



Chapters 1 – 4: Overview

- Photogrammetry: introduction, applications, and tools
- GNSS/INS-assisted photogrammetric and LiDAR mapping
- LiDAR mapping: principles, applications, mathematical model, and error sources and their impact.
- QA/QC of LiDAR mapping
- This chapter will be focusing on an alternative approach for the representation of rotation in 3D space: quaternions
 - Definition
 - Properties
 - Rotation axis and rotation angle representation of a rotation in 3D space



Chapter 5

QUATERNIONS & ROTATION IN 3D SPACE



Overview

- Quaternions: definition
- Quaternion properties
- Quaternions and rotation matrices
- Quaternion-rotation matrices relationship
- Spherical linear interpolation
- Concluding remarks

Quaternions

$$q = q_0 + q_x i + q_y j + q_z k$$

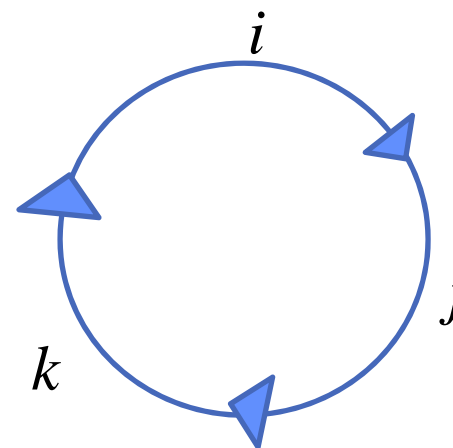
Real Part Imaginary Part

$$i^2 = j^2 = k^2 = ijk = -1$$

$$i = jk = -kj$$

$$j = ki = -ik$$

$$k = ij = -ji$$



- The real part for a “*Pure Quaternion*” is zero.



Quaternion Multiplication

$$\mathbf{q}_a = q_{o_a} + q_{x_a}\mathbf{i} + q_{y_a}\mathbf{j} + q_{z_a}\mathbf{k} = q_{o_a} + \vec{\mathbf{q}}_a = (q_{o_a}; \vec{\mathbf{q}}_a)$$

$$\mathbf{q}_b = q_{o_b} + q_{x_b}\mathbf{i} + q_{y_b}\mathbf{j} + q_{z_b}\mathbf{k} = q_{o_b} + \vec{\mathbf{q}}_b = (q_{o_b}; \vec{\mathbf{q}}_b)$$

$$\begin{aligned} \mathbf{q}_a\mathbf{q}_b &= q_{o_a} (q_{o_b} + q_{x_b}\mathbf{i} + q_{y_b}\mathbf{j} + q_{z_b}\mathbf{k}) \\ &+ q_{x_a}\mathbf{i} (q_{o_b} + q_{x_b}\mathbf{i} + q_{y_b}\mathbf{j} + q_{z_b}\mathbf{k}) \\ &+ q_{y_a}\mathbf{j} (q_{o_b} + q_{x_b}\mathbf{i} + q_{y_b}\mathbf{j} + q_{z_b}\mathbf{k}) \\ &+ q_{z_a}\mathbf{k} (q_{o_b} + q_{x_b}\mathbf{i} + q_{y_b}\mathbf{j} + q_{z_b}\mathbf{k}) \end{aligned}$$

- Using the rules in the previous slide, we can get the following definition for quaternion multiplication:

$$\mathbf{q}_a\mathbf{q}_b = (q_{o_a}q_{o_b} - \vec{\mathbf{q}}_a \cdot \vec{\mathbf{q}}_b; q_{o_a}\vec{\mathbf{q}}_b + q_{o_b}\vec{\mathbf{q}}_a + \vec{\mathbf{q}}_a \times \vec{\mathbf{q}}_b)$$



Quaternion Multiplication

$$\mathbf{q}_a = q_{o_a} + q_{x_a} \mathbf{i} + q_{y_a} \mathbf{j} + q_{z_a} \mathbf{k} = q_{o_a} + \vec{\mathbf{q}}_a = (q_{o_a}; \vec{\mathbf{q}}_a)$$

$$\mathbf{q}_b = q_{o_b} + q_{x_b} \mathbf{i} + q_{y_b} \mathbf{j} + q_{z_b} \mathbf{k} = q_{o_b} + \vec{\mathbf{q}}_b = (q_{o_b}; \vec{\mathbf{q}}_b)$$

$$\mathbf{q}_a \mathbf{q}_b = \mathbf{C}_{q_a} \mathbf{q}_b = \bar{\mathbf{C}}_{q_b} \mathbf{q}_a$$

$$\mathbf{C}_{q_a} = \begin{bmatrix} q_{o_a} & -q_{x_a} & -q_{y_a} & -q_{z_a} \\ q_{x_a} & q_{o_a} & -q_{z_a} & q_{y_a} \\ q_{y_a} & q_{z_a} & q_{o_a} & -q_{x_a} \\ q_{z_a} & -q_{y_a} & q_{x_a} & q_{o_a} \end{bmatrix}$$

$$\bar{\mathbf{C}}_{q_b} = \begin{bmatrix} q_{o_b} & -q_{x_b} & -q_{y_b} & -q_{z_b} \\ q_{x_b} & q_{o_b} & q_{z_b} & -q_{y_b} \\ q_{y_b} & -q_{z_b} & q_{o_b} & q_{x_b} \\ q_{z_b} & q_{y_b} & -q_{x_b} & q_{o_b} \end{bmatrix}$$

- \mathbf{C}_{q_a} & $\bar{\mathbf{C}}_{q_b}$ simplify the quaternion multiplication to matrix-vector multiplication – ortho-normal matrices.



Quaternion Multiplication

- Unit quaternions:

$$q_0^2 + q_x^2 + q_y^2 + q_z^2 = 1$$

- For unit quaternions:

$$\begin{aligned}
 & \mathbf{C}_{q_a} \mathbf{C}_{q_a}^T \\
 &= \begin{bmatrix} q_{0a} & -q_{xa} & -q_{ya} & -q_{za} \\ q_{xa} & q_{0a} & -q_{za} & q_{ya} \\ q_{ya} & q_{za} & q_{0a} & -q_{xa} \\ q_{za} & -q_{ya} & q_{xa} & q_{0a} \end{bmatrix} \begin{bmatrix} q_{0a} & q_{xa} & q_{ya} & q_{za} \\ -q_{xa} & q_{0a} & q_{za} & -q_{ya} \\ -q_{ya} & -q_{za} & q_{0a} & q_{xa} \\ -q_{za} & q_{ya} & -q_{xa} & q_{0a} \end{bmatrix} \\
 &= \mathbf{I}_4
 \end{aligned}$$

$$\bar{\mathbf{C}}_{q_a} \bar{\mathbf{C}}_{q_a}^T = \mathbf{I}_4$$



Quaternion Properties

- Quaternion conjugate:

$$\mathbf{q}_a = q_{o_a} + q_{x_a}\mathbf{i} + q_{y_a}\mathbf{j} + q_{z_a}\mathbf{k} = q_{o_a} + \vec{\mathbf{q}}_a = (q_{o_a}; \vec{\mathbf{q}}_a)$$

$$\mathbf{q}_a^* = q_{o_a} - q_{x_a}\mathbf{i} - q_{y_a}\mathbf{j} - q_{z_a}\mathbf{k} = q_{o_a} - \vec{\mathbf{q}}_a = (q_{o_a}; -\vec{\mathbf{q}}_a)$$

$$\mathbf{q}_a\mathbf{q}_a^* = (q_{o_a} + \vec{\mathbf{q}}_a)(q_{o_a} - \vec{\mathbf{q}}_a)$$

$$\mathbf{q}_a\mathbf{q}_a^* = (q_{o_a}^2 + \vec{\mathbf{q}}_a \cdot \vec{\mathbf{q}}_a; q_{o_a}\vec{\mathbf{q}}_a - q_{o_a}\vec{\mathbf{q}}_a - \vec{\mathbf{q}}_a \times \vec{\mathbf{q}}_a)$$

