Robust State Estimation with Applications to Networked Control Systems: An Unknown Input Observer Approach

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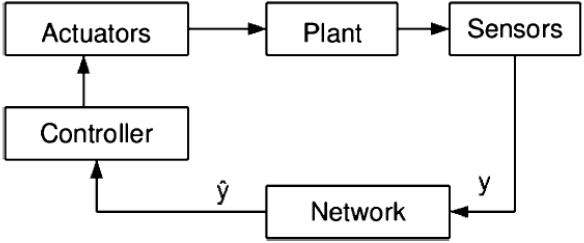
Some History, New Perspectives, and the Workshop Overview

Outline

- Motivation for state estimation/observation
- Asymptotic observer
- Uncertain system modeling; systems with unknown inputs
- An introduction to unknown input observer (UIO)
- The role of system zeros
- Workshop topics

Network Control System (NCS)—What Is It?

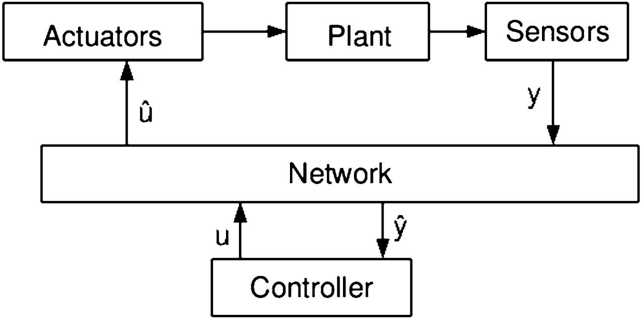
- When a feedback control system is closed via a communication channel, which may be shared with other nodes outside the control system, then the control system is called a NCS. A NCS can also be described as a feedback control system where the control loops are closed through a real-time communication network.



L. Bushnell and H. Ye, *Networked Control Systems: Architecture and Stability Issues*, in Encyclopedia of Systems and Control, Springer-Verlag London 2014

Another Look at a Network Control System (NCS)

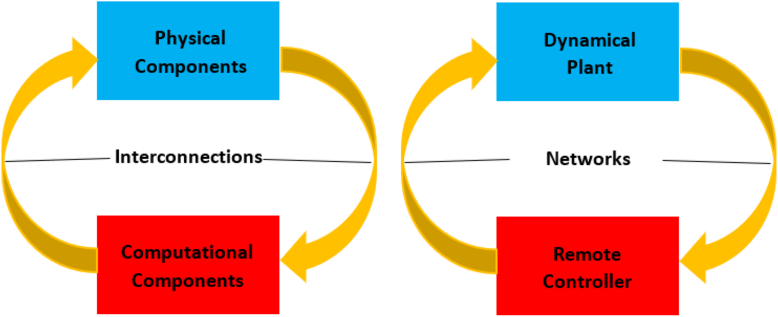
- A Networked Control System is a control system where the control loops are closed through a communication network. In a NCS feedback signals are exchanged among the system's components in the form of information packages through a network.



Cyber-Physical Systems

Definition

Cyber-Physical Systems combine cyber and physical components, that is, they are combinations of the physical world with the virtual world of information processing



More on the Term Cyber-Physical System

Definition

Cyber-Physical System (CPS)—a combination of virtual and physical components

- The term "cyber-physical system" (CPS)—first proposed in 2006 by Helen Gill of the US National Science Foundation

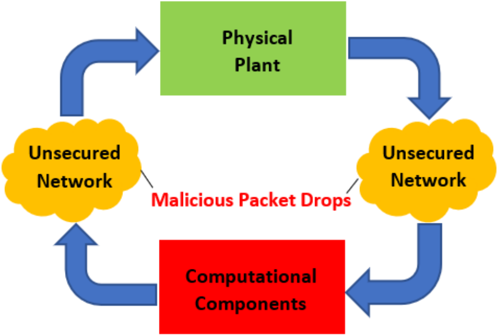
NCS versus CPS

- The CPS and "Networked System" Venn diagrams intersect, but neither is contained within the other.
  You could have CPS that are not necessarily networked in a graph (pacemaker, etc.) You could have networked systems that have no cyber component (human contact network, genetic networks) You could have systems that are both networked and CPS (power grid).

Shreyas Sundaram—private communication

Challenges in NCS

- Major challenge in the NCS design—security
- For example, malicious packet drop attacks in the communication networks



Need for the System State Estimate

- Many controllers call for the complete availability of the state vector of the controlled system
- It is frequently impossible to directly measure all the elements of the system state vector
- To retain the many useful properties of the state feedback control, one needs to overcome the problem of incomplete state vector information
- How to obtain good state estimate in the presence of modeling uncertainties and disturbances?
- What about system security operation?

A Short History of the Observer

My dear Watson, you see but you do not observe.

Sir Arthur Conan Doyle, *Scandal in Bohemia*, 1891

Beginnings of the observer

- Observer—a dynamical system that estimates the system state based on the system inputs and outputs
- The observer provides a solution to the problem of incomplete state vector information
- D. G. Luenberger initiated the theory of observers in 1963 in his Ph.D. thesis, *Determining the State of a Linear System with Observers of Low Dynamic Order*, at Stanford

D. G. Luenberger, *Observing the state of a linear system*, IEEE Transactions on Military Electronics, Vol. 8, Issue 2, pp. 74–80, April 1964

First page of the 1964 Luenberger's paper on observers

Observing the State of a Linear System

DAVID G. LUENBERGER, STUDENT MEMBER, IEEE

***Summary*—In much of modern control theory designs are based on the assumption that the state vector of the system to be controlled is available for measurement. In many practical situations only a few output quantities are available. Application of theories which assume that the state vector is known is severely limited in these cases. In this paper it is shown that the state vector of a linear system can be reconstructed from observations of the system inputs and outputs.**

It is shown that the observer, which reconstructs the state vector, is itself a linear system whose complexity decreases as the number of output quantities available increases. The observer may be incorporated in the control of a system which does not have its state vector available for measurement. The observer supplies the state vector, but at the expense of adding poles to the over-all system.

I. Introduction

IN THE PAST few years there has been an increasing percentage of control system literature written from the "state variable" point of view [1]–[8]. In the case of a continuous, time-invariant linear system the state variable representation of the system is of the form:

$$\dot{y}(t) = Ay(t) + Bx(t),$$

where

$y(t)$ is an $(n \times 1)$ state vector
$x(t)$ is an $(m \times 1)$ input vector
A is an $(n \times n)$ transition matrix
B is an $(n \times m)$ distribution matrix.

