Secure State Estimation of Networked Systems Under Arbitrary Malicious Error Attacks

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- Dr. Stefen Hui from San Diego State University

and with ourformer Ph.D. students:

- Dr. Badriah Alenezi from Kuwait University, and
- Dr. Mukai Zhang from Purdue University

Outline

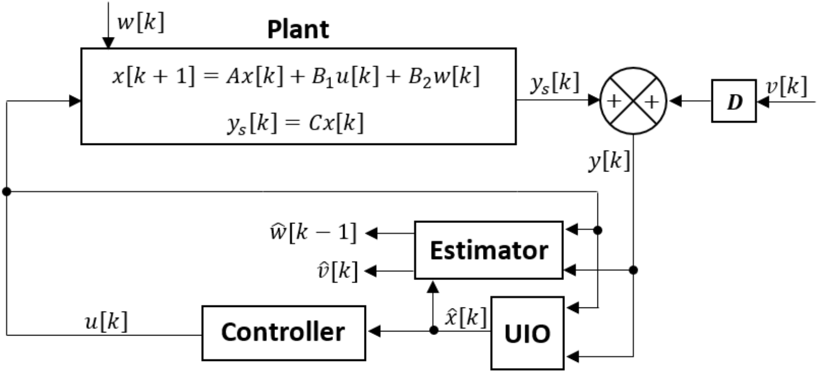
- Problem statement and motivation
- State estimation in the presence of bounded sensors and actuators' errors
- Unknown input observer (UIO) synthesis
- Stability of the observation error dynamics
- Unknown input estimators
- Output disturbance estimation

Problem Statement and Motivation

Networked Control System (NCS) Security

- Networked Control Systems depend on wireless communication—a major challenge in the NCS design is their security
- Actuators and sensor measurements exposed to malicious attacks in communication networks
- Methods of detecting sparse malicious packet drop attacks in the communication networks proposed
- Limitations of the previously presented methods—malicious attacks assumed to be sparse

Our Proposed Approach



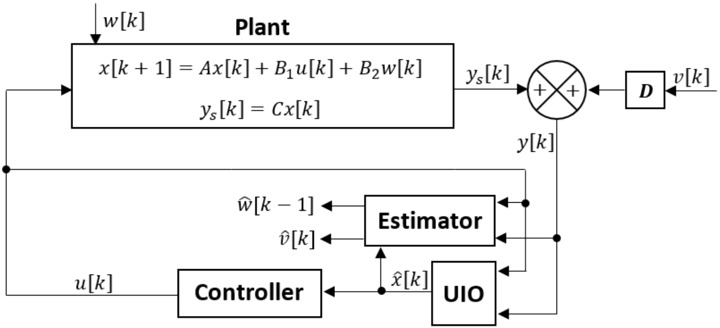
Plant Model

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}_1\boldsymbol{u}[k] + \boldsymbol{B}_2\boldsymbol{w}[k] \\ \boldsymbol{y}[k] &= \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}\boldsymbol{v}[k], \end{aligned}$$

where

- $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_1 \in \mathbb{R}^{n\times m_1}$, $\boldsymbol{B}_2 \in \mathbb{R}^{n\times m_2}$, rank $\boldsymbol{B}_2 = m_2$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, $\boldsymbol{D} \in \mathbb{R}^{p\times r}$, and rank $\boldsymbol{D} = r$
- Control input—$\boldsymbol{u}[k] \in \mathbb{R}^{m_1}$
- Unknown input—$\boldsymbol{w}[k] \in \mathbb{R}^{m_2}$
- Output disturbance—$\boldsymbol{v}[k] \in \mathbb{R}^{r}$
- $\boldsymbol{w}[k]$ and $\boldsymbol{v}[k]$ uniformly bounded as functions of k

Objectives



- Construct Unknown Input Observer (UIO) to estimate the plant state in the presence of unknown input $\boldsymbol{w}[k]$ and output disturbance $\boldsymbol{v}[k]$
- Estimate the unknown input and output disturbance

Unknown Input Observer (UIO) Synthesis

UIO Synthesis: Preliminaries

- Begin by representing $\boldsymbol{x}[k]$ as

$$
\begin{aligned}
\boldsymbol{x}[k] &= \boldsymbol{x}[k] - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}[k] \\
&= (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}[k] + \boldsymbol{M}(\boldsymbol{y}[k] - \boldsymbol{D}\boldsymbol{v}[k]) \\
&= (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}[k] + \boldsymbol{M}\boldsymbol{y}[k] - \boldsymbol{M}\boldsymbol{D}\boldsymbol{v}[k])
\end{aligned}
$$

where

- $\boldsymbol{M} \in \mathbb{R}^{n \times p}$ is to be determined
- Select $\boldsymbol{M}$ such that

$$\boldsymbol{M}\boldsymbol{D} = \boldsymbol{O}_{n \times r}$$

where $\boldsymbol{O}_{n \times r}$ is an n-by-r matrix of zeros

- We obtain:

$$\boldsymbol{x}[k] = (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}[k] + \boldsymbol{M}\boldsymbol{y}[k]$$

UIO Synthesis: Preliminaries

- Begin by representing $\boldsymbol{x}[k]$ as

$$\begin{aligned}\boldsymbol{x}[k] &= \boldsymbol{x}[k] - \boldsymbol{MCx}[k] + \boldsymbol{MCx}[k] \\ &= (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k] + \boldsymbol{M}(\boldsymbol{y}[k] - \boldsymbol{Dv}[k]) \\ &= (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k] + \boldsymbol{My}[k] - \boldsymbol{MDv}[k])\end{aligned}$$

where

- $\boldsymbol{M} \in \mathbb{R}^{n \times p}$ is to be determined
- Select $\boldsymbol{M}$ such that

$$\boldsymbol{MD} = \boldsymbol{O}_{n \times r}$$

where $\boldsymbol{O}_{n \times r}$ is an n-by-r matrix of zeros

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- We obtain:

$$\boldsymbol{x}[k] = (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}[k] + \boldsymbol{M}\boldsymbol{y}[k]$$

Manipulations

- We have: $\boldsymbol{x}[k] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k] + \boldsymbol{My}[k]$
- Let $\boldsymbol{z}[k] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k]$
- Hence

$$\boldsymbol{x}[k] = \boldsymbol{z}[k] + \boldsymbol{My}[k]$$

- We will now show that an estimate of the state $\boldsymbol{x}[k]$ can be obtained from

$$\hat{\boldsymbol{x}}[k] = \boldsymbol{z}[k] + \boldsymbol{My}[k]$$

- The signal $\boldsymbol{z}[k]$ is obtained from

$$\boldsymbol{z}[k+1] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k+1]$$

Manipulations

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- The signal $\boldsymbol{z}[k]$ is obtained from

$$\boldsymbol{z}[k+1] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k+1]$$

Manipulations—Contd.

- Substitute the state dynamics equation into $\boldsymbol{z}[k+1] = (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}[k+1]$ to obtain

$$\boldsymbol{z}[k+1] = (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}_1\boldsymbol{u}[k] + \boldsymbol{B}_2\boldsymbol{w}[k])$$

- Substitute $\boldsymbol{x}[k] = \boldsymbol{z}[k] + \boldsymbol{M}\boldsymbol{y}[k]$ into the above

$$\begin{aligned}\boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{z}[k] + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\ &\quad + (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2\boldsymbol{w}[k]\end{aligned}$$

- Select $\boldsymbol{M}$ so that

$$(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$$

Manipulations—Contd.