$$\mathbf{q}_a\mathbf{q}_a^* = (\|\mathbf{q}_a\|; \mathbf{0})$$

- For unit quaternions:

$$\mathbf{q}_a\mathbf{q}_a^* = (\mathbf{1}; \mathbf{0})$$



Quaternion Properties

- Quaternion conjugate:

$$C_{q_a^*} = \begin{bmatrix} q_{o_a} & q_{x_a} & q_{y_a} & q_{z_a} \\ -q_{x_a} & q_{o_a} & q_{z_a} & -q_{y_a} \\ -q_{y_a} & -q_{z_a} & q_{o_a} & q_{x_a} \\ -q_{z_a} & q_{y_a} & -q_{x_a} & q_{o_a} \end{bmatrix} = C_{q_a}^T$$

$$\bar{C}_{q_b^*} = \begin{bmatrix} q_{o_b} & q_{x_b} & q_{y_b} & q_{z_b} \\ -q_{x_b} & q_{o_b} & -q_{z_b} & q_{y_b} \\ -q_{y_b} & q_{z_b} & q_{o_b} & -q_{x_b} \\ -q_{z_b} & -q_{y_b} & q_{x_b} & q_{o_b} \end{bmatrix} = \bar{C}_{q_b}^T$$

$$\begin{aligned} q_a \cdot (q_b q_c q_b^*) &= q_a \cdot (\bar{C}_{q_b}^* q_b q_c) = (\bar{C}_{q_b}^{*T} q_a) \cdot (q_b q_c) = (\bar{C}_{q_b} q_a) \cdot (q_b q_c) \\ &= (q_a q_b) \cdot (q_b q_c) \end{aligned}$$



Quaternions & Rotation Matrices

\vec{u} is a unit vector

- Given the following quaternions:

$$q = \cos\theta + \sin\theta\vec{u}$$

$$q^* = \cos\theta - \sin\theta\vec{u}$$

- q is a unit quaternion.
- $a\vec{u}$ is a pure quaternion (real part is zero).

$$a\vec{u} \longleftrightarrow a\dot{u} = (0; a\vec{u})$$

$$q a \dot{u} q^* = a q \dot{u} q^*$$

$$q \dot{u} = (\cos\theta; \sin\theta\vec{u})(0; \vec{u}) = (-\sin\theta\vec{u} \cdot \vec{u}; \cos\theta\vec{u} + \sin\theta\vec{u} \times \vec{u})$$

$$q \dot{u} = (-\sin\theta; \cos\theta\vec{u})$$

Quaternions & Rotation Matrices



$$q\dot{u}q^* = (-\sin\theta; \cos\theta\vec{u})(\cos\theta; -\sin\theta\vec{u})$$

$$q\dot{u}q^* = (-\sin\theta\cos\theta + \sin\theta\cos\theta\vec{u}\cdot\vec{u}; \\ \cos^2\theta\vec{u} + \sin^2\theta\vec{u} - \sin\theta\cos\theta\vec{u} \times \vec{u})$$

$$q\dot{u}q^* = (\mathbf{0}; \vec{u})$$

$$qaiq^* = ai$$

1

- The product $qaiq^*$ produces the same vector ai .

Quaternions & Rotation Matrices

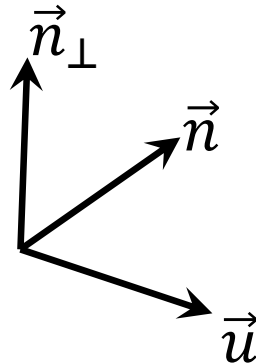


$$\vec{v} \longleftrightarrow v = (0; \vec{n} + a\vec{u})$$

- \vec{n} is perpendicular to \vec{u} .

$$q\dot{n} = (\cos\theta; \sin\theta\vec{u})(0; \vec{n}) = (-\sin\theta\vec{u} \cdot \vec{n}; \cos\theta\vec{n} + \sin\theta\vec{u} \times \vec{n})$$

$$q\dot{n} = (\cos\theta; \sin\theta\vec{u})(0; \vec{n}) = (0; \cos\theta\vec{n} + \sin\theta\vec{n}_{\perp})$$

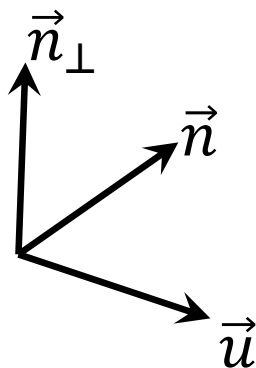




Quaternions & Rotation Matrices

$$q\dot{n}q^* = (0; \cos\theta\vec{n} + \sin\theta\vec{n}_\perp)(\cos\theta; -\sin\theta\vec{u})$$

$$\begin{aligned} q\dot{n}q^* &= (\sin\theta\cos\theta\vec{n} \cdot \vec{u} + \sin^2\theta\vec{n}_\perp \cdot \vec{u}; \cos^2\theta\vec{n} + \sin\theta\cos\theta\vec{n}_\perp \\ &\quad - \sin\theta\cos\theta\vec{n} \times \vec{u} - \sin^2\theta\vec{n}_\perp \times \vec{u}) \end{aligned}$$



$$q\dot{n}q^* = (0; \cos^2\theta\vec{n} + \sin\theta\cos\theta\vec{n}_\perp + \sin\theta\cos\theta\vec{n}_\perp - \sin^2\theta\vec{n})$$

$$q\dot{n}q^* = (0; [\cos^2\theta - \sin^2\theta]\vec{n} + 2\sin\theta\cos\theta\vec{n}_\perp)$$

Quaternions & Rotation Matrices

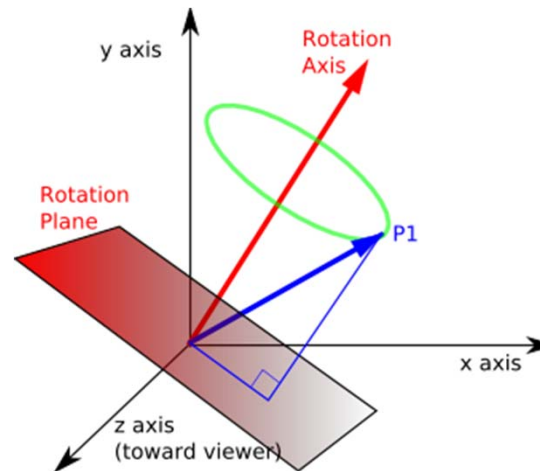
$$q\dot{n}q^* = (0; [\cos^2\theta - \sin^2\theta]\vec{n} + 2\sin\theta\cos\theta\vec{n}_\perp)$$