This state variable representation has some conceptual advantages over the more conventional transfer function representation. The state vector $y(t)$ contains enough information to completely summarize the past behavior of the system, and the future behavior is governed by a simple first-order differential equation. The properties of the system are determined by the constant matrices A and B. Thus the study of the system can be carried out in the field of matrix theory which is not only well developed, but has many notational and conceptual advantages over other methods.

When faced with the problem of controlling a system, some scheme must be devised to choose the input vector $x(t)$ so that the system behaves in an acceptable manner. Since the state vector $y(t)$ contains all the essential information about the system, it is reasonable to base the choice of $x(t)$ solely on the values of $y(t)$ and perhaps also t. In other words, x is determined by a relation of the form $x(t) = F[y(t), t]$.

This is, in fact, the approach taken in a large portion of present day control system literature. Several new techniques have been developed to find the function F for special classes of control problems. These techniques include dynamic programming [8]–[10], Pontryagin's maximum principle [11], and methods based on Lyapunov's theory [2], [12].

In most control situations, however, the state vector is not available for direct measurement. This means that it is not possible to evaluate the function $F[y(t), t]$. In these cases either the method must be abandoned or a reasonable substitute for the state vector must be found.

In this paper it is shown how the available system inputs and outputs may be used to construct an estimate of the system state vector. The device which reconstructs the state vector is called an observer. The observer itself as a time-invariant linear system driven by the inputs and outputs of the system it observes.

Kalman [3], [13], [14] has done some work on this problem, primarily for sampled-data systems. He has treated both the nonrandom problem and the problem of estimating the state when measurements of the outputs are corrupted by noise. In this paper only the nonstatistical problem is discussed but for that case a fairly complete theory is developed.

It is shown that the time constants of an observer can be chosen arbitrarily and that the number of dynamic elements required by the observer decreases as more output measurements become available. The novel point of view taken in this paper leads to a simple conceptual understanding of the observer process.

II. Observation of a Free Dynamic System

As a first step toward the construction of an observer it is useful to consider a slightly more general problem. Instead of requiring that the observer reconstruct the state vector itself, require only that it reconstruct some constant linear transformation of the state vector. This problem is simpler than the previous problem and its solution provides a great deal of insight into the theory of observers.

Assuming it were possible to build a system which reconstructs some constant linear transformation T of the state vector y, it is clear that it would then be possible to reconstruct the state vector itself, provided that the transformation T were invertible. This is the approach taken in this paper. It is first shown that it is relatively simple to build a system which will reconstruct some linear transformation of the state vector and then it is shown how to guarantee that the transformation obtained is invertible.

The first result concerns systems which have no inputs. (Such systems are called free systems.) The situa-

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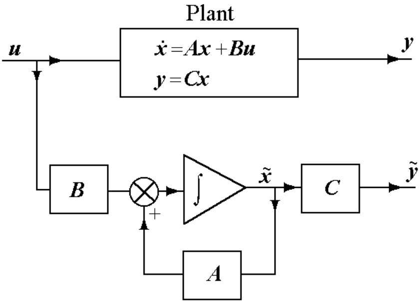
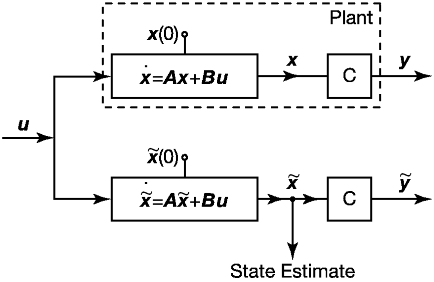
Trivial observer



- Trivial observer (open-loop observer)—the system model copy as an observer
- Observation error, $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$

Trivial observer—A different look



- Recall: Trivial observer (open-loop observer)—the plant model copy as an observer
- Observation error, $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$

Problems with the open-loop observer

- Observation error dynamics,

$$(\dot{\boldsymbol{x}} - \dot{\tilde{\boldsymbol{x}}}) = \boldsymbol{A}(\boldsymbol{x} - \tilde{\boldsymbol{x}})$$

- The observation error tends to zero only if the observed system is stable
- There is no control over the observation error dynamics
- There is a fix—add observer innovation to get the closed-loop observer

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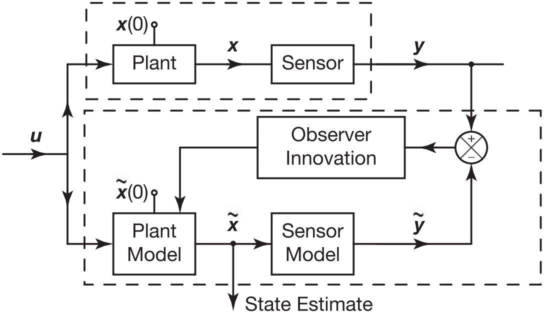
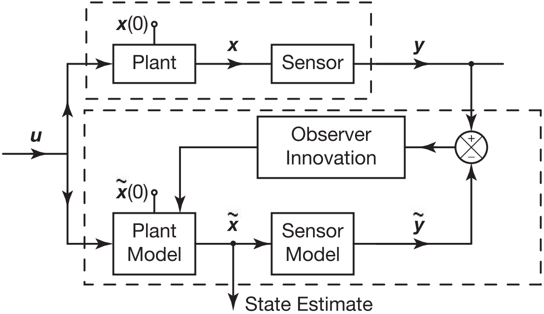
Closed-loop observer



- Luenberger's Innovation to obtain the closed-loop observer

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \tilde{\boldsymbol{y}})$$

Closed-loop observer



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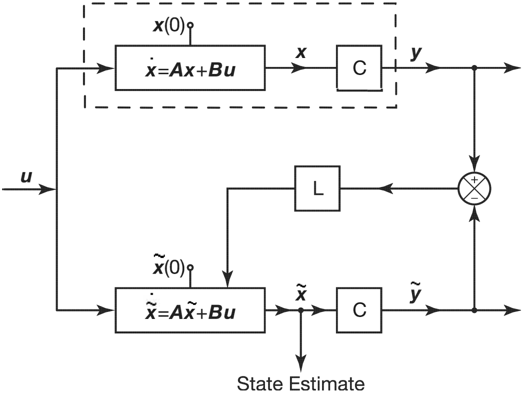
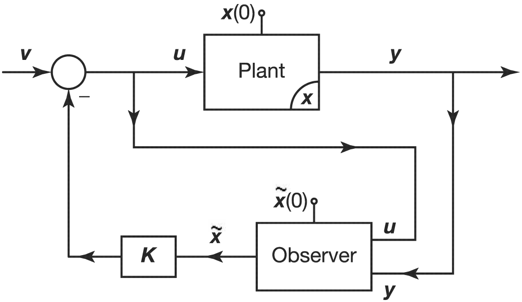
Luenberger's closed-loop observer