- Substitute the state dynamics equation into $\boldsymbol{z}[k+1] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k+1]$ to obtain

$$\boldsymbol{z}[k+1] = (\boldsymbol{I}_n - \boldsymbol{MC})(\boldsymbol{Ax}[k] + \boldsymbol{B}_1\boldsymbol{u}[k] + \boldsymbol{B}_2\boldsymbol{w}[k])$$

- Substitute $\boldsymbol{x}[k] = \boldsymbol{z}[k] + \boldsymbol{My}[k]$ into the above

$$\begin{aligned} \boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{MC})(\boldsymbol{Az}[k] + \boldsymbol{AMy}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\ &\quad + (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{w}[k] \end{aligned}$$

- Select $\boldsymbol{M}$ so that

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Open-Loop UIO

$$
\begin{aligned}
\boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{z}[k] + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\
\hat{\boldsymbol{x}}[k] &= \boldsymbol{z}[k] + \boldsymbol{M}\boldsymbol{y}[k]
\end{aligned}
$$

- Observation error $\boldsymbol{e}[k] = \boldsymbol{x}[k] - \hat{\boldsymbol{x}}[k]$
- Observation error dynamics

$$
\boldsymbol{e}[k+1] = (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}\boldsymbol{e}[k]
$$

- Add innovation term—the closed-loop UIO

Synthesis of the Closed-Loop UIO

- Observation error dynamics of the open-loop UIO

$$\begin{aligned} \boldsymbol{e}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}\boldsymbol{e}[k] \\ &= \boldsymbol{A}_1\boldsymbol{e}[k] \end{aligned}$$

- Add $-\boldsymbol{L}(\boldsymbol{y}[k] - \hat{\boldsymbol{y}}[k])$, where $\boldsymbol{L} \in \mathbb{R}^{n\times p}$ and

$$\hat{\boldsymbol{y}}[k] = \boldsymbol{C}\hat{\boldsymbol{x}}[k] = \boldsymbol{C}(\boldsymbol{z}[k] + \boldsymbol{M}\boldsymbol{y}[k])$$

- Observation error dynamics of the closed-loop UIO

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k] - \boldsymbol{L}\boldsymbol{D}\boldsymbol{v}[k]$$

Closed-Loop UIO

- Observation error dynamics of the closed-loop UIO

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$$

- The closed-loop UIO

$$\begin{aligned} \boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{MC})(\boldsymbol{Az}[k] + \boldsymbol{AMy}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\ &\quad + \boldsymbol{L}(\boldsymbol{y}[k] - \hat{\boldsymbol{y}}[k]) \\ \hat{\boldsymbol{x}}[k] &= \boldsymbol{z}[k] + \boldsymbol{My}[k] \end{aligned}$$

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UIO Synthesis: Solving for $\boldsymbol{M}$

Theorem

There exists a solution $\boldsymbol{M}$ *to*

$$\begin{aligned}(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 &= \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{M}\boldsymbol{D} &= \boldsymbol{O}_{n\times r}\end{aligned}$$

if and only if

$$rank\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r}\end{bmatrix} = rank\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}$$

Solving for $\boldsymbol{M}$—Proof of Theorem

- Represent

$$\begin{aligned}(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 &= \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{M}\boldsymbol{D} &= \boldsymbol{O}_{n\times r}\end{aligned}$$

as

$$\boldsymbol{M}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}$$

- A necessary and sufficient condition (NASC) for $\boldsymbol{M}$ to solve the above matrix equation is that the space spanned by the rows of the matrix $\begin{bmatrix}\boldsymbol{B}_2 & \boldsymbol{O}_{n\times r}\end{bmatrix}$ is in the range of the space spanned by the rows of the matrix $\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}$
- This is equivalent to

$$\operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}$$

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Solving for $\boldsymbol{M}$—Proof of Theorem

- Represent

$$\begin{aligned} (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 &= \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{M}\boldsymbol{D} &= \boldsymbol{O}_{n\times r} \end{aligned}$$

  as

$$\boldsymbol{M}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}$$

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Solving for $\boldsymbol{M}$—Another NASC

Theorem

There exists a solution $\boldsymbol{M}$ to

$$\boldsymbol{M}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}$$

if and only if

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \operatorname{rank}(\boldsymbol{B}_2) + \operatorname{rank}(\boldsymbol{D})$$

We have

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{B}_2 & \boldsymbol{O} \end{bmatrix} \\
&= \operatorname{rank}\left(\begin{bmatrix} \boldsymbol{I}_p & -\boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{I}_n \end{bmatrix}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{B}_2 & \boldsymbol{O} \end{bmatrix}\right) \\
&= \operatorname{rank}\begin{bmatrix} \boldsymbol{O} & \boldsymbol{D} \\ \boldsymbol{B}_2 & \boldsymbol{O} \end{bmatrix} = \operatorname{rank}(\boldsymbol{B}_2) + \operatorname{rank}(\boldsymbol{D})
\end{aligned}$$

Solving for $\boldsymbol{M}$—Another NASC

Theorem

There exists a solution $\boldsymbol{M}$ to

$$\boldsymbol{M}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}$$

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We have

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&= \operatorname{rank}\left(\begin{bmatrix} \boldsymbol{I}_p & -\boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{I}_n \end{bmatrix}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{B}_2 & \boldsymbol{O} \end{bmatrix}\right) \\
&= \operatorname{rank}\begin{bmatrix} \boldsymbol{O} & \boldsymbol{D} \\ \boldsymbol{B}_2 & \boldsymbol{O} \end{bmatrix} = \operatorname{rank}(\boldsymbol{B}_2) + \operatorname{rank}(\boldsymbol{D})
\end{aligned}$$

A Formula to Compute $\boldsymbol{M}$

- Represent

$$\begin{aligned}(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 &= \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{M}\boldsymbol{D} &= \boldsymbol{O}_{n\times r}\end{aligned}$$

as

$$\boldsymbol{M}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}$$

- If $\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \operatorname{rank}(\boldsymbol{B}_2) + \operatorname{rank}(\boldsymbol{D})$ then

$$\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}$$

has full column rank and therefore it is left invertible

Computing $\boldsymbol{M}$—Contd.

- We are solving

$$\boldsymbol{M} \begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}$$

- We obtain

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}^{\dagger}$$

- A general class of solutions

$$\begin{aligned} \boldsymbol{M} &= \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix} \left(\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}^{\dagger} \right. \\ &\quad \left. + \boldsymbol{H}_0 \left(\boldsymbol{I}_p - \begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}^{\dagger} \right) \right) \end{aligned}$$

where $\boldsymbol{H}_0 \in \mathbb{R}^{(m_2+r)\times p}$ is a design parameter matrix

More on the Synthesis of the UIO

- Proposed UIO

$$\begin{aligned}\boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{MC})(\boldsymbol{Az}[k] + \boldsymbol{AMy}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\ &\quad + \boldsymbol{L}(\boldsymbol{y}[k] - \hat{\boldsymbol{y}}[k]) \\ \hat{\boldsymbol{x}}[k] &= \boldsymbol{z}[k] + \boldsymbol{My}[k]\end{aligned}$$

- Observation error dynamics:

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$$

where $\boldsymbol{A}_1 = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{A}$

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More on the Synthesis of the UIO—Contd.