$$q\dot{n}q^* = (0; \cos(2\theta)\vec{n} + \sin(2\theta)\vec{n}_\perp)$$

2

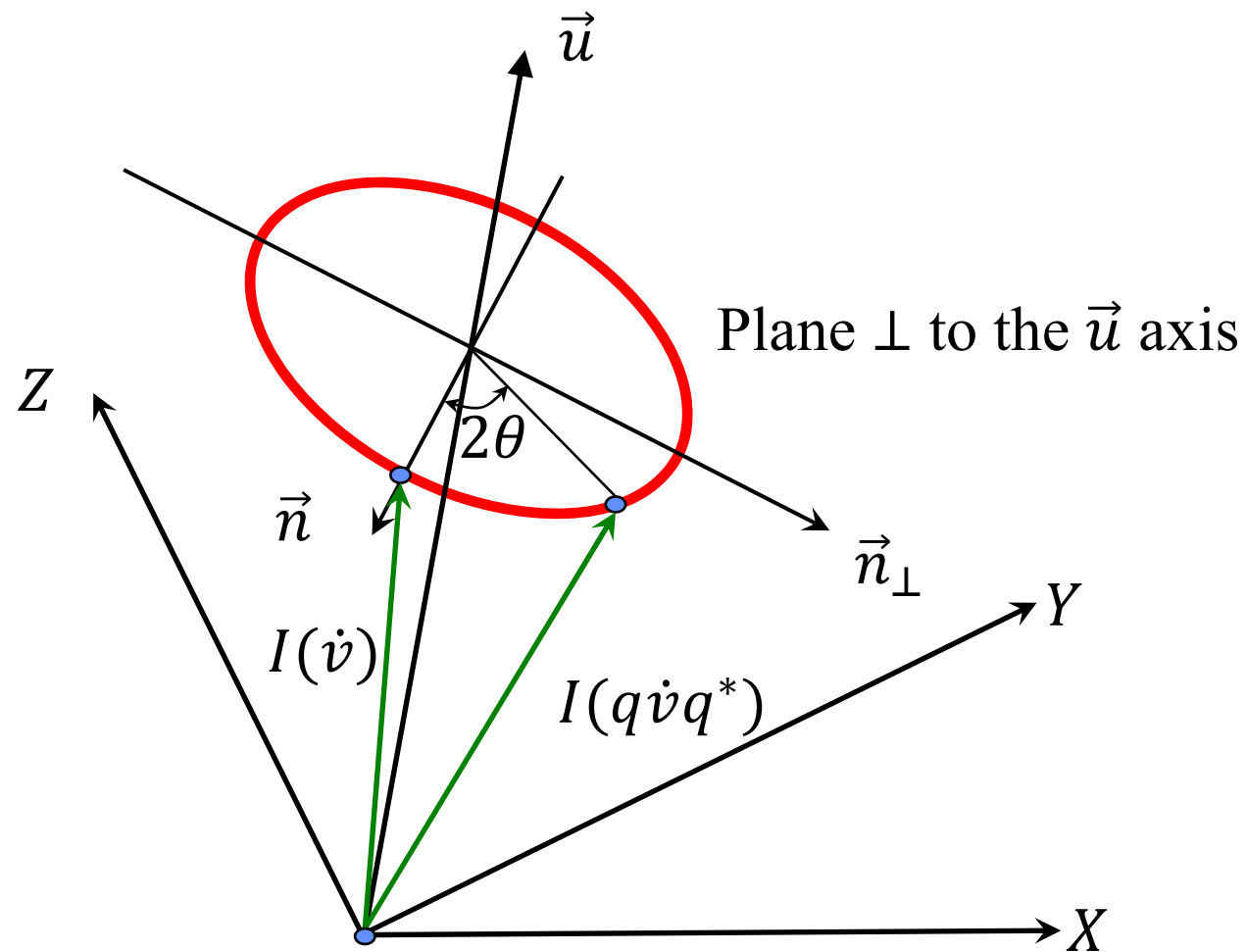
- From 1 & 2, one can conclude that:

$$q\dot{v}q^* = q(\dot{n} + a\dot{u})q^* = (0; a\vec{u} + \cos(2\theta)\vec{n} + \sin(2\theta)\vec{n}_\perp)$$



<http://www.euclideanspace.com>

Quaternions & Rotation Matrices

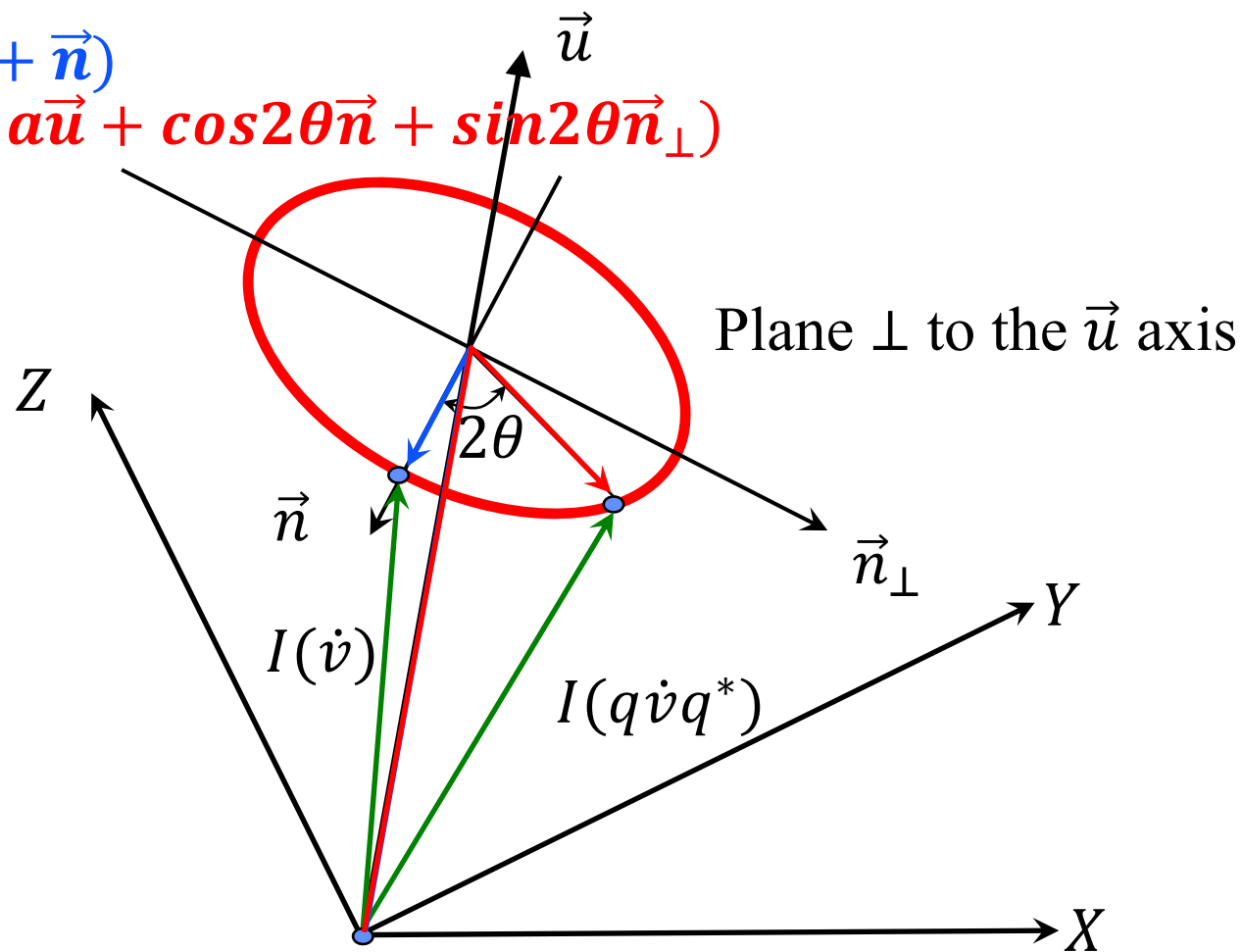


\dot{v} & $q\dot{v}q^*$ are pure quaternions
 $I(\dot{v})$ & $I(q\dot{v}q^*)$ are the imaginary components of \dot{v} & $q\dot{v}q^*$.

Quaternions & Rotation Matrices

$$\dot{v} = (\mathbf{0}; a\vec{u} + \vec{n})$$

$$q\dot{v}q^* = (\mathbf{0}; a\vec{u} + \cos 2\theta \vec{n} + \sin 2\theta \vec{n}_\perp)$$



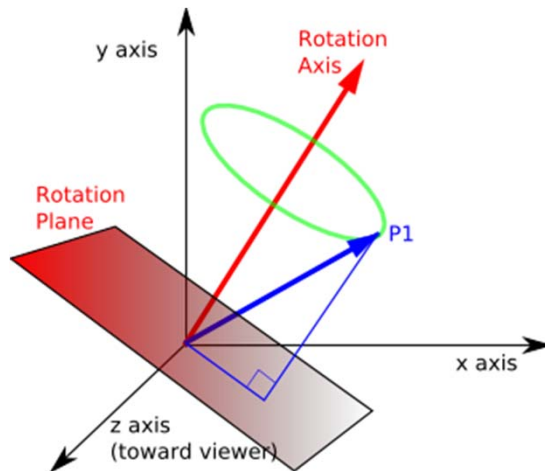
\dot{v} & $q\dot{v}q^*$ are pure quaternions

$I(\dot{v})$ & $I(q\dot{v}q^*)$ are the imaginary components of \dot{v} & $q\dot{v}q^*$.