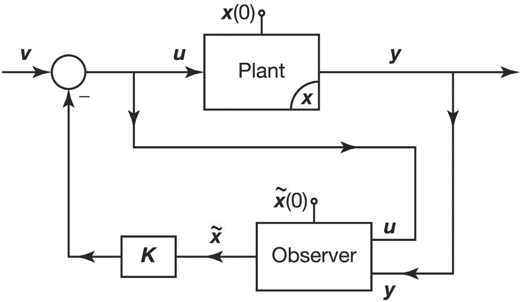
- Luenberger's observer, $\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \tilde{\boldsymbol{y}})$
- Observation error dynamics, $(\dot{\boldsymbol{x}} - \dot{\tilde{\boldsymbol{x}}}) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})(\boldsymbol{x} - \tilde{\boldsymbol{x}})$

Combined observer-controller compensator



- Works well for systems without uncertainties
- What about systems with uncertainties?

Combined observer-controller compensator



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Observers for Systems With Unknown Inputs

Plant Model

Standard linear dynamical system model:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x}, \end{aligned}$$

where $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$

- Parameters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are known
- m_1 of inputs are known and $m_2 = m - m_1$ are unknown
- Re-arrange the order of the inputs if necessary, partition the input matrix $\boldsymbol{B}$ corresponding to the known, $\boldsymbol{u}_1$, and unknown inputs, $\boldsymbol{u}_2$, as

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \end{bmatrix},$$

with $\boldsymbol{B}_1 \in \mathbb{R}^{n\times m_1}$ and $\boldsymbol{B}_2 \in \mathbb{R}^{n\times m_2}$ and

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System Model—Contd.

The system model

$$\boxed{\begin{array}{rcl} \dot{\boldsymbol{x}} & = & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2 \\ \boldsymbol{y} & = & \boldsymbol{C}\boldsymbol{x} \end{array}}$$

- The vector function $\boldsymbol{u}_2$ may also model lumped uncertainties or nonlinearities in the plant
- Similar notation as in Basile and Marro, where $\boldsymbol{u}_2$ is called the disturbance vector
- The output matrix is $\boldsymbol{C} \in \mathbb{R}^{p \times n}$
- The pair $(\boldsymbol{A}, \boldsymbol{C})$ detectable

G. Basile and G. Marro, *On the observability of linear, time-invariant systems with unknown inputs*, Journal of Optimization Theory and Applications, Vol. 3, No. 6, pp. 410–415, Nov. 1969

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Second page of the 1969 Basile and Marro's paper

We deal with a linear, purely dynamical, time-invariant system described by the equations

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in R^n$ is the *state vector*, $u_1 \in R^m$ is the *control vector*, $u_2 \in R^l$ is the *disturbance vector*, $y \in R^s$ is the *output vector*, and A, B_1, B_2, C are real, constant matrices of proper sizes. We call $\mathscr{F}_1 = \mathscr{R}(B_1)$ the subspace of control actions and $\mathscr{F}_2 = \mathscr{R}(B_2)$ the subspace of disturbance actions.

It is well known that, in the particular case where $B_1 \neq 0$, $B_2 = 0$, from the observation of input and output functions in a finite interval of time, it is possible to recognize the orthogonal projection of the state on the least subspace which is invariant under A^T and contains $\mathscr{R}(C^T)$. The word *least* is justified because the intersection of two invariants is an invariant. This subspace is sometimes called *observability subspace* and its orthogonal complement *unobservability subspace*.

In this particular case, when the input functions are completely known, the observation of the system (1)–(2) reduces to the observation of the corresponding autonomous system; that is, since

$$y(t) = C\Phi(t, 0)\, x_0 + C \int_0^t \Phi(t, \tau)\, B_1 u_1(\tau)\, d\tau \tag{3}$$

where $\Phi(t, \tau)$ is the state-transition matrix, it is possible to determine by a simple subtraction the output functions of the corresponding autonomous system, namely, the zero-input output functions.

By similar reasoning, the general case in which a part of the input is known and a part is unknown can be reduced to the case of completely unknown input. Thus, it is sufficient to consider only this last case. In the next section, we state a theorem that provides the observability subspace as the least conditioned invariant under the matrix A^T, with respect to the subspace $\mathscr{F}_2^{\perp}$, containing $\mathscr{R}(C^T)$, and which includes the previous results, corresponding to $B_2 = 0$.

2. Observability Subspace for Systems with Unknown Inputs

First, we recall some definitions and results given in a previous paper (Ref. 6) which provide a background for the analysis presented here. Consider an $n \times n$ matrix A and a subspace $\mathscr{F} \subset R^n$. We use the following definitions

Projection Operator UIO—Idea

- Decompose the state $\boldsymbol{x}$ as

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{x} - \boldsymbol{M}\boldsymbol{y} + \boldsymbol{M}\boldsymbol{y}\\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x} + \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}\\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x} + \boldsymbol{M}\boldsymbol{y},\end{aligned}$$

  where $\boldsymbol{M}$: $n \times p$ real matrix to be determined

- $\boldsymbol{q} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}$: Unknown part of the decomposition

S. Hui and S. H. Żak, *Observer design for systems with unknown inputs*, International Journal of Applied Mathematics and Computer Science, Vol. 15, No. 4, pp. 431–446, 2005

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Projection Operator UIO—Decomposed Dynamics

- Some manipulations:

$$\begin{aligned}\dot{\boldsymbol{q}} &= (\boldsymbol{I}-\boldsymbol{MC})\dot{\boldsymbol{x}} \\ &= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Ax}+\boldsymbol{B}_1\boldsymbol{u}_1+\boldsymbol{B}_2\boldsymbol{u}_2) \\ &= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Ax}+\boldsymbol{B}_1\boldsymbol{u}_1)+(\boldsymbol{I}-\boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2\end{aligned}$$

- Recall: $\boldsymbol{x}=\boldsymbol{q}+\boldsymbol{My}$

$$\dot{\boldsymbol{q}} = (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)+(\boldsymbol{I}-\boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2$$

- Choose $\boldsymbol{M}$ to make $(\boldsymbol{I}-\boldsymbol{MC})\boldsymbol{B}_2=\boldsymbol{O}$
- Then

$$\boxed{\dot{\boldsymbol{q}} = (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)}$$