- Observation error dynamics:

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$$

  where $\boldsymbol{A}_1 = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{A}$

- Note that if an $\boldsymbol{L}$ exists such that $(\boldsymbol{A}_1 - \boldsymbol{LC})$ is Schur stable and

$$\boldsymbol{LD} = \boldsymbol{O},$$

  then the error dynamics become

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k]$$

UIO Synthesis—Example 1

System model matrices

$$\boldsymbol{A} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \boldsymbol{B}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}$$

- The matrix rank condition satisfied
- Solve for $\boldsymbol{M} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0 \end{bmatrix}$
- Construct

$$\boldsymbol{A}_1 = (\boldsymbol{I}_3 - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A} = \begin{bmatrix} 0.5 & -0.5 & -0.25 \\ 0 & 0 & -0.25 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Example 1 Contd.

- Can we find an $\boldsymbol{L}$ such that $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ is Schur stable and

$$\boldsymbol{L}\boldsymbol{D} = \boldsymbol{O}$$

  so that the error dynamics would become

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k]?$$

- Used `cvx` to obtain $\boldsymbol{L} = \begin{bmatrix} -0.25 & 0 \\ -0.05 & 0 \\ 0.10 & 0 \end{bmatrix}$
- Eigenvalues of $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ at

$$0.5,\ 0.0,\ 0.5$$

UIO Synthesis—Example 2

System model matrices

$$
\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.3 \end{bmatrix} \boldsymbol{B}_2 = \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix}, \\
\boldsymbol{C} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}
\end{aligned}
$$

- The matrix rank condition satisfied
- Solve for $\boldsymbol{M} = \begin{bmatrix} -2 & 2 \\ -3 & 3 \\ -4 & 4 \end{bmatrix}$
- Construct

$$
\boldsymbol{A}_1 = (\boldsymbol{I}_3 - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A} = \begin{bmatrix} -3 & 4 & 0 \\ -3 & 4 & 0 \\ -4 & 8 & -0.3 \end{bmatrix}
$$

Example 2 Contd.

- Can we find an $\boldsymbol{L}$ such that $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ is Schur stable and

$$\boldsymbol{L}\boldsymbol{D} = \boldsymbol{O},$$

so that the error dynamics would become

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k]?$$

- Used `cvx` to obtain $\boldsymbol{L} = \begin{bmatrix} -3.4974 & 3.4974 \\ -3.4936 & 3.4936 \\ -5.9137 & 5.9137 \end{bmatrix}$
- Eigenvalues of $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ at

$$0.3000, \;\; 1.0000, \;\; 0.0038$$

- No such luck in this example

Conclusions from the examples

- In general it may not be possible find an $\boldsymbol{L}$ such that $(\boldsymbol{A}_1 - \boldsymbol{LC})$ is Schur stable and

$$\boldsymbol{LD} = \boldsymbol{O},$$

so that the error dynamics would become

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k]$$

- We thus need to analyze the error dynamics

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$$

Stability of the Observation Error Dynamics

l_∞-stability with performance level (p.l.) γ

- Recall the observation error dynamics of the closed-loop UIO

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$$

Notation

- For any vector $\boldsymbol{v} \in \mathbb{R}^n$, denote $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}^\top \boldsymbol{v}}$
- For a sequence of vectors $\boldsymbol{v}_{k=k_0}^{\infty}$, denote

$$\|\boldsymbol{v}\|_\infty \triangleq \sup_{k \geq k_0} \|\boldsymbol{v}_k\|$$

B. Alenezi, M. Zhang, S. Hui, and S. H. Żak, Simultaneous Estimation of the State, Unknown Input, and Output Disturbance in Discrete-Time Linear Systems, *IEEE Transactions on Automatic Control*, Date of Publication: 24 February 2021

l_∞-stability definition

The system $\boldsymbol{e}[k+1] = \boldsymbol{f}(k, \boldsymbol{e}[k], \boldsymbol{v}[k])$ is globally uniformly l_∞-stable with performance level γ if

1. $\boldsymbol{e}[k+1] = \boldsymbol{f}(k, \boldsymbol{e}[k], \boldsymbol{0})$ globally uniformly exponentially stable with respect to the origin
2. for $\boldsymbol{e}[k_0] = \boldsymbol{0}$, and every bounded unknown input $\boldsymbol{v}[k]$, $\|\boldsymbol{e}[k]\| \leq \gamma \|\boldsymbol{v}[k]\|_\infty \ \forall k \geq k_0$
3. for any $\boldsymbol{e}[k_0] = \boldsymbol{e}_0$ and $\boldsymbol{v}[\cdot]$,

$$\limsup_{k \to \infty} \|\boldsymbol{e}[k]\| \leq \gamma \|\boldsymbol{v}[k]\|_\infty$$

A. Chakrabarty, S. H. Żak, and S. Sundaram, *State and unknown input observers for discrete-time nonlinear systems*, 2016 IEEE 55th CDC, Las Vegas, Dec 12–14, 2016, pp. 7111–7116

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3. for any $\boldsymbol{e}[k_0] = \boldsymbol{e}_0$ and $\boldsymbol{v}[\cdot]$,

$$\limsup_{k\to\infty} \|\boldsymbol{e}[k]\| \leq \gamma \|\boldsymbol{v}[k]\|_\infty$$

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A. Chakrabarty, S. H. Żak, and S. Sundaram, *State and unknown input observers for discrete-time nonlinear systems*, 2016 IEEE 55th CDC, Las Vegas, Dec 12–14, 2016, pp. 7111–7116

Sufficient condition for l_∞-stability

Lemma

Suppose that for $\boldsymbol{e}[k+1] = \boldsymbol{f}(k, \boldsymbol{e}[k], \boldsymbol{v}[k])$, *there exists* $V : \mathbb{R}^n \to \mathbb{R}$ *and scalars* $\delta \in (0,1)$, $\beta_1, \beta_2 > 0$ *and* $\mu_1, \mu_2 \geq 0$ *such that*

$$\beta_1 \|\boldsymbol{e}[k]\|^2 \leq V(\boldsymbol{e}[k]) \leq \beta_2 \|\boldsymbol{e}[k]\|^2$$

and

$$\begin{aligned} \Delta V[k] &\leq -\delta(V(\boldsymbol{e}[k]) - \mu_1 \|\boldsymbol{v}[k]\|^2) \\ \|\boldsymbol{e}[k]\|^2 &\leq \mu_2 V(\boldsymbol{e}[k]) \end{aligned}$$

for all $k \geq 0$, *where* $\Delta V[k] = V(\boldsymbol{e}[k+1]) - V(\boldsymbol{e}[k])$. *Then, the error system is globally uniformly* l_∞*-stable with performance level* $\gamma = \sqrt{\mu_1 \mu_2}$ *with respect to the disturbance input sequence* $\boldsymbol{v}[k]$

Proof of the sufficient condition

- Expand $\Delta V[k] \leq -\delta(V(\boldsymbol{e}[k]) - \mu_1\|\boldsymbol{v}[k]\|^2)$
- Use $\Delta V[k] = V(\boldsymbol{e}[k+1]) - V(\boldsymbol{e}[k])$ to obtain

$$V(\boldsymbol{e}[k+1]) \leq (1-\delta)V(\boldsymbol{e}[k]) + \delta\mu_1\|\boldsymbol{v}[k]\|^2$$

- Hence

$$\begin{aligned} V(\boldsymbol{e}[k]) &\leq (1-\delta)^k V(\boldsymbol{e}[0]) + \delta\mu_1 \sum_{j=0}^{k-1} \|\boldsymbol{v}[k]\|^2 \\ &\leq (1-\delta)^k V(\boldsymbol{e}[0]) + \mu_1\|\boldsymbol{v}\|_\infty^2 \end{aligned}$$

for any $k \geq 0$ since $0 < \delta < 1$

Proof of the sufficient condition—Contd.