Quaternions & Rotation Matrices

- Any 3D rotation matrix can be represented by a rotation (θ) around a unit vector (\vec{u}).
- This rotation can be defined by the following unit quaternion:

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)u_x i + \sin\left(\frac{\theta}{2}\right)u_y j + \sin\left(\frac{\theta}{2}\right)u_z k$$



<http://www.euclideanspace.com>



Quaternions & Rotation Matrices

- Rotation maintains the magnitude of a vector:

$$\begin{aligned} & (q\dot{v}q^*). (q\dot{v}q^*) \\ &= (\bar{C}_q^* C_q \dot{v}). (\bar{C}_q^* C_q \dot{v}) \end{aligned}$$

$$(q\dot{v}q^*). (q\dot{v}q^*) = \dot{v}^T C_q^T \bar{C}_q^T \bar{C}_q^* C_q \dot{v} = \dot{v}^T \dot{v}$$



Quaternions & Rotation Matrices

- Rotation maintains the angular deviation between two vectors:

$$(q\dot{v}_a q^*) \cdot (q\dot{v}_b q^*)$$

$$= (\bar{C}_{q^*} C_q \dot{v}_a) \cdot (\bar{C}_{q^*} C_q \dot{v}_b)$$

$$(q\dot{v}_a q^*) \cdot (q\dot{v}_b q^*) = \dot{v}_a^T C_q^T \bar{C}_{q^*}^T \bar{C}_{q^*} C_q \dot{v}_b = \dot{v}_a^T \dot{v}_b$$



Quaternions & Rotation Matrices

- Rotation maintains the magnitude of a triple product:

$$[v_a, v_b, v_c] = v_a \cdot (v_b \times v_c)$$

- Since:
 - Quaternion rotation maintains vector magnitude.
 - Quaternion rotation maintains angular deviation between two vectors.
- Then:
 - Quaternion rotation maintains the magnitude of the triple product.

$$[v_a, v_b, v_c] = [qv_aq^*, qv_bq^*, qv_cq^*]$$



Quaternions & Rotation Matrices

- Quaternion/rotation matrix relationship:

$$R_C^m \vec{v} \longleftrightarrow q \dot{v} q^*$$

$$q \dot{v} q^* = \bar{C}_{q^*} C_q \dot{v}$$

$$\bar{C}_{q^*} C_q = \begin{bmatrix} q_0 & q_x & q_y & q_z \\ -q_x & q_0 & -q_z & q_y \\ -q_y & q_z & q_0 & -q_x \\ -q_z & -q_y & q_x & q_0 \end{bmatrix} \begin{bmatrix} q_0 & -q_x & -q_y & -q_z \\ q_x & q_0 & -q_z & q_y \\ q_y & q_z & q_0 & -q_x \\ q_z & -q_y & q_x & q_0 \end{bmatrix}$$

Quaternions & Rotation Matrices



- Quaternion/rotation matrix relationship:

$$\bar{C}_q * C_q = \begin{bmatrix} q_0 & q_x & q_y & q_z \\ -q_x & q_0 & -q_z & q_y \\ -q_y & q_z & q_0 & -q_x \\ -q_z & -q_y & q_x & q_0 \end{bmatrix} \begin{bmatrix} q_0 & -q_x & -q_y & -q_z \\ q_x & q_0 & -q_z & q_y \\ q_y & q_z & q_0 & -q_x \\ q_z & -q_y & q_x & q_0 \end{bmatrix}$$

$$\bar{C}_q * C_q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r_{11} & r_{12} & r_{13} \\ 0 & r_{21} & r_{22} & r_{23} \\ 0 & r_{31} & r_{31} & r_{33} \end{bmatrix}$$



Quaternions & Rotation Matrices

Quaternion to Rotation Transformation

$$r_{11} = q_x^2 + q_o^2 - q_z^2 - q_y^2$$

$$r_{12} = 2q_xq_y - 2q_oq_z$$

$$r_{13} = 2q_xq_z + 2q_oq_y$$

$$r_{21} = 2q_xq_y + 2q_oq_z$$

$$r_{22} = q_y^2 - q_z^2 + q_o^2 - q_x^2$$

$$r_{23} = 2q_yq_z - 2q_oq_x$$

$$r_{31} = 2q_xq_z - 2q_oq_y$$

$$r_{32} = 2q_yq_z + 2q_oq_x$$

$$r_{33} = q_z^2 - q_y^2 - q_x^2 + q_o^2$$

q & $-q$ define the same rotation matrix

Quaternions & Rotation Matrices



Rotation to Quaternion Transformation (Option # 1)

$$r_{11} + r_{22} + r_{33} = 3q_0^2 - q_x^2 - q_y^2 - q_z^2$$

$$r_{11} + r_{22} + r_{33} = 4q_0^2 - 1$$

$$q_0 = \sqrt{(r_{11} + r_{22} + r_{33} + 1)/2}$$

$$r_{32} - r_{23} = 4q_0q_x$$

$$q_x = (r_{32} - r_{23})/4q_0$$

$$r_{13} - r_{31} = 4q_0q_y$$

$$q_y = (r_{13} - r_{31})/4q_0$$

$$r_{21} - r_{12} = 4q_0q_z$$

$$q_z = (r_{21} - r_{12})/4q_0 \quad \text{Assumption: } (r_{11} + r_{22} + r_{33} + 1) > 0$$

Quaternions & Rotation Matrices



Rotation to Quaternion Transformation (Option # 2)

$$r_{11} - r_{22} - r_{33} = 3q_x^2 - q_o^2 - q_y^2 - q_z^2 = 4q_x^2 - 1$$

$$q_x = \sqrt{(r_{11} - r_{22} - r_{33} + 1)/2}$$

$$r_{12} + r_{21} = 4q_x q_y$$

$$q_y = (r_{12} + r_{21})/4q_x$$

$$r_{13} + r_{31} = 4q_x q_z$$

$$q_z = (r_{13} + r_{31})/4q_x$$

$$r_{32} - r_{23} = 4q_o q_x$$

$$q_o = (r_{32} - r_{23})/4q_x$$

Assumption: $(r_{11} - r_{22} - r_{33} + 1) > 0$

Quaternions & Rotation Matrices



Rotation to Quaternion Transformation (Option # 3)

$$r_{22} - r_{11} - r_{33} = 3q_y^2 - q_0^2 - q_x^2 - q_z^2 = 4q_y^2 - 1$$

$$q_y^2 = (r_{22} - r_{11} - r_{33} + 1)/4$$

$$q_y = \sqrt{(r_{22} - r_{11} - r_{33} + 1)}/2$$

$$r_{12} + r_{21} = 4q_x q_y$$

$$q_x = (r_{12} + r_{21})/4q_y$$

$$r_{23} + r_{32} = 4q_y q_z$$

$$q_z = (r_{23} + r_{32})/4q_y$$

$$r_{13} - r_{31} = 4q_0 q_y$$

$$q_0 = (r_{13} - r_{31})/4q_y$$

Assumption: $(r_{22} - r_{11} - r_{33} + 1) > 0$

Quaternions & Rotation Matrices



Rotation to Quaternion Transformation (Option # 4)

$$r_{33} - r_{11} - r_{22} = 3q_z^2 - q_0^2 - q_x^2 - q_y^2 = 4q_z^2 - 1$$

$$4q_z^2 = (r_{33} - r_{11} - r_{22} + 1)$$

$$q_z = \sqrt{(r_{33} - r_{11} - r_{22} + 1)}/2$$

$$r_{13} + r_{31} = 4q_x q_z$$

$$q_x = (r_{13} + r_{31})/4q_z$$

$$r_{23} + r_{32} = 4q_y q_z$$

$$q_y = (r_{23} + r_{32})/4q_z$$

$$r_{21} - r_{12} = 4q_0 q_z$$

$$q_0 = (r_{21} - r_{12})/4q_z \quad \text{Assumption: } (r_{33} - r_{11} - r_{22} + 1) > 0$$

Quaternions & Rotation Matrices



Rotation to Quaternion Transformation

- Among the options, choose the one that ensures the highest numerical stability.
- Option # 1: q_o is the largest among (q_o , q_x , q_y , and q_z).
- Option # 2: q_x is the largest among (q_o , q_x , q_y , and q_z).
- Option # 3: q_y is the largest among (q_o , q_x , q_y , and q_z).
- Option # 4: q_z is the largest among (q_o , q_x , q_y , and q_z).