- Important: $\boldsymbol{u}_1$ and $\boldsymbol{y}$ are known

Projection Operator UIO—Decomposed Dynamics

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- Important: $\boldsymbol{u}_1$ and $\boldsymbol{y}$ are known

Projection Operator UIO—The Rank Condition

- Need:

$$(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$$

- Linear Algebra:

$$\text{rank}\ (\boldsymbol{MCB}_2) \le \text{rank}\ (\boldsymbol{CB}_2) \le \text{rank}\ (\boldsymbol{B}_2)$$

- Necessary and Sufficient Condition:

$$\text{rank}\ (\boldsymbol{CB}_2) = \text{rank}(\boldsymbol{B}_2)$$

- Implication: At least as many independent outputs as unknown inputs for the method to work

Projection Operator UIO—The Rank Condition

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$$(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$$

- Linear Algebra:

$$\text{rank }(\boldsymbol{MCB}_2) \leq \text{rank }(\boldsymbol{CB}_2) \leq \text{rank }(\boldsymbol{B}_2)$$

- Necessary and Sufficient Condition:

$$\text{rank }(\boldsymbol{CB}_2) = \text{rank}(\boldsymbol{B}_2)$$

- Implication: At least as many independent outputs as unknown inputs for the method to work

Projection Operator UIO—The Rank Condition

- Need:

$$(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$$

- Linear Algebra:

$$\text{rank}\ (\boldsymbol{M}\boldsymbol{C}\boldsymbol{B}_2) \leq \text{rank}\ (\boldsymbol{C}\boldsymbol{B}_2) \leq \text{rank}\ (\boldsymbol{B}_2)$$

- Necessary and Sufficient Condition:

$$\text{rank}\ (\boldsymbol{C}\boldsymbol{B}_2) = \text{rank}(\boldsymbol{B}_2)$$

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Projection Operator UIO Dynamics

- If we know $\boldsymbol{q}$ and the initial condition

$$\boldsymbol{q}(0) = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}(0),$$

then

$$\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{C}\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y}$$

is known for all $t \geq 0$

- Indeed, integrate both sides of $\dot{\boldsymbol{q}} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\dot{\boldsymbol{x}}$ to obtain

$$\boldsymbol{q}(t)\text{-}\boldsymbol{q}(0)= (\boldsymbol{I}\text{-}\boldsymbol{M}\boldsymbol{C})(\boldsymbol{x}(t)\text{-}\boldsymbol{x}(0))$$

- Hence

$$\boldsymbol{q}(t) = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}(t) - (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}(0) + \boldsymbol{q}(0)$$

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Projection Operator UIO Dynamics

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Projection Operator UIO Dynamics—Contd.

- But we do not know $\boldsymbol{x}(0)$
- We have

$$\boldsymbol{q}(t) = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}(t) - (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}(0) + \boldsymbol{q}(0)$$

- So we only get an approximation

$$\boxed{\tilde{\boldsymbol{x}} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y}}$$

where $\boldsymbol{q}$ is obtained from

$$\boxed{\dot{\boldsymbol{q}} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1)}$$

Projection Operator UIO Dynamics—Contd.

- But we do not know $\boldsymbol{x}(0)$
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Open-Loop UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{Cx}$

Then we have

$$\begin{aligned}
\frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \\
&= \frac{d}{dt}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= \frac{d}{dt}((\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x} - \boldsymbol{q}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
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&= (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{Ae}
\end{aligned}$$

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Open-Loop UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$

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Closed-Loop UIO

- We add the innovation term to obtain the closed-loop UIO:

$$
\begin{aligned}
\dot{\boldsymbol{q}} &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})((\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{L}(\boldsymbol{y}-\tilde{\boldsymbol{y}})) \\
&= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})((\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{L}(\boldsymbol{y}-\boldsymbol{C}\boldsymbol{q}-\boldsymbol{C}\boldsymbol{M}\boldsymbol{y})) \\
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\end{aligned}
$$

- State estimate is

$$\tilde{\boldsymbol{x}} = \boldsymbol{q}+\boldsymbol{M}\boldsymbol{y}$$

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Closed-Loop UIO Analysis

- Let $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$
- We will show that

$$\dot{e} = (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{e}$$

  and $\boldsymbol{e}(t) \to \boldsymbol{0}$ as $t \to \infty$ under mild conditions
- Note that $(\boldsymbol{A} - \boldsymbol{LC})$ asymptotically stable does not guarantee that $(\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})$ is asymptotically stable
- It is possible for a product of a projection matrix and an asymptotically stable matrix to be unstable

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Example

- Let
$$\boldsymbol{\Pi} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{A} = \begin{bmatrix} 1 & -3 \\ 3 & -2 \end{bmatrix}$$
- $\boldsymbol{A}$ is asymptotically stable
- $\boldsymbol{\Pi A}$ is unstable
- The system $\dot{\boldsymbol{x}} = \boldsymbol{A x}$ restricted to the range of $\boldsymbol{\Pi}$ is governed by $\dot{z} = z$, which is also unstable

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Projection Operator UIO Structure

- We will analyze convergence properties of the proposed full-order observer
- We will show $\tilde{\boldsymbol{x}} \to \boldsymbol{x}$ as $t \to \infty$ for

$$\begin{aligned} \dot{\boldsymbol{q}} &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})((\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{y})) \\ \tilde{\boldsymbol{x}} &= \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y} \end{aligned}$$

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\tilde{\boldsymbol{x}} &= \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y}
\end{aligned}
$$

Projection Operator UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$. Then we have

$$
\begin{aligned}
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&\quad +\boldsymbol{L}(\boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x})) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}) \\
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\frac{de}{dt} &= \frac{d}{dt}(x - \tilde{x}) \\
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&= \frac{d}{dt}((I - MC)x - q) \\
&= (I - MC)(Ax + B_1u_1 + B_2u_2) \\
&\quad -(I - MC)((Aq + AMy + B_1u_1) + L(y - Cq - CMy)) \\
&= (I - MC)(Ax + B_1u_1) + (I - MC)B_2u_2 \\
&\quad -(I - MC)((Aq + AMCx + B_1u_1) \\
&\quad +L(Cx - Cq - CMCx)) \\
&= (I - MC)(A - LC)(x - q - MCx) \\
&= (I - MC)\,(A - LC)\,e
\end{aligned}
$$

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&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad -(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})((\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{y})) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad -(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})((\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad +\boldsymbol{L}(\boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x})) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\,\boldsymbol{e}
\end{aligned}
$$

