec{d} = 0 \qquad \text{A.1}$$

2. LSA Target Function $\phi(\delta\vec{x}, \vec{d})$

$$\phi(\delta\vec{x}, \vec{d}) = \vec{e}^T P^{`} \vec{e} = \left(\vec{y} - A\,\delta\vec{x} - \vec{d}\right)^T P^{`} \left(\vec{y} - A\,\delta\vec{x} - \vec{d}\right) = min|_{\delta\vec{x},\vec{d}} \qquad \text{A.2}$$

Since $P^{`}\vec{d} = 0$:

$$\phi(\delta\vec{x}, \vec{d}) = \phi(\delta\vec{x}) = \vec{e}^T P^{`} \vec{e} = (\vec{y} - A\delta\vec{x})^T P^{`} (\vec{y} - A\delta\vec{x}) = min|_{\delta\vec{x}} \qquad \text{A.3}$$

Expanding Equation A.3 we get:

$$\begin{aligned}\phi(\delta\vec{x}) &= (\vec{y} - A\,\delta\vec{x})^T P^{`} (\vec{y} - A\,\delta\vec{x}) \\ &= \vec{y}^T P^{`} \vec{y} - \vec{y}^T P^{`} A\,\delta\vec{x} - \delta\vec{x}^T A^T P^{`} \vec{y} + \delta\vec{x}^T A^T P^{`} A\,\delta\vec{x} = min|_{\delta\vec{x}}\end{aligned} \qquad \text{A.4}$$

Equation A.3 can be simplified to:

$$\phi(\delta\vec{x}) = \vec{y}^T P^{`} \vec{y} + \delta\vec{x}^T A^T P^{`} A\,\delta\vec{x} - 2\delta\vec{x}^T A^T P^{`} \vec{y} = min|_{\delta\vec{x}} \qquad \text{A.5}$$

3. Solution Vector $(\delta\hat{\vec{x}})$

The solution vector $(\delta\hat{\vec{x}})$ that minimizes $\phi(\delta\vec{x})$ can be obtained by differentiating $\phi(\delta\vec{x})$ w.r.t. $\delta\vec{x}$ and equating it to zero:

$$\frac{\partial\phi}{\partial\delta\vec{x}} = 2A^T P^{`} A\,\delta\vec{x} - 2A^T P^{`} \vec{y} = 0 \qquad \text{A.6}$$

$$\delta\hat{\vec{x}} = \left(A^T P^{`} A\right)^{-1} A^T P^{`} \vec{y} = N^{-1} A^T P^{`} \vec{y} \quad \text{where } N = A^T P^{`} A \qquad \text{A.7}$$

4. Variance-covariance matrix of the solution vector $(\Sigma\{\delta\hat{\vec{x}}\})$

Using the law of error propagation, the variance-covariance matrix of the solution vector $(\Sigma\{\delta\hat{\vec{x}}\})$ can be obtained as follows:

$$\Sigma\{\delta\hat{\vec{x}}\} = \sigma_o^2 N^{-1} A^T P^{`} P^{`+} P^{`} A N^{-1} \qquad \text{A.8}$$

Since for a Moore-Penrose pseudo-inverse, $P^{`} P^{`+} P^{`} = P^{`}$ (Koch, 1988):

$$\Sigma\{\widehat{\delta\vec{x}}\} = \sigma_o^2 N^{-1} N N^{-1} = \sigma_o^2 N^{-1} \qquad \text{A.9}$$

5. A-posteriori variance factor ($\hat{\sigma}_o^2$)

The a-posteriori variance factor ($\hat{\sigma}_o^2$) is obtained by deriving the expected value of the sum of squares of the weighted predicted residuals:

$$E\left(\tilde{\vec{e}}^T P^{`} \tilde{\vec{e}}\right) = E\{(\vec{y} - A\widehat{\delta\vec{x}} - \vec{d})^T P^{`} \left(\vec{y} - A\widehat{\delta\vec{x}} - \vec{d}\right)\} \qquad \text{A.10}$$

Since $P^{`}\vec{d} = 0$, Equation A.10 gets the form:

$$E\left(\tilde{\vec{e}}^T P^{`} \tilde{\vec{e}}\right) = E\left\{\left(\vec{y} - A\widehat{\delta\vec{x}}\right)^T P^{`} \left(\vec{y} - A\widehat{\delta\vec{x}}\right)\right\} \qquad \text{A.11}$$

Expanding Equation A.11 while using the derived solution for $\widehat{\delta\vec{x}}$ in Equation A.7 we get (while considering that $\left(I_n - AN^{-1}A^T P^{`}\right)$ is an idempotent matrix):

$$E\left(\tilde{\vec{e}}^T P^{`} \tilde{\vec{e}}\right) = E\left\{\vec{y}^T P^{`} \vec{y} - \vec{y}^T P^{`} A N^{-1} A^T P^{`} \vec{y}\right\} \qquad \text{A.12}$$