Quaternions & Rotation Matrices

- The product of two quaternions:

$$q_\alpha = \cos\alpha + \sin\alpha \vec{u} \qquad q_\beta = \cos\beta + \sin\beta \vec{u}$$

$$q_\alpha q_\beta = (\cos\alpha; \sin\alpha \vec{u})(\cos\beta; \sin\beta \vec{u})$$

$$q_\alpha q_\beta = (\cos\alpha \cos\beta - \sin\alpha \sin\beta \vec{u} \cdot \vec{u}; \\ \cos\alpha \sin\beta \vec{u} + \sin\alpha \cos\beta \vec{u} + \sin\alpha \sin\beta \vec{u} \times \vec{u})$$

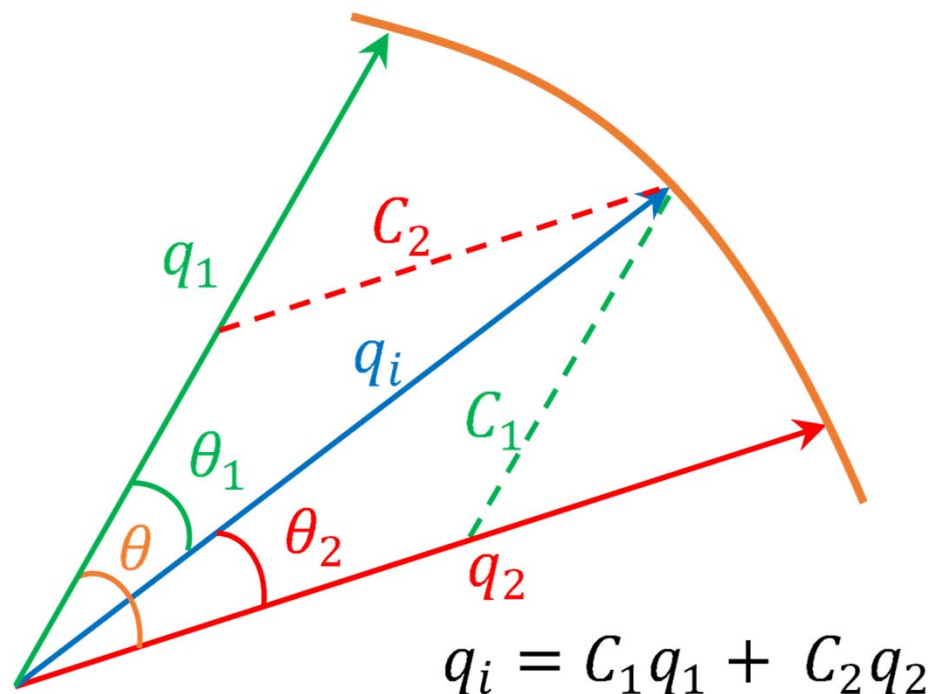
$$q_\alpha q_\beta = (\cos\alpha \cos\beta - \sin\alpha \sin\beta; \\ [\cos\alpha \sin\beta + \sin\alpha \cos\beta] \vec{u})$$

$$q_\alpha q_\beta = (\cos[\alpha + \beta]; \sin[\alpha + \beta] \vec{u})$$

- This product is equivalent to rotation angle $(2[\alpha + \beta])$ around the axis \vec{u} .

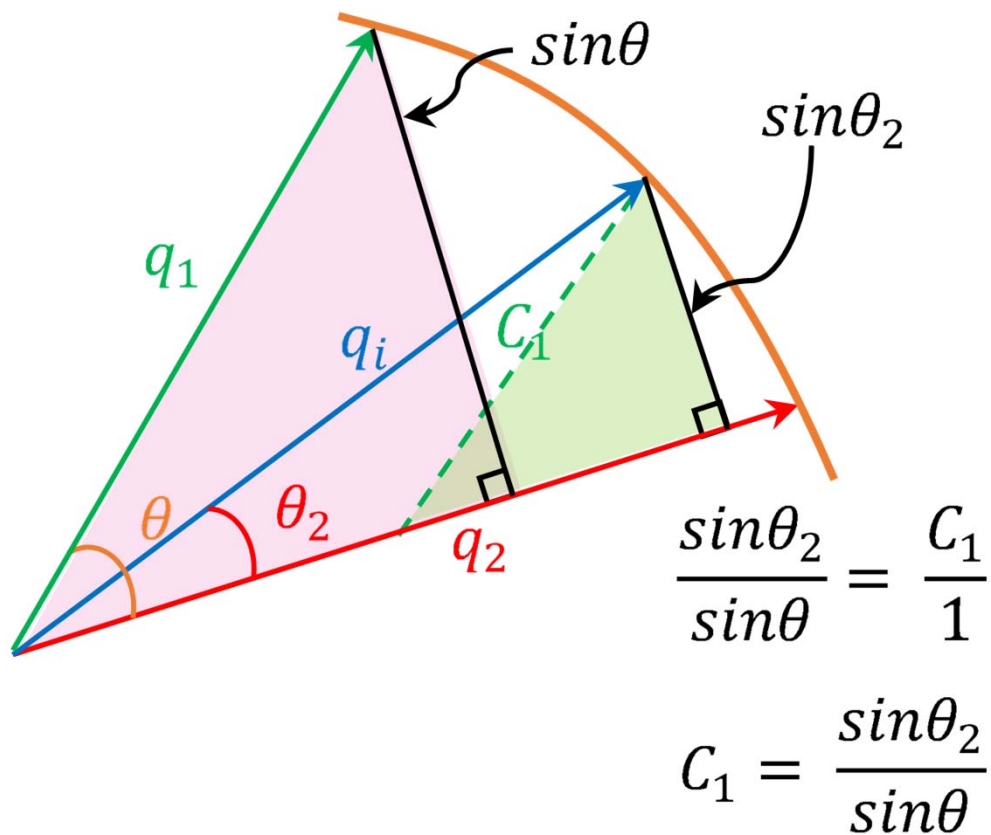
Spherical Linear Interpolation

- **Problem Statement:** Given the rotations represented by q_1 and q_2 , whose angular deviation is θ , we need to evaluate the interpolated quaternion rotation q_i , whose angular deviations to q_1 and q_2 are θ_1 and θ_2 , respectively.