- We have $V(\boldsymbol{e}[k]) \leq (1-\delta)^k V(\boldsymbol{e}[0]) + \mu_1 \|\boldsymbol{v}\|_\infty^2$
- Hence

$$\begin{array}{rcl} \|\boldsymbol{e}[k]\|^2 & \leq & \mu_2 V(\boldsymbol{e}[k]) \\ & \leq & \mu_2 (1-\delta)^k V(\boldsymbol{e}[0]) + \mu_1 \mu_2 \|\boldsymbol{v}\|_\infty^2 \end{array}$$

- This implies

$$\limsup_{k \to \infty} \|\boldsymbol{e}[k]\|^2 \leq \mu_1 \mu_2 \|\boldsymbol{v}\|_\infty^2$$

- In sum, the error dynamics are l_∞-stable with performance level $\gamma = \sqrt{\mu_1 \mu_2}$

Stability of the error dynamics

Recall the observation error dynamics of the closed-loop UIO

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$$

Theorem

The observation error dynamics are globally uniformly l_∞-stable with performance level γ if $(\boldsymbol{A}_1 - \boldsymbol{LC})$ is Schur stable and either of the conditions of the definition of the l_∞-stability is satisfied

In sum: We proved stability of the error dynamics lemma that we will use next

Observation error: $\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$

Lemma

Suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ *and scalars* $\delta \in (0, 1)$, $\beta_1, \beta_2 > 0$ *and* $\mu_1, \mu_2 \geq 0$ *such that*

$$\beta_1 \|\boldsymbol{e}[k]\|^2 \leq V(\boldsymbol{e}[k]) \leq \beta_2 \|\boldsymbol{e}[k]\|^2,$$

$$\Delta V[k] \leq -\delta(V(\boldsymbol{e}[k]) - \mu_1 \|\boldsymbol{v}[k]\|^2), \quad \|\boldsymbol{e}[k]\|^2 \leq \mu_2 V(\boldsymbol{e}[k])$$

for all $k \geq 0$*. Then, the observation error is globally uniformly* l_∞*-stable with performance level* $\gamma = \sqrt{\mu_1 \mu_2}$ *with respect to the output disturbance* $\boldsymbol{v}[k]$

Observation error stability test

$$e[k+1] = (A_1 - LC)e[k] - LDv[k] := Ee[k] + Nv[k]$$

Theorem

If there exist matrices $P = P^\top \succ 0$ and L, and $\alpha \in (0,1)$ such that

$$\begin{bmatrix} E^\top PE - (1-\alpha)P & * \\ N^\top PE & N^\top PN - \alpha I \end{bmatrix} \preceq 0$$

then the state observation error is l_∞-stable with performance level $\gamma = 1/\sqrt{\lambda_{\min}(P)}$

Error stability test proof

- Since $P = P^\top \succ 0$, conditions of the lemma are satisfied with $\beta_1 = \lambda_{\text{min}}(P)$, $\beta_2 = \lambda_{\text{max}}(P)$, and $\mu_2 = 1/\lambda_{\text{min}}(P)$
- Let $V[k] = \boldsymbol{e}[k]^\top \boldsymbol{P}\boldsymbol{e}[k]$ be a Lyapunov function candidate for the estimation error dynamics
- Evaluate the first forward difference $\Delta V[k] = V[k+1] - V[k]$ on the trajectories of the error dynamics

$$\begin{aligned}\Delta V[k] &= \boldsymbol{e}[k]^\top(\boldsymbol{E}^\top\boldsymbol{P}\boldsymbol{E} - \boldsymbol{P})\boldsymbol{e}[k] + 2\boldsymbol{e}[k]^\top\boldsymbol{E}^\top\boldsymbol{P}\boldsymbol{N}\boldsymbol{v}[k] \\ &\quad + \boldsymbol{v}[k]^\top\boldsymbol{N}^\top\boldsymbol{P}\boldsymbol{N}\boldsymbol{v}[k]\end{aligned}$$

- Let $\boldsymbol{\zeta} = \begin{bmatrix} \boldsymbol{e}[k]^\top & \boldsymbol{v}[k]^\top \end{bmatrix}^\top$
- Pre-multiplying and post-multiplying the "big" matrix inequality by $\boldsymbol{\zeta}^\top$ and $\boldsymbol{\zeta}$, respectively, gives

$$\Delta V[k] + \alpha(V[k] - \|\boldsymbol{v}[k]\|^2) \preceq 0$$

Error stability test proof—Contd.

- Condition of the lemma holds with $\mu_1 = 1$
- The observer error satisfies

$$\limsup_{k\to\infty} \|e[k]\| \leq \gamma \limsup_{k\to\infty} \|v[k]\|_\infty$$

  where

$$\gamma = 1/\sqrt{\lambda_{\min}(P)}$$

- In summary, the state error dynamics are ℓ_∞-stable with performance level γ

From matrix inequality to an LMI

- Let $\boldsymbol{Z} = \boldsymbol{P}\boldsymbol{L}$, then solving the "big" matrix inequality is equivalent to solving the LMI

$$\begin{bmatrix} -\boldsymbol{P} & * \\ \boldsymbol{\Omega_{21}} & \boldsymbol{\Omega_{22}} \end{bmatrix} \preceq 0,$$

for $\boldsymbol{P}$ and $\boldsymbol{Z}$, where

$$\boldsymbol{\Omega_{21}}^\top = \begin{bmatrix} \boldsymbol{P}\boldsymbol{A}_1 - \boldsymbol{Z}\boldsymbol{C} & -\boldsymbol{Z}\boldsymbol{D} \end{bmatrix}$$

and

$$\boldsymbol{\Omega_{22}} = \begin{bmatrix} -(1-\alpha)\boldsymbol{P} & \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{O}_{m_2\times n} & -\alpha\boldsymbol{I} \end{bmatrix}$$

- Take the Schur complement

$$\boldsymbol{\Omega_{22}} + \boldsymbol{\Omega_{21}}\boldsymbol{P}^{-1}\boldsymbol{\Omega_{21}}^\top \preceq 0$$

which yields the "big" matrix inequality

Sufficient condition for UIO existence

Theorem

The UIO exists if

1. *there exists* $\boldsymbol{M}$ *such that*

$$(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}_{n\times m_2} \quad \textit{and} \quad \boldsymbol{M}\boldsymbol{D} = \boldsymbol{O}_{n\times r}$$

2. *the pair* $(\boldsymbol{A}_1, \boldsymbol{C})$ *is detectable*

If $(\boldsymbol{A}_1, \boldsymbol{C})$ detectable, then we can find the observer gain matrix $\boldsymbol{L}$ such that $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ is Schur stable

Interpretation of the UIO Conditions

Lemma

If the pair $((\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})\boldsymbol{A},\boldsymbol{M}\boldsymbol{C}) := (\boldsymbol{A}_1,\boldsymbol{M}\boldsymbol{C})$ *is detectable, then the pair* $(\boldsymbol{A}_1,\boldsymbol{C})$ *is detectable*

Proof by contraposition

- Suppose $(\boldsymbol{A}_1,\boldsymbol{C})$ is non-detectable
- Then there exists an eigenvalue $|z_1|\geq 1$, which is an unobservable mode of the pair $(\boldsymbol{A}_1,\boldsymbol{C})$ such that

$$\operatorname{rank}\begin{bmatrix} z_1\boldsymbol{I}-\boldsymbol{A}_1 \\ \boldsymbol{C} \end{bmatrix} < n$$

Lemma proof—Contd.