Given that the trace of a scalar equals to the scalar, i.e., *tr(S)* = *S* and that the trace operation is commutative, i.e., *tr(AB)* = *tr(BA)* (Koch, 1988), Equation A.12 can be manipulated as follows:

$$E\left(\tilde{\vec{e}}^T P^{`} \tilde{\vec{e}}\right) = E\left\{tr\left(P^{`} \vec{y}\vec{y}^T\right) - tr(P^{`} A N^{-1} A^T P^{`} \vec{y}\vec{y}^T)\right\} \qquad \text{A.13}$$

Based on the properties that *tr(A)* + *tr(B)* = *tr(A+B)* and that $E\{tr(A)\} = trE(A)$ (Koch, 1988), Equation A.13 can be rewritten as follows:

$$\begin{aligned} E\left(\tilde{\vec{e}}^T P^{`} \tilde{\vec{e}}\right) &= trP^{`}\left[E(\vec{y}\vec{y}^T) - AN^{-1}A^T P^{`} E(\vec{y}\vec{y}^T)\right] \\ &= trP^{`}\left(I_n - AN^{-1}A^T P^{`}\right)E(\vec{y}\vec{y}^T) \end{aligned} \qquad \text{A.14}$$

where:

I_n is an nxn identity matrix.

The term $E(\vec{y}\vec{y}^T)$ can be derived from the variance-covariance matrix of the observations vector ($\Sigma\{\vec{y}\}$) as follows:

$$\Sigma\{\vec{y}\} = \sigma_o^2 P^{`+} = E\left\{\left(\vec{y} - A\,\delta\vec{x} - \vec{d}\right)\left(\vec{y} - A\,\delta\vec{x} - \vec{d}\right)^T\right\} \qquad \text{A.15}$$

Expanding Equation A.15, we get:

$$E(\vec{y}\vec{y}^T) = \sigma_o^2 P^{`+} + \left(A\,\delta\vec{x} + \vec{d}\right)\left(A\,\delta\vec{x} + \vec{d}\right)^T$$
$$= \sigma_o^2 P^{`+} + A\delta\vec{x}\delta\vec{x}^T A^T + A\delta\vec{x}\vec{d}^T + \vec{d}\delta\vec{x}^T A^T + \vec{d}\vec{d}^T \quad \text{A.16}$$

Substituting Equation A.16 in Equation A.14 yields:

$$E\left(\tilde{\vec{e}}^T P^` \tilde{\vec{e}}\right) = trP^`\left(I - AN^{-1}A^T P^`\right)\left[\sigma_o^2 P^{`+} + A\delta\vec{x}\delta\vec{x}^T A^T + A\delta\vec{x}\vec{d}^T + \vec{d}\delta\vec{x}^T A^T + \vec{d}\vec{d}^T\right] \quad \text{A.17}$$

Given that $P^`\vec{d} = 0$ and $\left(I - AN^{-1}A^T P^`\right)A = 0$, Equation A.17 can be simplified to:

$$E\left(\tilde{\vec{e}}^T P^` \tilde{\vec{e}}\right) = \sigma_o^2 trP^`\left(I - AN^{-1}A^T P^`\right)P^{`+} = \sigma_o^2 trP^` P^{`+} - \sigma_o^2 trN^{-1}A^T P^` P^{`+} P^` A \quad \text{A.18}$$

Based on the property that $tr(AB) = rank(AB)$ (given that AB is idempotent) and $rank(AB) \leq \min(rankA, rankB)$ (Koch, 1988), the following can be stated:

$$tr(P^` P^{`+}) = rank(P^` P^{`+}) = min\left(rankP^`, rankP^{`+}\right) = rankP^` = q \quad \text{A.19}$$

Given that $tr(P^` P^{`+}) = q$ (as shown in Equation A.19) and that $P^` P^{`+} P^` = P^`$, Equation A.18 can be simplified to:

$$E\left(\tilde{\vec{e}}^T P^` \tilde{\vec{e}}\right) = \sigma_o^2 q - \sigma_o^2 trN^{-1}N = \sigma_o^2 q - \sigma_o^2 trI_m = \sigma_o^2 q - \sigma_o^2 m \quad \text{A.20}$$

where,

m $is\ the\ number\ of\ unknwon\ parameters.$

Finally, we can get the expression for the estimated a-posteriori variance factor ($\hat{\sigma}_o^2$) from Equation A.20 as follows:

$$\hat{\sigma}_o^2 = \frac{\tilde{\vec{e}}^T P^` \tilde{\vec{e}}}{(rankP^` - m)} \quad \text{A.21}$$