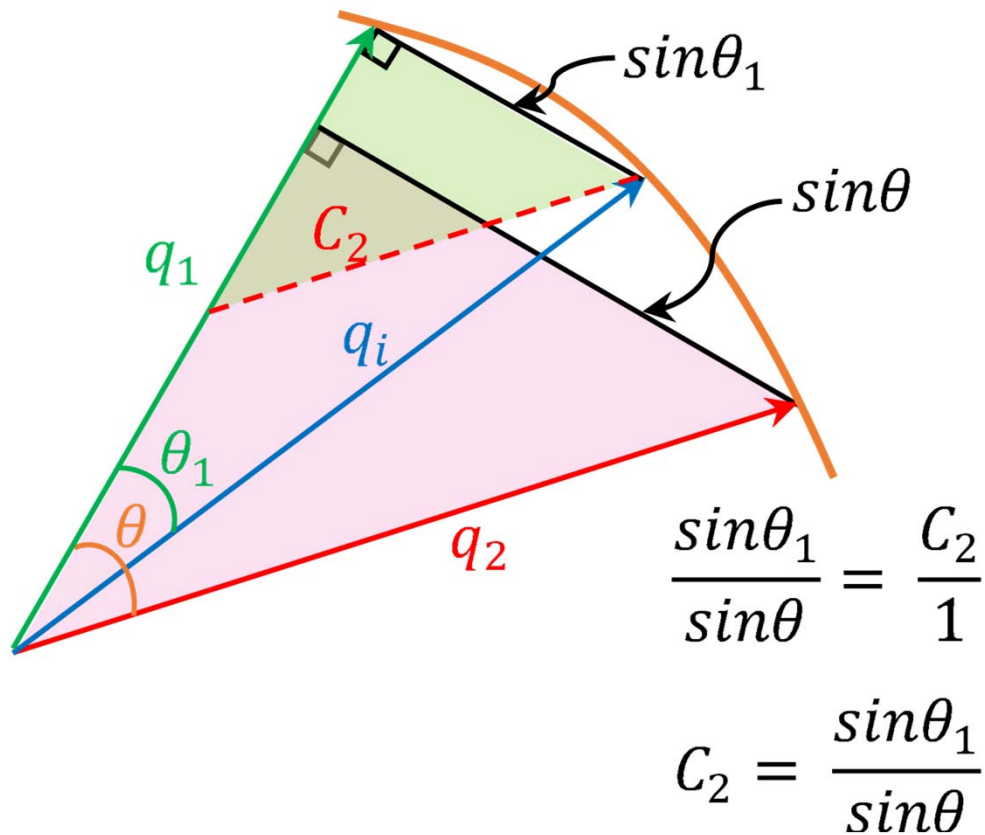


- As per the figure above: $q_i = C_1 q_1 + C_2 q_2$

Spherical Linear Interpolation



Spherical Linear Interpolation

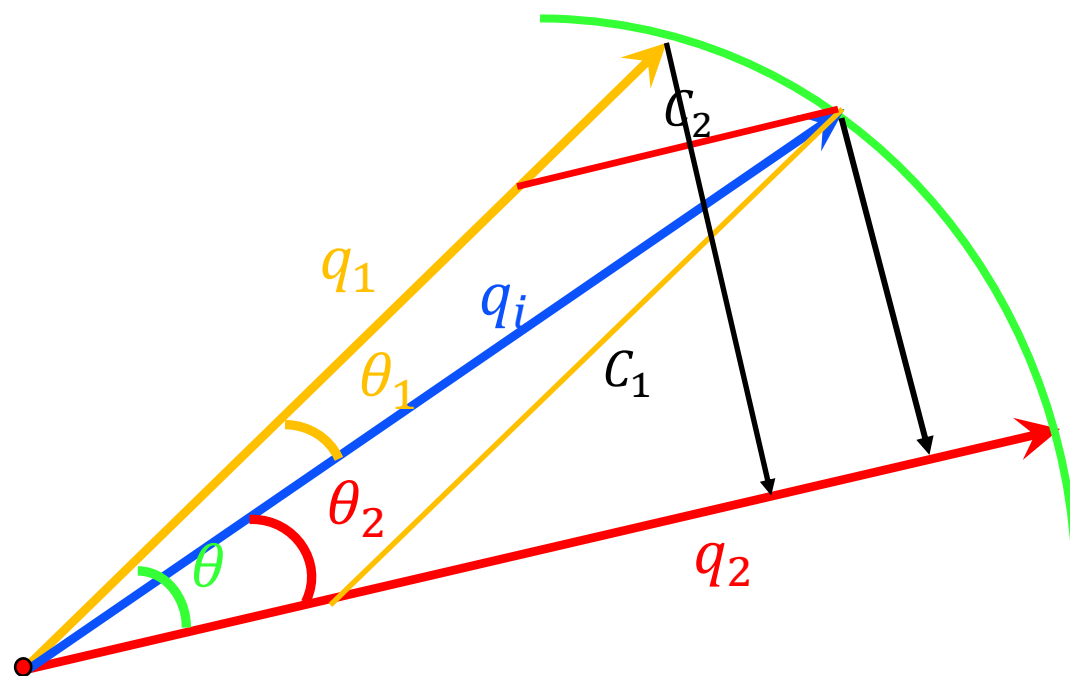


Spherical Linear Interpolation

$$q_1 \cdot q_2 = \cos\theta$$

$$C_1 / \sin\theta = C_1 = \sin\theta_2 / \sin\theta$$

$$C_2 / \sin\theta = C_2 = \sin\theta_1 / \sin\theta$$



$$q_i = C_1 q_1 + C_2 q_2$$

$$q_i = \frac{\sin\theta_2}{\sin\theta} q_1 + \frac{\sin\theta_1}{\sin\theta} q_2$$



Spherical Linear Interpolation

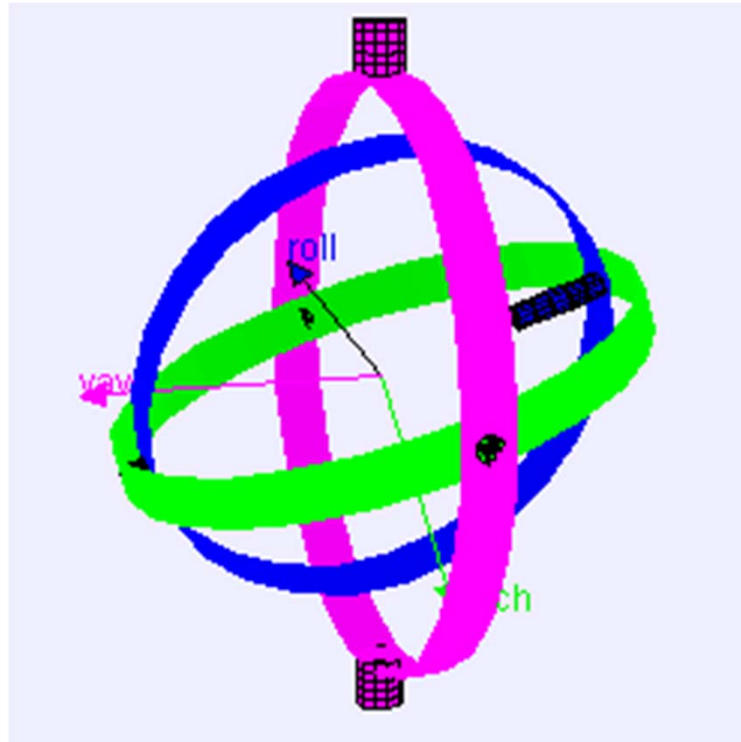
- Spherical Linear Interpolation is useful for:
 - Interpolation of derived rotation matrices from integrated GNSS/INS attitude – This is the case when deriving the rotation matrices at much higher rate than that derived from GNSS/INS unit (LiDAR & Line Camera systems)
 - Modeling variation of the rotation matrices as time dependent values for Line Camera Systems

Quaternions & Rotation Matrices



- Quaternions characteristics compared to rotation matrices:
 - It avoids the gimbal lock problem.
 - Happens whenever the secondary rotation is 90°
 - Two rotations take place around the same axis in space.
 - Quaternion multiplication requires fewer operations compared to multiplication of two rotation matrices.
 - Quaternion-based rotation requires more operations when compared to traditional rotation of vectors.
 - Quaternions has one constraint while rotation matrices has 6 orthogonality constraints.
 - Interpolation of quaternion rotations is much more straight forward than 3D rotation matrices.