- There exists a vector $\boldsymbol{v}_1 \in \mathbb{C}^n$ such that

$$\text{rank}\begin{bmatrix} z_1\boldsymbol{I} - \boldsymbol{A}_1 \\ \boldsymbol{C} \end{bmatrix}\boldsymbol{v}_1 = \boldsymbol{0}$$

- Thus $\boldsymbol{C}\boldsymbol{v}_1 = \boldsymbol{0}$
- Pre-multiply the above by $\boldsymbol{M}$ to obtain $\boldsymbol{M}\boldsymbol{C}\boldsymbol{v}_1 = \boldsymbol{0}$
- Therefore rank$\begin{bmatrix} z_1\boldsymbol{I} - \boldsymbol{A}_1 \\ \boldsymbol{C} \end{bmatrix} < n$
- Thus, z_1 also corresponds to a non-detectable mode of the pair $(\boldsymbol{A}_1, \boldsymbol{M}\boldsymbol{C})$, that is, the pair $(\boldsymbol{A}_1, \boldsymbol{M}\boldsymbol{C})$ is non-detectable

Another lemma

Lemma

If

- $rank(\boldsymbol{C}\boldsymbol{B}_2) = rank(\boldsymbol{B}_2) = m_2$
- $rank(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C}) = n - m_2$,

then, the following are equivalent:

1. $(\boldsymbol{A}_1, \boldsymbol{M}\boldsymbol{C})$ *is detectable*
2. $rank \begin{bmatrix} z(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C}) - \boldsymbol{A}_1 \\ \boldsymbol{M}\boldsymbol{C} \end{bmatrix} = n$ *for all* $|z| \geq 1$
3. $rank \begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B}_2 \\ \boldsymbol{M}\boldsymbol{C} & \boldsymbol{O}_{n\times m_2} \end{bmatrix} = n + m_2$ *for all* $|z| \geq 1$

Proof of the second lemma

- First, we prove that conditions 1 and 2 are equivalent
- $(\boldsymbol{A}_1, \boldsymbol{MC})$ being detectable is equivalent to

$$\operatorname{rank}\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A}_1 \\ \boldsymbol{MC} \end{bmatrix} = n \text{ for all } |z| \geq 1$$

- This is equivalent to

$$\begin{aligned} &\operatorname{rank}\left(\begin{bmatrix} \boldsymbol{I}_n & -z\boldsymbol{I}_n \\ \boldsymbol{O} & \boldsymbol{I}_n \end{bmatrix}\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A}_1 \\ \boldsymbol{MC} \end{bmatrix}\right) \\ &= \operatorname{rank}\begin{bmatrix} z(\boldsymbol{I}_n - \boldsymbol{MC}) - \boldsymbol{A}_1 \\ \boldsymbol{MC} \end{bmatrix} \text{ for all } |z| \geq 1 \end{aligned}$$

- Thus conditions 1 and 2 are equivalent

Proof of the second lemma—Contd.

- We will show that conditions 2 and 3 are equivalent
- Since $\boldsymbol{B}_2$ has full column rank, it is left invertible
- Take, for example, ${\boldsymbol{B}_2}^{\dagger}\boldsymbol{B}_2 = \boldsymbol{I}_{m_2}$
- Then, $\ker\left({\boldsymbol{B}_2}^{\dagger}\right) \cap \ker(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C}) = \{\boldsymbol{0}\}$ and

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C} \\ {\boldsymbol{B}_2}^{\dagger} \end{bmatrix} = n$$

Let

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C} & \boldsymbol{O}_{n\times p} \\ {\boldsymbol{B}_2}^{\dagger} & \boldsymbol{O}_{m_2\times p} \\ \boldsymbol{O}_{p\times n} & \boldsymbol{I}_p \end{bmatrix}, \quad \boldsymbol{T} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{O}_{n\times m_2} \\ -(z{\boldsymbol{B}_2}^{\dagger} - {\boldsymbol{B}_2}^{\dagger}\boldsymbol{A}) & \boldsymbol{I}_{m_2} \end{bmatrix}$$

where $\boldsymbol{S} \in \mathbb{R}^{(n+p+m_2)\times(n+p)}$, $\boldsymbol{T} \in \mathbb{R}^{(n+m_2)\times(n+m_2)}$, and $\operatorname{rank}(\boldsymbol{S}) = n + p$

Conditions 2 and 3 equivalent

We have

$$
\begin{aligned}
&\operatorname{rank}\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B}_2 \\ \boldsymbol{M}\boldsymbol{C} & \boldsymbol{O} \end{bmatrix} = \operatorname{rank}\left(\boldsymbol{S}\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B}_2 \\ \boldsymbol{M}\boldsymbol{C} & \boldsymbol{O} \end{bmatrix}\boldsymbol{T}\right) \\
&= \operatorname{rank}\begin{bmatrix} z(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C}) - \boldsymbol{A}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{m_2} \\ \boldsymbol{M}\boldsymbol{C} & \boldsymbol{O} \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} z(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C}) - \boldsymbol{A}_1 \\ \boldsymbol{M}\boldsymbol{C} \end{bmatrix} + m_2 \\
&= n + m_2
\end{aligned}
$$

This concludes that conditions 2 and 3 are equivalent

The role of system zeros

Theorem

If

- $rank\begin{bmatrix} \boldsymbol{CB}_2 & \boldsymbol{D} \end{bmatrix} = rank(\boldsymbol{B}_2) + rank(\boldsymbol{D})$
- $\begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}$ *is defined and has full column rank*
- $rank\begin{bmatrix} \boldsymbol{I} - \boldsymbol{MC} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{M} \end{bmatrix} = n$
- $rank\begin{bmatrix} z(\boldsymbol{I}_n - \boldsymbol{MC}) - \boldsymbol{A}_1 \\ \boldsymbol{MC} \end{bmatrix} = n$ *for all* $|z| \geq 1$

then

$$rank\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = n + m_2 \text{ for all} |z| \geq 1$$

Proof of theorem

- If the matrix rank condition satisfied then there exists a solution $\boldsymbol{M}$ that satisfies

$$\begin{bmatrix} \boldsymbol{I} - \boldsymbol{M}\boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{M} \end{bmatrix} \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix} = \boldsymbol{O}$$

- Let $\tilde{\boldsymbol{M}} = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{M}\boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{M} \end{bmatrix}$
- There exists $\boldsymbol{M}_1 \in \mathbb{R}^{(p-m_2)\times(n+p)}$ such that

$$\boldsymbol{M}_1 \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix} = \boldsymbol{O}$$

and rank$\begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \end{bmatrix} = n + p - m_2$

Proof of theorem—Contd.