Projection Operator UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{Cx}$. Then we have

$$
\begin{aligned}
\frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}\left(\boldsymbol{x} - \tilde{\boldsymbol{x}}\right) \\
&= \frac{d}{dt}\left(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}\right) \\
&= \frac{d}{dt}\left((\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x} - \boldsymbol{q}\right) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMCx} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad +\boldsymbol{L}(\boldsymbol{Cx} - \boldsymbol{Cq} - \boldsymbol{CMCx})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})\,(\boldsymbol{A} - \boldsymbol{LC})\,\boldsymbol{e}
\end{aligned}
$$

Projection Operator UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{Cx}$. Then we have

$$
\begin{aligned}
\frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \\
&= \frac{d}{dt}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= \frac{d}{dt}((\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x} - \boldsymbol{q}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMCx} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad +\boldsymbol{L}(\boldsymbol{Cx} - \boldsymbol{Cq} - \boldsymbol{CMCx})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{e}
\end{aligned}
$$

Projection Operator UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{Cx}$. Then we have

$$
\begin{aligned}
\frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \\
&= \frac{d}{dt}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= \frac{d}{dt}((\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x} - \boldsymbol{q}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMCx} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad +\boldsymbol{L}(\boldsymbol{Cx} - \boldsymbol{Cq} - \boldsymbol{CMCx})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{e}
\end{aligned}
$$

Projection Operator UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{Cx}$. Then we have

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\begin{aligned}
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&= \frac{d}{dt}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= \frac{d}{dt}((\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x} - \boldsymbol{q}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad\; -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad\; -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMCx} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad\; +\boldsymbol{L}(\boldsymbol{Cx} - \boldsymbol{Cq} - \boldsymbol{CMCx})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})\,\boldsymbol{e}
\end{aligned}
$$

Projection Operator UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{Cx}$. Then we have

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\begin{aligned}
\frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \\
&= \frac{d}{dt}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= \frac{d}{dt}((\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x} - \boldsymbol{q}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMCx} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad +\boldsymbol{L}(\boldsymbol{Cx} - \boldsymbol{Cq} - \boldsymbol{CMCx})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{e}
\end{aligned}
$$

Projection Operator UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{Cx}$. Then we have

$$
\begin{aligned}
\frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \\
&= \frac{d}{dt}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= \frac{d}{dt}((\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x} - \boldsymbol{q}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMCx} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad +\boldsymbol{L}(\boldsymbol{Cx} - \boldsymbol{Cq} - \boldsymbol{CMCx})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{MCx}) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})\,\boldsymbol{e}
\end{aligned}
$$

Projection Operator UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$. Then we have

$$
\begin{aligned}
\frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}\left(\boldsymbol{x} - \tilde{\boldsymbol{x}}\right) \\
&= \frac{d}{dt}\left(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}\right) \\
&= \frac{d}{dt}\left((\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x} - \boldsymbol{q}\right) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad -(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})((\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{y})) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad -(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})((\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad +\boldsymbol{L}(\boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x})) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}
\end{aligned}
$$

Projection Operator UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$. Then we have

$$
\begin{aligned}
\frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}\left(\boldsymbol{x} - \tilde{\boldsymbol{x}}\right) \\
&= \frac{d}{dt}\left(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}\right) \\
&= \frac{d}{dt}\left((\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x} - \boldsymbol{q}\right) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad -(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})((\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{y})) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad -(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})((\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad +\boldsymbol{L}(\boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x})) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\,(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\,\boldsymbol{e}
\end{aligned}
$$

Projection Operator UIO Analysis

- Objective: Specify $\boldsymbol{M}$ and $\boldsymbol{L}$ and a set of initial conditions so that $\boldsymbol{e}(t) \to \mathbf{0}$ as $t \to \infty$
- A class of solutions to $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$

$$\boldsymbol{M} = \boldsymbol{B}_2\left((\boldsymbol{C}\boldsymbol{B}_2)^{\dagger} + \boldsymbol{H}_0\left(\boldsymbol{I}_p - (\boldsymbol{C}\boldsymbol{B}_2)(\boldsymbol{C}\boldsymbol{B}_2)^{\dagger}\right)\right)$$

- $\dagger$ denotes the Moore-Penrose pseudo-inverse
- $\boldsymbol{H}_0 \in \mathbb{R}^{m_2 \times p}$ is a design parameter matrix
- We have $(\boldsymbol{C}\boldsymbol{B}_2)^{\dagger}(\boldsymbol{C}\boldsymbol{B}_2) = \boldsymbol{I}_{m_2}$ because rank $(\boldsymbol{C}\boldsymbol{B}_2) =$ rank $\boldsymbol{B}_2$ and $\boldsymbol{B}_2$ has full rank
- If $\boldsymbol{C}\boldsymbol{B}_2$ is square, $\boldsymbol{M}$ reduces to $\boldsymbol{B}_2(\boldsymbol{C}\boldsymbol{B}_2)^{-1}$
- $\boldsymbol{M}\boldsymbol{C}$ is a projection (not necessarily orthogonal): $(\boldsymbol{M}\boldsymbol{C})^2 = \boldsymbol{M}\boldsymbol{C}$
- $\tilde{\boldsymbol{\Pi}} = \boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}$ is also a projection

Projection Operator UIO Analysis

- Objective: Specify $\boldsymbol{M}$ and $\boldsymbol{L}$ and a set of initial conditions so that $\boldsymbol{e}(t) \to \mathbf{0}$ as $t \to \infty$
- A class of solutions to $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$

$$\boldsymbol{M} = \boldsymbol{B}_2 \left((\boldsymbol{C}\boldsymbol{B}_2)^\dagger + \boldsymbol{H}_0 \left(\boldsymbol{I}_p - (\boldsymbol{C}\boldsymbol{B}_2)(\boldsymbol{C}\boldsymbol{B}_2)^\dagger \right) \right)$$

- $\dagger$ denotes the Moore-Penrose pseudo-inverse
- $\boldsymbol{H}_0 \in \mathbb{R}^{m_2 \times p}$ is a design parameter matrix
- We have $(\boldsymbol{C}\boldsymbol{B}_2)^\dagger(\boldsymbol{C}\boldsymbol{B}_2) = \boldsymbol{I}_{m_2}$ because rank $(\boldsymbol{C}\boldsymbol{B}_2) =$ rank $\boldsymbol{B}_2$ and $\boldsymbol{B}_2$ has full rank
- If $\boldsymbol{C}\boldsymbol{B}_2$ is square, $\boldsymbol{M}$ reduces to $\boldsymbol{B}_2(\boldsymbol{C}\boldsymbol{B}_2)^{-1}$
- $\boldsymbol{M}\boldsymbol{C}$ is a projection (not necessarily orthogonal): $(\boldsymbol{M}\boldsymbol{C})^2 = \boldsymbol{M}\boldsymbol{C}$
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Projection Operator UIO Analysis