Gimbal Lock

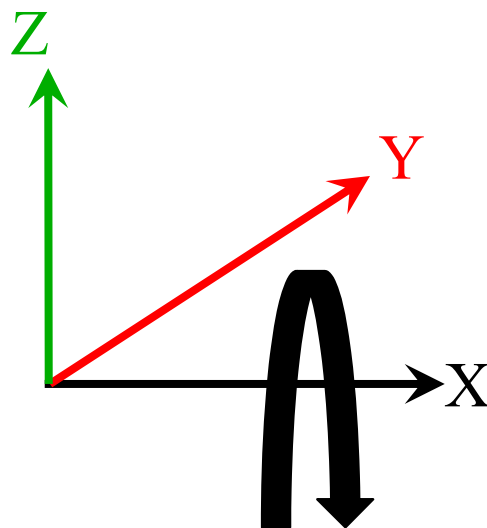


http://en.wikipedia.org/wiki/Gimbal_lock

- A set of three gimbals mounted together to allow three degrees of freedom: roll, pitch and yaw.
- When two gimbals rotate around the same axis, the system loses one degree of freedom.

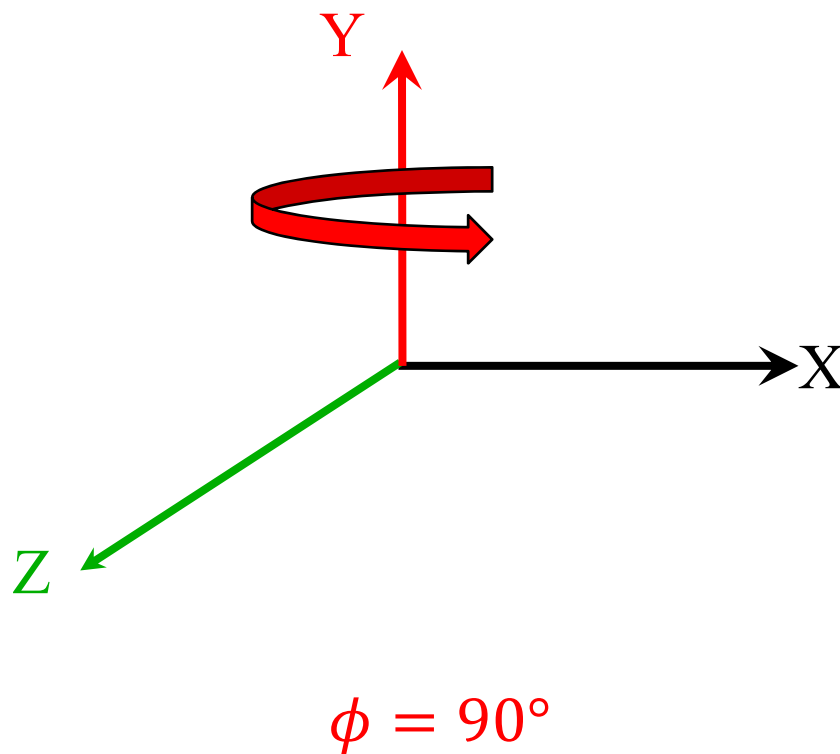


Gimbal Lock

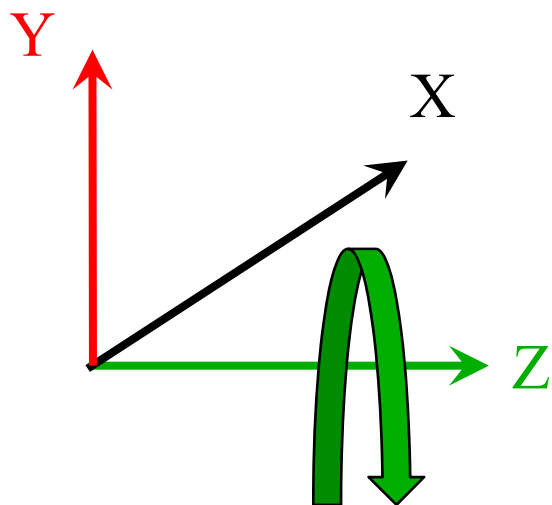


$$\omega = 90^\circ$$

Gimbal Lock



Gimbal Lock

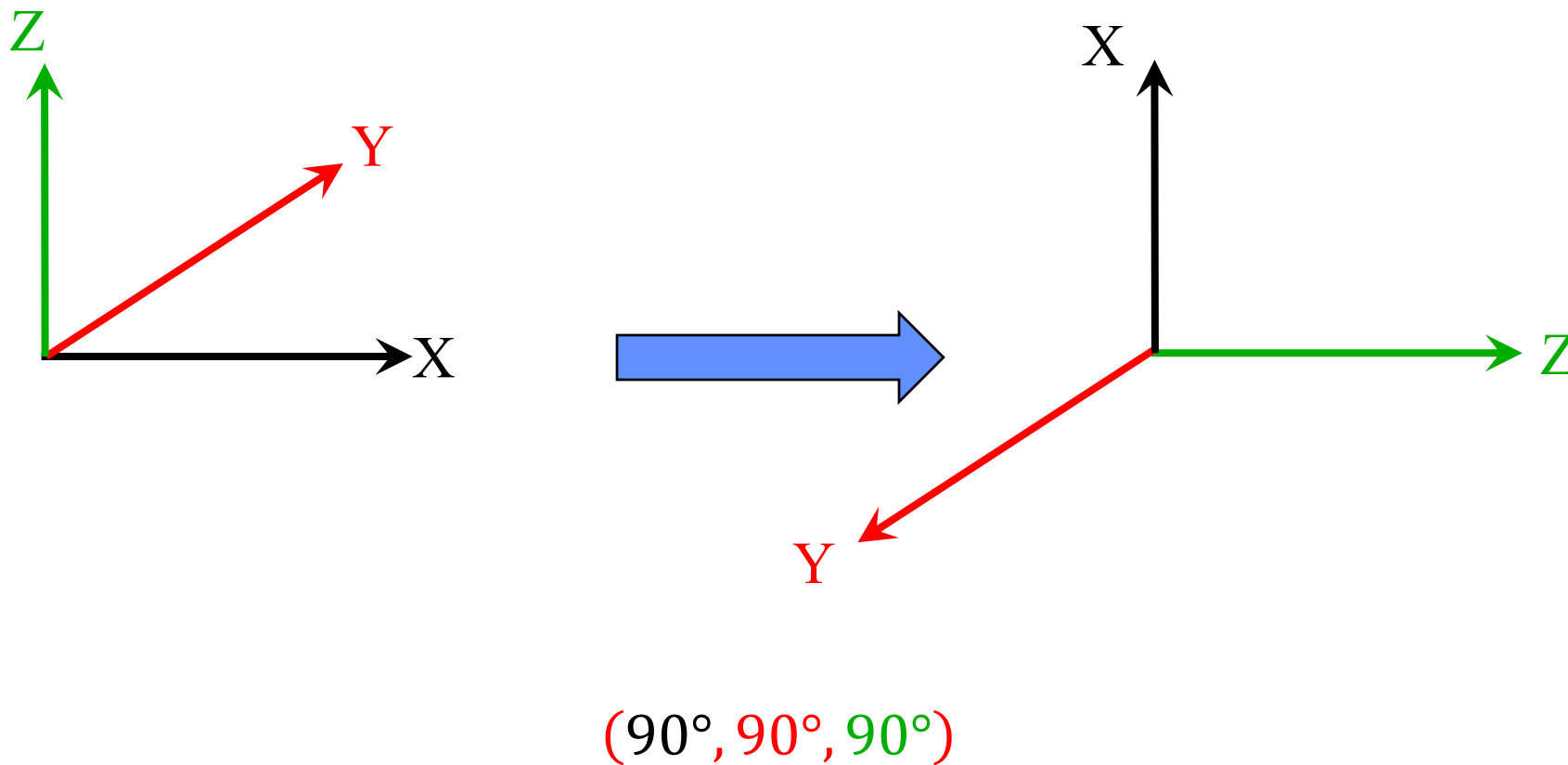