- Since $\begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}$ has full column rank, it is left invertible, that is,

$$\begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}^{\dagger} \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix} = \boldsymbol{I}_{m_2}$$

- Therefore,

$$\ker \begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \end{bmatrix} \cap \ker \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}^{\dagger} = \{\boldsymbol{0}\}$$

  and

$$\operatorname{rank} \begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \\ \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}^{\dagger} \end{bmatrix} = n + p$$

Proof of theorem—Almost there

- Let $\boldsymbol{S} = \begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \\ \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}^\dagger \end{bmatrix}$ and

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{O} \\ -\begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}^\dagger \begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} & \boldsymbol{I}_{m_2} \end{bmatrix}$$

- Then,

$$\begin{aligned} &\operatorname{rank} \boldsymbol{S} \begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \boldsymbol{T} \\ &= \operatorname{rank} \begin{bmatrix} \tilde{\boldsymbol{M}} \begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} & \boldsymbol{O} \\ \boldsymbol{M}_1 \begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{m_2} \end{bmatrix} \end{aligned}$$

Proof of theorem—Two more steps

We continue

$$\operatorname{rank}\begin{bmatrix} \tilde{\boldsymbol{M}}\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} & \boldsymbol{O} \\ \boldsymbol{M}_1\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{m_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} \tilde{\boldsymbol{M}}\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} \\ \boldsymbol{M}_1\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} \end{bmatrix} + m_2$$

Note that

$$\tilde{\boldsymbol{M}}\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} z(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}) - \boldsymbol{A}_1 \\ \boldsymbol{M}\boldsymbol{C} \end{bmatrix}$$

Proof of theorem—Finally!

- Therefore, $\text{rank}\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = n + m_2$ for all $|z| \geq 1$ if
  $\text{rank}\begin{bmatrix} z(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}) - \boldsymbol{A}_1 \\ \boldsymbol{M}\boldsymbol{C} \end{bmatrix} = n, \;$ for all $|z| \geq 1$
- Recall that $\text{rank}\begin{bmatrix} z(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}) - \boldsymbol{A}_1 \\ \boldsymbol{M}\boldsymbol{C} \end{bmatrix} = n, \;$ for all $|z| \geq 1$ implies detectability of the pair $(\boldsymbol{A}_1, \boldsymbol{C})$

Constructing $\boldsymbol{S}$

- Recall, $\boldsymbol{S} = \begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \\ \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}^{\dagger} \end{bmatrix}$
- How to compute $\boldsymbol{M}_1$?
- Note that $\boldsymbol{M}_1$ is such that

$$\begin{aligned} \begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \end{bmatrix} \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{O}_{2n\times m_2} \\ \boldsymbol{O}_{(p-m_2)\times m_2} \end{bmatrix} \\ \operatorname{rank}\begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \end{bmatrix} &= n+p-m_2 \end{aligned}$$

- Can compute $\boldsymbol{M}_1$, using MATLAB, as

$$\boldsymbol{M}_1^{\top} = \operatorname{null}\begin{bmatrix} \tilde{\boldsymbol{M}} \\ \begin{bmatrix} -\boldsymbol{B}_2^{\top} & \boldsymbol{D}^{\top} \end{bmatrix} \end{bmatrix}$$

What about rank $\boldsymbol{S}$?

- Recall, $\operatorname{rank} \boldsymbol{S} = \operatorname{rank} \begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \\ \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}^\dagger \end{bmatrix} = n + p$
- We also have,

$$\begin{aligned} \ker \begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \end{bmatrix} \quad &\cap \quad \ker \begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}^\dagger = \{\mathbf{0}\} \\ \operatorname{rank} \begin{bmatrix} \tilde{\boldsymbol{M}} \\ \boldsymbol{M}_1 \end{bmatrix} \quad &= \quad n + p - m_2 \end{aligned}$$

- Hence, we have to have

$$\operatorname{rank} \boldsymbol{S} = n + p$$

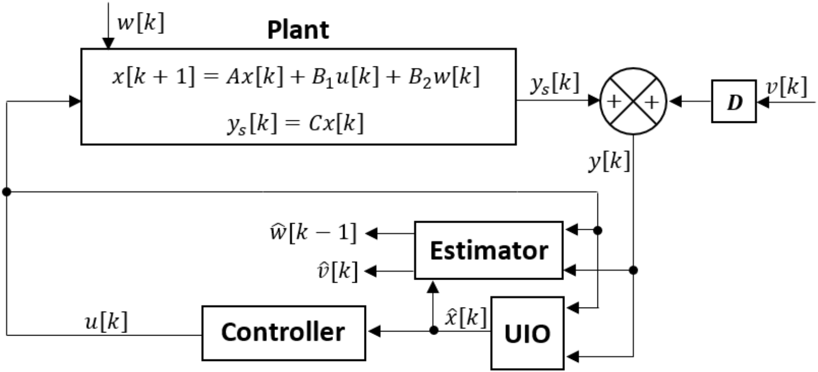
Simple MATLAB code to compute $\boldsymbol{S}$

```
function[]=trans_mat_S()
clc
clear
% Example 1:
A=[-1 0 0;0 -2 0;0 0 -0.3];
B2=[-2 -3 -4]';
C=[1 0 0;0 1 0];
D=[2 2]';
% Example 2
A=[1 0;1 1];
B2=[0 1]';
C=[2 1;0 1];
D=[1 0]';
```

MATLAB code to compute $\boldsymbol{S}$—Contd.

```
% Dimensions
n=size(A,1);
m2=size(B2,2);
p=size(C,1);
r=size(D,2);
% Solving for M
M=[B2 zeros(n,r)]*pinv([C*B2 D]);
Mtilde=[(eye(n)- M*C)  zeros(n,m2+r);zeros(n,n) M];
M1=null([-B2'  D';Mtilde])';
% Transformation matrix
S=[Mtilde;M1;pinv([-B2;D])]
```

Unknown input and output disturbance estimators



Unknown input estimator

- Pre-multiply both sides of the state dynamic $\boldsymbol{B}_2^\dagger$

$$\boldsymbol{B}_2^\dagger \boldsymbol{x}[k+1] = \boldsymbol{B}_2^\dagger \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}_2^\dagger \boldsymbol{B}_1 \boldsymbol{u}[k] + \boldsymbol{B}_2^\dagger \boldsymbol{B}_2 \boldsymbol{w}[k]$$

- Use $\boldsymbol{B}_2^\dagger \boldsymbol{B}_2 = \boldsymbol{I}_{m_2}$ to obtain

$$\boldsymbol{w}[k] = \boldsymbol{B}_2^\dagger \boldsymbol{x}[k+1] - \boldsymbol{B}_2^\dagger \boldsymbol{A}\boldsymbol{x}[k] - \boldsymbol{B}_2^\dagger \boldsymbol{B}_1 \boldsymbol{u}[k]$$

- The unknown input estimator:

$$\hat{\boldsymbol{w}}[k] = \boldsymbol{B}_2^\dagger \hat{\boldsymbol{x}}[k+1] - \boldsymbol{B}_2^\dagger \boldsymbol{A}\hat{\boldsymbol{x}}[k] - \boldsymbol{B}_2^\dagger \boldsymbol{B}_1 \boldsymbol{u}[k]$$

- The above estimator depends on $\hat{\boldsymbol{x}}[k+1]$
- Can estimate the unknown input with one sampling period time-delay

$$\hat{\boldsymbol{w}}[k-1] = \boldsymbol{B}_2^\dagger \hat{\boldsymbol{x}}[k] - \boldsymbol{B}_2^\dagger \boldsymbol{A}\hat{\boldsymbol{x}}[k-1] - \boldsymbol{B}_2^\dagger \boldsymbol{B}_1 \boldsymbol{u}[k-1]$$