- Objective: Specify $\boldsymbol{M}$ and $\boldsymbol{L}$ and a set of initial conditions so that $\boldsymbol{e}(t) \to \mathbf{0}$ as $t \to \infty$
- A class of solutions to $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$

$$\boldsymbol{M} = \boldsymbol{B}_2 \left((\boldsymbol{C}\boldsymbol{B}_2)^{\dagger} + \boldsymbol{H}_0 \left(\boldsymbol{I}_p - (\boldsymbol{C}\boldsymbol{B}_2)(\boldsymbol{C}\boldsymbol{B}_2)^{\dagger} \right) \right)$$

- $\dagger$ denotes the Moore-Penrose pseudo-inverse
- $\boldsymbol{H}_0 \in \mathbb{R}^{m_2 \times p}$ is a design parameter matrix
- We have $(\boldsymbol{C}\boldsymbol{B}_2)^{\dagger}(\boldsymbol{C}\boldsymbol{B}_2) = \boldsymbol{I}_{m_2}$ because rank $(\boldsymbol{C}\boldsymbol{B}_2) =$ rank $\boldsymbol{B}_2$ and $\boldsymbol{B}_2$ has full rank
- If $\boldsymbol{C}\boldsymbol{B}_2$ is square, $\boldsymbol{M}$ reduces to $\boldsymbol{B}_2(\boldsymbol{C}\boldsymbol{B}_2)^{-1}$
- $\boldsymbol{M}\boldsymbol{C}$ is a projection (not necessarily orthogonal): $(\boldsymbol{M}\boldsymbol{C})^2 = \boldsymbol{M}\boldsymbol{C}$
- $\tilde{\boldsymbol{\Pi}} = \boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}$ is also a projection

Projection Operator UIO Analysis

- Objective: Specify $\boldsymbol{M}$ and $\boldsymbol{L}$ and a set of initial conditions so that $\boldsymbol{e}(t) \to \mathbf{0}$ as $t \to \infty$
- A class of solutions to $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$

$$\boldsymbol{M} = \boldsymbol{B}_2 \left((\boldsymbol{CB}_2)^\dagger + \boldsymbol{H}_0 \left(\boldsymbol{I}_p - (\boldsymbol{CB}_2)(\boldsymbol{CB}_2)^\dagger \right) \right)$$

- $\dagger$ denotes the Moore-Penrose pseudo-inverse
- $\boldsymbol{H}_0 \in \mathbb{R}^{m_2 \times p}$ is a design parameter matrix
- We have $(\boldsymbol{CB}_2)^\dagger(\boldsymbol{CB}_2) = \boldsymbol{I}_{m_2}$ because rank $(\boldsymbol{CB}_2) =$ rank $\boldsymbol{B}_2$ and $\boldsymbol{B}_2$ has full rank
- If $\boldsymbol{CB}_2$ is square, $\boldsymbol{M}$ reduces to $\boldsymbol{B}_2(\boldsymbol{CB}_2)^{-1}$
- $\boldsymbol{MC}$ is a projection (not necessarily orthogonal): $(\boldsymbol{MC})^2 = \boldsymbol{MC}$
- $\tilde{\boldsymbol{\Pi}} = \boldsymbol{I} - \boldsymbol{MC}$ is also a projection

Projection Operator UIO Analysis

- Objective: Specify $\boldsymbol{M}$ and $\boldsymbol{L}$ and a set of initial conditions so that $\boldsymbol{e}(t) \to \mathbf{0}$ as $t \to \infty$
- A class of solutions to $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$

$$\boldsymbol{M} = \boldsymbol{B}_2 \left((\boldsymbol{C}\boldsymbol{B}_2)^\dagger + \boldsymbol{H}_0 \left(\boldsymbol{I}_p - (\boldsymbol{C}\boldsymbol{B}_2)(\boldsymbol{C}\boldsymbol{B}_2)^\dagger \right) \right)$$

- $\dagger$ denotes the Moore-Penrose pseudo-inverse
- $\boldsymbol{H}_0 \in \mathbb{R}^{m_2 \times p}$ is a design parameter matrix
- We have $(\boldsymbol{C}\boldsymbol{B}_2)^\dagger(\boldsymbol{C}\boldsymbol{B}_2) = \boldsymbol{I}_{m_2}$ because rank $(\boldsymbol{C}\boldsymbol{B}_2) =$ rank $\boldsymbol{B}_2$ and $\boldsymbol{B}_2$ has full rank
- If $\boldsymbol{C}\boldsymbol{B}_2$ is square, $\boldsymbol{M}$ reduces to $\boldsymbol{B}_2(\boldsymbol{C}\boldsymbol{B}_2)^{-1}$
- $\boldsymbol{M}\boldsymbol{C}$ is a projection (not necessarily orthogonal): $(\boldsymbol{M}\boldsymbol{C})^2 = \boldsymbol{M}\boldsymbol{C}$
- $\tilde{\boldsymbol{\Pi}} = \boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}$ is also a projection

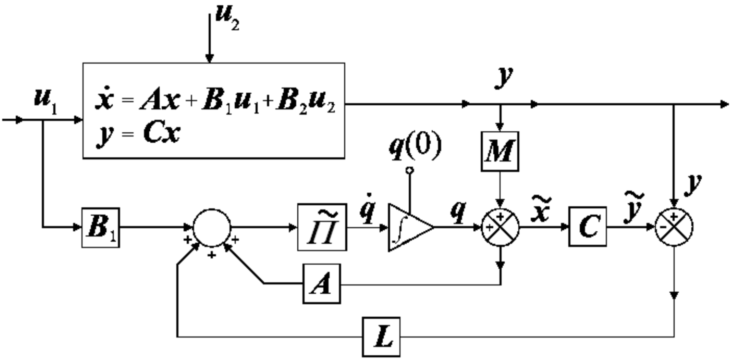
Projection Operator UIO Analysis

- Objective: Specify $\boldsymbol{M}$ and $\boldsymbol{L}$ and a set of initial conditions so that $\boldsymbol{e}(t) \to \mathbf{0}$ as $t \to \infty$
- A class of solutions to $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$

$$\boldsymbol{M} = \boldsymbol{B}_2\left((\boldsymbol{CB}_2)^{\dagger} + \boldsymbol{H}_0\left(\boldsymbol{I}_p - (\boldsymbol{CB}_2)(\boldsymbol{CB}_2)^{\dagger}\right)\right)$$