$$\kappa = 90^\circ$$

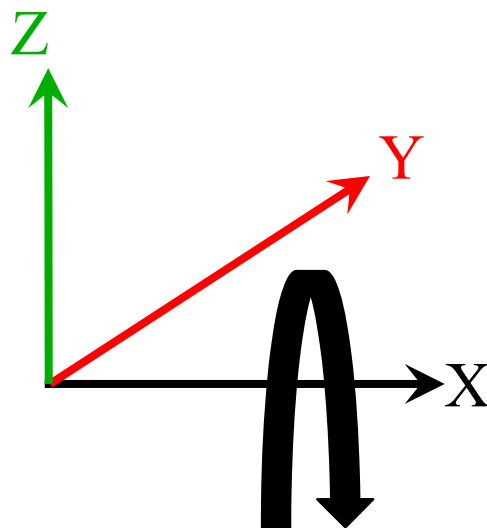
ω & κ rotation angles are around the same axis in space.



Gimbal Lock



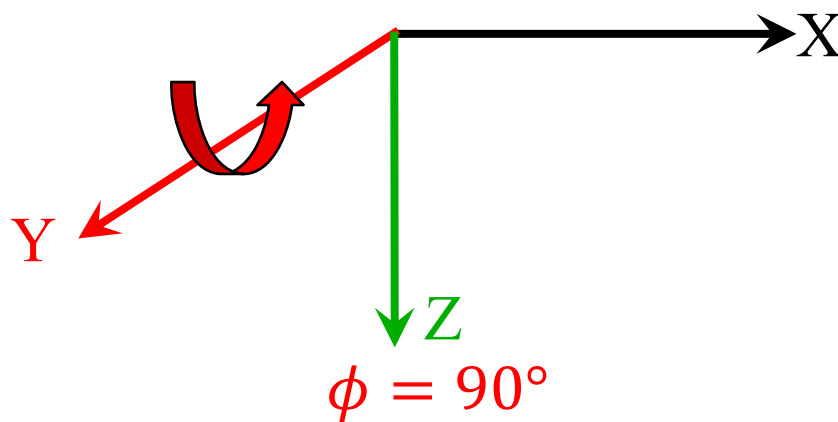
Gimbal Lock



$$\omega = 180^\circ$$

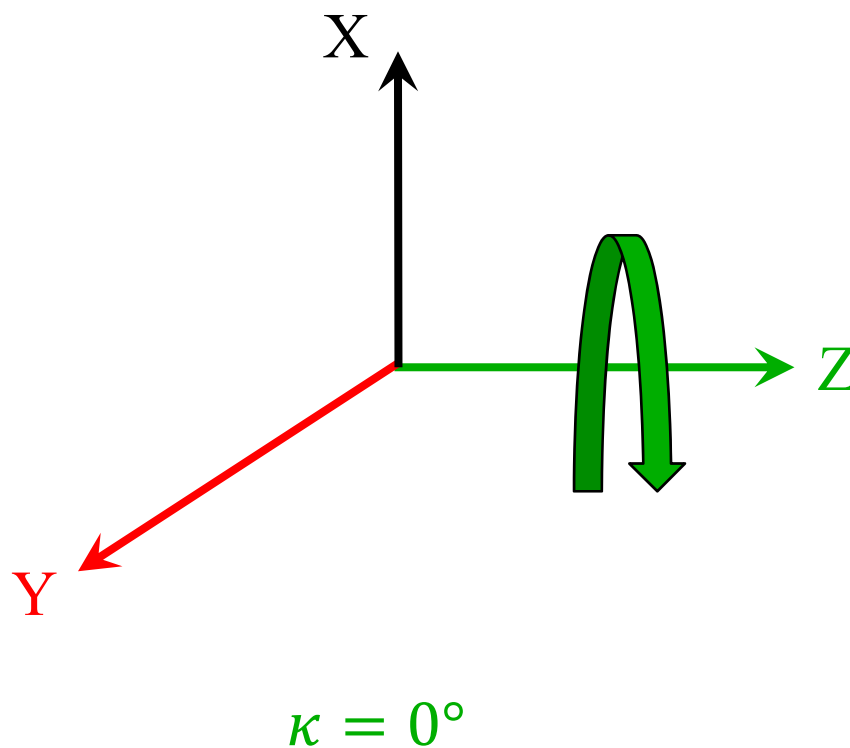


Gimbal Lock

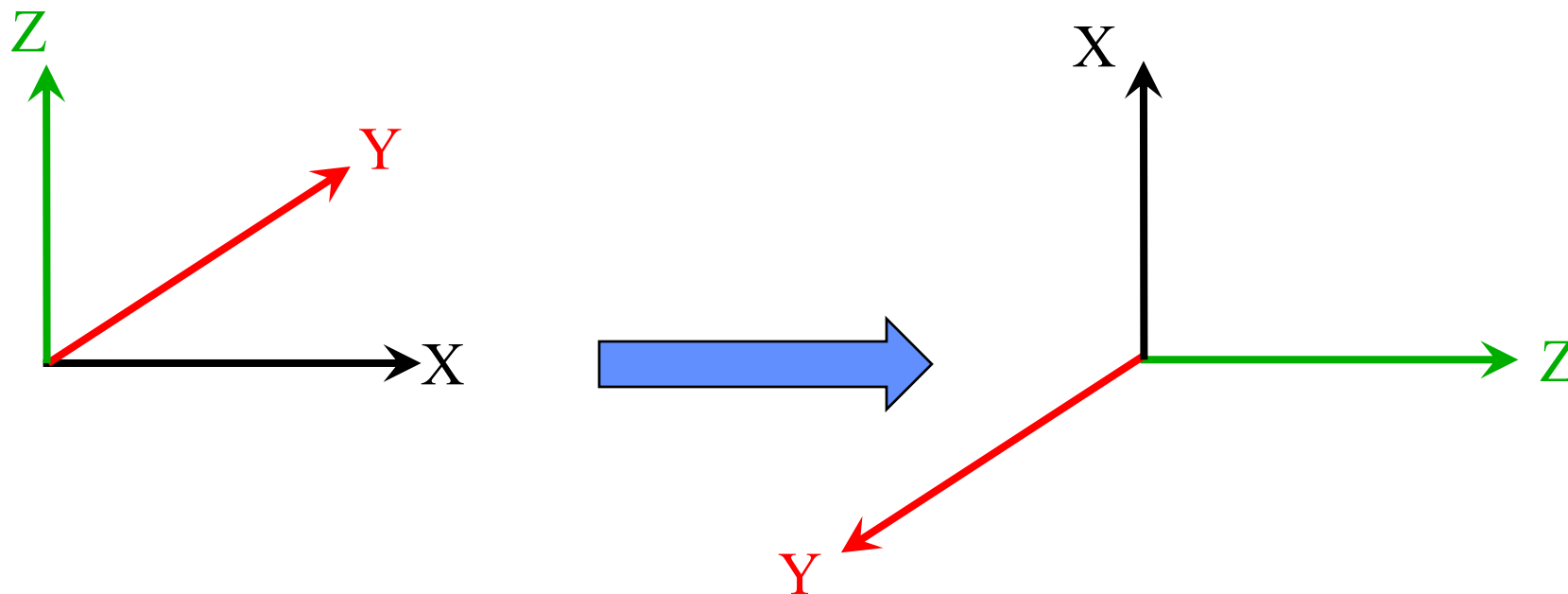




Gimbal Lock



Gimbal Lock



$(90^\circ, 90^\circ, 90^\circ)$ & $(180^\circ, 90^\circ, 0^\circ)$ are equivalent!!!

Singularity in the derivation of the rotation angles