Unknown input estimator performance

- Unknown input estimation error, $\boldsymbol{e}_w[k] = \boldsymbol{w}[k] - \hat{\boldsymbol{w}}[k]$
- Then, $\boldsymbol{e}_w[k] = \boldsymbol{B}_2^\dagger \boldsymbol{e}[k+1] - \boldsymbol{B}_2^\dagger \boldsymbol{A}\boldsymbol{e}[k]$
- We have, $\limsup_{k\to\infty} \|\boldsymbol{e}[k]\| \le \gamma \|\boldsymbol{v}[k]\|_\infty$
- Unknown input estimation error bound

$$\begin{aligned}\limsup_{k\to\infty} \|\boldsymbol{e}_w[k]\| &\le \|\boldsymbol{B}_2^\dagger\|(\gamma\|\boldsymbol{v}[k+1]\|_\infty + \|\boldsymbol{A}\|\gamma\|\boldsymbol{v}[k]\|_\infty) \\ &\le \|\boldsymbol{B}_2^\dagger\|(1+\|\boldsymbol{A}\|)\sqrt{\mu}\|\boldsymbol{v}[k]\|_\infty\end{aligned}$$

- Let $\gamma_w = \|\boldsymbol{B}_2^\dagger\|(1+\|\boldsymbol{A}\|)\sqrt{\mu}$
- Then, $\limsup_{k\to\infty} \|\boldsymbol{e}_w[k]\| \le \gamma_w \|\boldsymbol{v}[k]\|_\infty$
- Unknown input estimator performance level γ_w

Output disturbance estimator

- Pre-multiply output equation by $\boldsymbol{D}^\dagger$

$$\boldsymbol{D}^\dagger \boldsymbol{y}[k] = \boldsymbol{D}^\dagger \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}^\dagger \boldsymbol{D}\boldsymbol{v}[k]$$

- Rearrange to obtain, $\boldsymbol{v}[k] = \boldsymbol{D}^\dagger \boldsymbol{y}[k] - \boldsymbol{D}^\dagger \boldsymbol{C}\boldsymbol{x}[k]$
- Output disturbance estimator

$$\hat{\boldsymbol{v}}[k] = \boldsymbol{D}^\dagger \boldsymbol{y}[k] - \boldsymbol{D}^\dagger \boldsymbol{C}\hat{\boldsymbol{x}}[k]$$

- Output disturbance estimation error:

$$\boldsymbol{e}_v[k] = \boldsymbol{v}[k] - \hat{\boldsymbol{v}}[k]$$

- Hence, $\boldsymbol{e}_v[k] = -\boldsymbol{D}^\dagger \boldsymbol{C}\boldsymbol{e}[k]$

Output disturbance estimator performance

- We have

$$\boldsymbol{e}_v[k] = -\boldsymbol{D}^\dagger \boldsymbol{C}\boldsymbol{e}[k]$$

- Recall that $\limsup_{k\to\infty} \|\boldsymbol{e}[k]\| \leq \gamma \|\boldsymbol{v}[k]\|_\infty$
- Output disturbance estimation error bound

$$\begin{aligned} \limsup_{k\to\infty} \|\boldsymbol{e}_v[k]\| &\leq \|\boldsymbol{D}^\dagger\|\|\boldsymbol{C}\|\gamma\|\boldsymbol{v}[k]\|_\infty \\ &\leq \|\boldsymbol{D}^\dagger\|\|\boldsymbol{C}\|\sqrt{\mu}\|\boldsymbol{v}[k]\|_\infty \end{aligned}$$

- Output disturbance estimator performance level,

$$\gamma_v = \|\boldsymbol{D}^\dagger\|\|\boldsymbol{C}\|\sqrt{\mu}$$

Relations With the Strong Observer of Hautus

- System considered by Hautus

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}_2\boldsymbol{w}[k] \\ \boldsymbol{y}[k] &= \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}\boldsymbol{w}[k] \end{aligned}$$

- UIO (strong observer) exists $\iff$
  1. $\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{O} \end{bmatrix} = \operatorname{rank}\boldsymbol{D} + \operatorname{rank}\begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}$
  2. the system zeros of the system defined by quadruple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C}, \boldsymbol{D})$ are in the open unit disc

M. L. J. Hautus, *Strong Detectability and Observers*, Linear Algebra and Its Applications, Vol. 50, pp. 353–368, 1983

From our model into the Hautus model

- Need the same unknown input and output disturbance

$$
\begin{aligned}
\boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}_1\boldsymbol{u}[k] + \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{w}[k] \\ \boldsymbol{v}[k] \end{bmatrix} \\
\boldsymbol{y}[k] &= \boldsymbol{C}\boldsymbol{x}[k] + \begin{bmatrix} \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{w}[k] \\ \boldsymbol{v}[k] \end{bmatrix}
\end{aligned}
$$

- Apply the Hautus matrix rank condition

$$
\begin{aligned}
\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{D} \\ \boldsymbol{O} & \boldsymbol{D} & \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} \\
&\quad + \operatorname{rank}\begin{bmatrix} -\boldsymbol{B}_2 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix}
\end{aligned}
$$

Manipulations

- The Hautus matrix rank condition

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{D} \\ \boldsymbol{O} & \boldsymbol{D} & \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} = \operatorname{rank}[\ \boldsymbol{O} \quad \boldsymbol{D}\ ] + \operatorname{rank}\begin{bmatrix} -\boldsymbol{B}_2 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix}$$

- We obtain

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \operatorname{rank}(\boldsymbol{B}_2) + \operatorname{rank}(\boldsymbol{D})$$

- The matrix rank conditions equivalent for the augmented system

System zeros conditions

- The Hautus system zeros condition applied to the augmented system

$$\operatorname{rank}\begin{bmatrix} z\boldsymbol{I}-\boldsymbol{A} & -\boldsymbol{B}_2 & \boldsymbol{O} \\ \boldsymbol{C} & \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} = n + m_2 + r \text{ for all } |z| \geq 1$$

- If the rank condition

$$\operatorname{rank}\begin{bmatrix} z\boldsymbol{I}_n-\boldsymbol{A} & -\boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = n + m_2 \text{ for all } |z| \geq 1$$

not satisfied, then there are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, not both zero, such that

$$\begin{bmatrix} z\boldsymbol{I}_n-\boldsymbol{A} & -\boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

System zeros conditions—Contd

- Then

$$\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B}_2 & \boldsymbol{O} \\ \boldsymbol{C} & \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

The system zero condition for the augmented system implies the system zero condition for the original system

Matrix rank conditions

Theorem

The matrix rank condition

$$\mathit{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \mathit{rank}\,\boldsymbol{B}_2 + \mathit{rank}\,\boldsymbol{D}$$

implies the matrix rank condition of Hautus

$$\mathit{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{O} \end{bmatrix} = \mathit{rank}\,\boldsymbol{D} + \mathit{rank}\begin{bmatrix} -\boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}$$