- $\dagger$ denotes the Moore-Penrose pseudo-inverse
- $\boldsymbol{H}_0 \in \mathbb{R}^{m_2 \times p}$ is a design parameter matrix
- We have $(\boldsymbol{CB}_2)^{\dagger}(\boldsymbol{CB}_2) = \boldsymbol{I}_{m_2}$ because rank $(\boldsymbol{CB}_2) =$ rank $\boldsymbol{B}_2$ and $\boldsymbol{B}_2$ has full rank
- If $\boldsymbol{CB}_2$ is square, $\boldsymbol{M}$ reduces to $\boldsymbol{B}_2(\boldsymbol{CB}_2)^{-1}$
- $\boldsymbol{MC}$ is a projection (not necessarily orthogonal): $(\boldsymbol{MC})^2 = \boldsymbol{MC}$
- $\tilde{\boldsymbol{\Pi}} = \boldsymbol{I} - \boldsymbol{MC}$ is also a projection

Block diagram of the Full-Order UIO



Projection Operator UIO—Second Method

- Adding the innovation term to obtain the closed-loop UIO without being premultiplied by $(\boldsymbol{I}-\boldsymbol{MC})$:

$$\begin{aligned}\dot{\boldsymbol{q}} &= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{L}(\boldsymbol{y}-\tilde{\boldsymbol{y}}) \\ &= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{L}(\boldsymbol{y}-\boldsymbol{Cq}-\boldsymbol{CMy}) \\ &= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{LC}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{My})\end{aligned}$$

- $\tilde{\boldsymbol{x}}=\boldsymbol{q}+\boldsymbol{My}$

Projection Operator UIO—Second Method

- Adding the innovation term to obtain the closed-loop UIO without being premultiplied by $(\boldsymbol{I}-\boldsymbol{MC})$:

$$
\begin{aligned}
\dot{\boldsymbol{q}} &= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{L}(\boldsymbol{y}-\tilde{\boldsymbol{y}}) \\
&= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{L}(\boldsymbol{y}-\boldsymbol{Cq}-\boldsymbol{CMy}) \\
&= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{LC}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{My})
\end{aligned}
$$

- $\tilde{\boldsymbol{x}}=\boldsymbol{q}+\boldsymbol{My}$

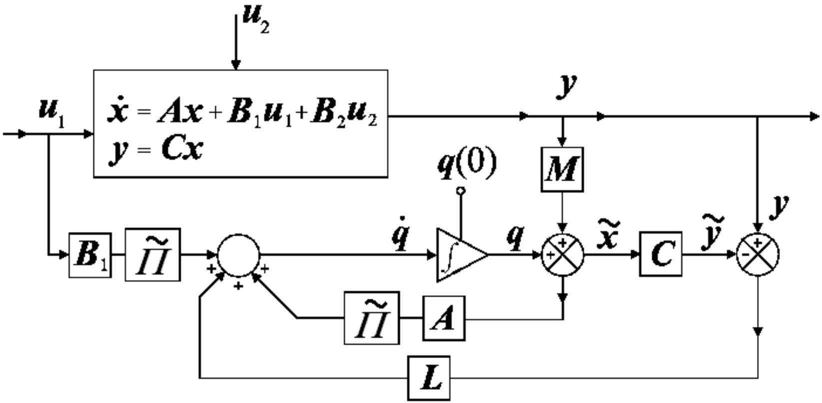
Projection Operator UIO—Second Method

- Adding the innovation term to obtain the closed-loop UIO without being premultiplied by $(\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})$:

$$
\begin{aligned}
\dot{\boldsymbol{q}} &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{L}(\boldsymbol{y}-\tilde{\boldsymbol{y}}) \\
&= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{L}(\boldsymbol{y}-\boldsymbol{C}\boldsymbol{q}-\boldsymbol{C}\boldsymbol{M}\boldsymbol{y}) \\
&= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{u}_1)+\boldsymbol{L}\boldsymbol{C}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{M}\boldsymbol{y})
\end{aligned}
$$

- $\tilde{\boldsymbol{x}}=\boldsymbol{q}+\boldsymbol{M}\boldsymbol{y}$

Block diagram of the second full-order UIO



Projection Operator UIO—Second Method Contd.

- Let $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$
- Let $\boldsymbol{A}_1 = (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{A}$
- Easy to show that

$$\dot{\boldsymbol{e}} = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}$$

- $\boldsymbol{e}(t) \to \boldsymbol{0}$ as $t \to \infty \iff$ the pair $(\boldsymbol{A}_1, \boldsymbol{C})$ is detectable

Projection Operator UIO—Second Method Contd.

- Let $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$
- Let $\boldsymbol{A}_1 = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}$
- Easy to show that

$$\dot{\boldsymbol{e}} = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}$$

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Zeros and Unknown Input Observers

Zeros of SISO Systems

- For a SISO system model $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$, its zeros are defined to be the zeros of the polynomial

$$\boldsymbol{c}\ \mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\ \boldsymbol{b}$$

  where adj denotes the classical adjoint

- For a MIMO system model $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$, the product

$$\boldsymbol{C}\ \mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\ \boldsymbol{B}$$

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- The collection of zeros of the polynomials would seem to be a natural generalization of the definition of system zeros
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System Zeros of SISO Systems: A Different Look

- Note that

$$\begin{bmatrix} \boldsymbol{I}_n & -\boldsymbol{0} \\ -\boldsymbol{c}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1} & 1 \end{bmatrix} \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{c} & 0 \end{bmatrix} = \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{0} & \boldsymbol{c}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b} \end{bmatrix}$$

- Hence

$$\begin{aligned} \det \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{c} & 0 \end{bmatrix} &= \det(s\boldsymbol{I}_n - \boldsymbol{A}) \det(\boldsymbol{c}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b}) \\ &= \det(s\boldsymbol{I}_n - \boldsymbol{A}) \frac{\boldsymbol{c}\,\mathrm{adj}(s\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{b}}{\det(s\boldsymbol{I}_n - \boldsymbol{A})} \end{aligned}$$

- Thus, $\boldsymbol{c}\,\mathrm{adj}(s\boldsymbol{I}_n - \boldsymbol{A})\,\boldsymbol{b} = \det \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{c} & 0 \end{bmatrix}$

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System Matrix and Normal Rank

- Rosenbrock's system matrix

$$\boldsymbol{P}(s) = \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

- The normal rank of a matrix valued function $\boldsymbol{M}$ defined on the complex plane $\mathbb{C}$ is

$$\operatorname{normalrank} \boldsymbol{M} = \max \{\operatorname{rank} \boldsymbol{M}(s) : s \in \mathbb{C}\}$$

- In other words, the normal rank of a matrix function is the largest possible rank among the collection of matrices in the range $\{\boldsymbol{M}(s) : s \in \mathbb{C}\}$ of $\boldsymbol{M}$.
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System Matrix and the Laplace Transform

- Linear time invariant plant model:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}\end{aligned}$$

- Take the Laplace transforms

$$\begin{aligned}s\boldsymbol{X}(s) - \boldsymbol{x}(0) &= \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \\ \boldsymbol{Y}(s) &= \boldsymbol{C}\boldsymbol{X}(s) + \boldsymbol{D}\boldsymbol{U}(s)\end{aligned}$$

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General Definition of System Zeros

A complex number z_0 is a system zero of the system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ if

$$\operatorname{rank} \begin{bmatrix} z_0 \boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} < \operatorname{normalrank} \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

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The Meaning of System Zeros—SISO Case

- Consider the second order input-output system model

$$\ddot{y} + y = \dot{u} - z_0 u \quad \Rightarrow \quad \text{Transfer function } = G(s) = \frac{s - z_0}{s^2 + 1}$$

- Only one system zero at $s = z_0$
- If $u(t) = e^{z_0 t}$, then $\dot{u} - z_0 u = 0$ and so this input has the same effect as the zero input.
- In other words, the system does not see the input $u(t) = e^{z_0 t}$
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The Meaning of System Zeros—MIMO Case

- Consider the case when $m \leq p$ and

$$\boldsymbol{P}(s) = \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

has rank less than $n + m$ for $s = z_0$

- Then there exists $\begin{bmatrix} \boldsymbol{x}_0^\top & \boldsymbol{u}_0^\top \end{bmatrix}^\top \neq \mathbf{0}$ such that

$$\boldsymbol{P}(z_0) \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{u}_0 \end{bmatrix} = \begin{bmatrix} z_0\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{u}_0 \end{bmatrix} = \mathbf{0}$$

- We have

$$\boldsymbol{P}(s) - \boldsymbol{P}(z_0) = (s - z_0) \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}$$

M. L. J. Hautus, *Strong detectability and observers*, Linear Algebra and Its Applications, Vol. 50, pp. 353–368, 1983; see page 366

The Meaning of System Zeros—MIMO Case Contd.

- Let
$$\boldsymbol{X}(s) = \frac{1}{s - z_0}\boldsymbol{x}_0 \text{ and } \boldsymbol{U}(s) = \frac{1}{s - z_0}\boldsymbol{u}_0$$
Then
$$\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}(s) \\ \boldsymbol{U}(s) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \end{bmatrix}$$

- Hence
$$\boldsymbol{x}(t) = \mathcal{L}^{-1}(\boldsymbol{X}(s)) = e^{z_0 t}\boldsymbol{x}_0 \text{ and } \boldsymbol{u}(t) = \mathcal{L}^{-1}(\boldsymbol{U}(s)) = e^{z_0 t}\boldsymbol{u}_0$$
satisfy
$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{aligned}$$
and the corresponding output equals 0.

The Importance of System Zeros in the UIO Synthesis

- System $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ with a system zero not in the open LHP will ignore certain unbounded or persistent inputs

- Conclusion: It is impossible to design a general unknown input estimator if there are system zeros not in the open LHP

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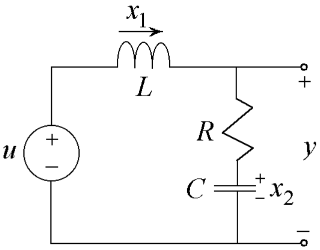
Full-Order UIO Example: Model



$x_1 =$ Current through L
$x_2 =$ Voltage across C

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_2 \boldsymbol{u}_2 \\
y &= \begin{bmatrix} R & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{C}\boldsymbol{x}
\end{aligned}$$

Full-Order UIO Example: Numerical Values

- Let $R = 2$, $L = 2$, and $C = 1/2$.
- The model

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} -1 & -0.5 \\ 2 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u \\
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Full-Order UIO Numerical Example: Dynamics

- Since $\boldsymbol{CB}$ is square,

$$\boldsymbol{M} = \boldsymbol{B}(\boldsymbol{CB})^{-1} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

- Then

$$\tilde{\boldsymbol{\Pi}} = \boldsymbol{I} - \boldsymbol{MC} = \begin{bmatrix} 0 & -0.5 \\ 0 & 1 \end{bmatrix}$$

- The dynamics of the observer are

$$\begin{aligned} \dot{\boldsymbol{q}} &= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Aq} + \boldsymbol{AM}\boldsymbol{y}) + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CM}\boldsymbol{y}) \\ &= \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} \boldsymbol{q} + \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (y - \tilde{y}) \\ \tilde{\boldsymbol{x}} &= \boldsymbol{q} + \boldsymbol{M}y = \boldsymbol{q} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} y \\ \tilde{y} &= \boldsymbol{C}\tilde{\boldsymbol{x}} \end{aligned}$$

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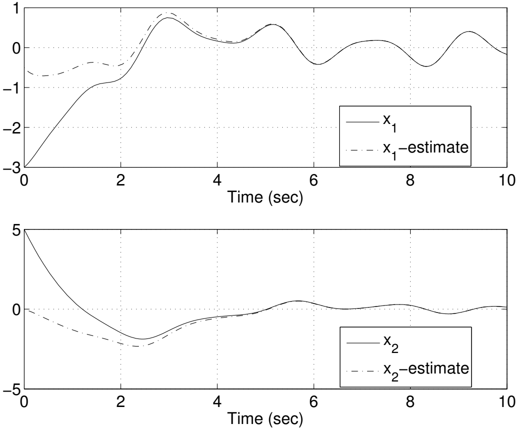
Full-Order UIO Numerical Example: Simulation Result



Unknown input:
$u(t) = \cos(5t) + 2\sin(3t)$

Initial conditions:
$\boldsymbol{x}(0) = [-3 \quad 5]^\top$
$\boldsymbol{q}(0) = \mathbf{0}$

Workshop Overview

Challenges in NCS

- NCS depend on wireless communication
- Major challenge in the NCS design—security
- For example, malicious packet drop attacks in the communication networks