Proof of Theorem: Notation for the various matrix rank conditions

1. $\mathcal{S} \iff \operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2\end{bmatrix} = \operatorname{rank}\boldsymbol{B}_2$
2. $\mathcal{G} \iff \operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix} = \operatorname{rank}\boldsymbol{B}_2 + \operatorname{rank}\boldsymbol{D}$
3. $\mathcal{M} \iff \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D}) = \operatorname{rank}\begin{bmatrix}\boldsymbol{B}_2 \\ \boldsymbol{D}\end{bmatrix}$, when $\boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D}$ is defined
4. $\mathcal{H} \iff$ Hautus' matrix rank condition:

$$\operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{O}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\boldsymbol{B}_2 \\ \boldsymbol{D}\end{bmatrix} + \operatorname{rank}\boldsymbol{D}$$

Well known equalities

- For any matrix $\boldsymbol{M}$, let

$$\begin{aligned} \mathfrak{c}(\boldsymbol{M}) &= \text{Number of columns of } \boldsymbol{M} \\ \ker \boldsymbol{M} &= \{\boldsymbol{v} : \boldsymbol{M}\boldsymbol{v} = \boldsymbol{0}\} \end{aligned}$$

- We will make use of the well known equality

$$\mathfrak{c}(\boldsymbol{M}) = \text{rank}\boldsymbol{M} + \dim\ker \boldsymbol{M}$$

- Note that $\mathcal{G}$ implies that $\text{rank}(\boldsymbol{C}\boldsymbol{B}_2) = \text{rank}\boldsymbol{B}_2$ and thus $\boldsymbol{C}\boldsymbol{B}_2\boldsymbol{v} = \boldsymbol{0} \iff \boldsymbol{B}_2\boldsymbol{v} = \boldsymbol{0}$
- Equivalently, $\ker(\boldsymbol{C}\boldsymbol{B}_2) = \ker \boldsymbol{B}_2$

More notation

- If $\boldsymbol{u}$, $\boldsymbol{v}$ are column vectors, we let

$$\boldsymbol{u} \oplus \boldsymbol{v} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$$

- If U, V are vector spaces of column vectors, we let

$$U \oplus V = \{\boldsymbol{u} \oplus \boldsymbol{v} : \boldsymbol{u} \in U, \boldsymbol{v} \in V\}$$

- It is easy to see that

$$\dim(U \oplus V) = \dim U + \dim V$$

- Just note that if $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p\}$ and $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_q\}$ are bases of U, V, respectively, then $\{\boldsymbol{u}_1 \oplus \mathbf{0}, \ldots, \boldsymbol{u}_p \oplus \mathbf{0}, \mathbf{0} \oplus \boldsymbol{v}_1, \ldots, \mathbf{0} \oplus \boldsymbol{v}_q\}$ is a basis for $U \oplus V$

Lemma 1

- Recall

$$\mathcal{G} \iff \mathrm{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix} = \mathrm{rank}\boldsymbol{B}_2 + \mathrm{rank}\boldsymbol{D}$$

If $\mathcal{G}$, then

$$\ker\left(\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}\right) = \ker \boldsymbol{B}_2 \oplus \ker \boldsymbol{D}$$

Proof of Lemma 1

- $\mathcal{G}$ and equality $\mathfrak{c}(\boldsymbol{M}) = \mathrm{rank}\boldsymbol{M} + \dim\ker\boldsymbol{M}$ imply

$$\begin{aligned}
& \dim\ker(\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}) \\
= \; & \mathfrak{c}(\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}) - \mathrm{rank}(\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}) \\
= \; & \mathfrak{c}(\boldsymbol{C}\boldsymbol{B}_2) + \mathfrak{c}(\boldsymbol{D}) - \mathrm{rank}(\boldsymbol{B}_2) - \mathrm{rank}\boldsymbol{D} \\
= \; & \dim\ker\boldsymbol{B}_2 + \dim\ker\boldsymbol{D} \\
= \; & \dim(\ker\boldsymbol{B}_2 \oplus \ker\boldsymbol{D})
\end{aligned}$$

- It is immediate that

$$\ker\boldsymbol{B}_2 \oplus \ker\boldsymbol{D} \subset \ker(\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix})$$

- Hence, $\ker(\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}) = \ker\boldsymbol{B}_2 \oplus \ker\boldsymbol{D}$

Lemma 2

If $\mathcal{G}$, then $\ker(\boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D}) = \ker \boldsymbol{B}_2 \cap \ker \boldsymbol{D}$

- Proof: We have

$$\ker \boldsymbol{B}_2 \cap \ker \boldsymbol{D} \subset \ker(\boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D})$$

- Suppose $(\boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D})\boldsymbol{v} = \boldsymbol{0}$, then

$$\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{v} \end{bmatrix} = \boldsymbol{0}$$

- It follows from Lemma 1 that

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{v} \end{bmatrix} \in \ker \boldsymbol{B}_2 \oplus \ker \boldsymbol{D}$$

- Therefore $\boldsymbol{B}_2\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{D}\boldsymbol{v} = \boldsymbol{0}$ and the lemma follows

Lemma 3

If $\mathcal{G}$ then $\mathcal{M}$

- Proof: Observe that

$$\ker \begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix} = \ker \boldsymbol{B}_2 \cap \ker \boldsymbol{D}$$

- Thus by Lemma 2, we have

$$\ker(\boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D}) = \ker \begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix}$$

- This is equivalent to the claim of the lemma since the two matrices have the same number of columns

Proof of the Theorem

- Assume $\mathcal{G}$
- Then the above lemmas hold and all will be used
- The Hautus matrix rank condition is

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{O} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{D} \end{bmatrix} + \operatorname{rank}\boldsymbol{D}$$

- By Lemma 3, the above is equivalent to

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{O} \end{bmatrix} = \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D}) + \operatorname{rank}\boldsymbol{D}$$

- This is equivalent to

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{O} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix}$$

Proof of the Theorem—Contd

- We prove

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{O} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix}$$

  by showing that the two matrices have the same kernel

- Suppose

$$\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \boldsymbol{0}$$

  Then

$$\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \boldsymbol{0}, \;\; \boldsymbol{D}\boldsymbol{u} = \boldsymbol{0}$$

Proof of the Theorem—Use Lemma 1

- By Lemma 1, $\boldsymbol{C}\boldsymbol{B}_2\boldsymbol{u} = \mathbf{0}$, $\boldsymbol{D}\boldsymbol{v} = \mathbf{0}$, $\boldsymbol{D}\boldsymbol{u} = \mathbf{0}$
- It follows that

$$\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \mathbf{0}$$

- Conversely, suppose

$$\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \mathbf{0}$$

- Then

$$(\boldsymbol{C}\boldsymbol{B}_2 + \boldsymbol{D})\boldsymbol{u} = \mathbf{0}, \; \boldsymbol{D}\boldsymbol{v} = \mathbf{0}$$

  By Lemma 6,

$$\boldsymbol{C}\boldsymbol{B}_2\boldsymbol{u} = \mathbf{0}, \; \boldsymbol{D}\boldsymbol{u} = \mathbf{0}, \; \boldsymbol{D}\boldsymbol{v} = \mathbf{0}$$

Conclusion of Proof of Theorem

- Represent

$$\boldsymbol{C}\boldsymbol{B}_2\boldsymbol{u} = \mathbf{0},\ \boldsymbol{D}\boldsymbol{u} = \mathbf{0},\ \boldsymbol{D}\boldsymbol{v} = \mathbf{0}$$

  in matrix format
- Thus

$$\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \mathbf{0}$$

- Therefore the two matrices in question have the same kernel and therefore the same rank since they clearly have the same number of columns

Conclusions

- Unknown Input Observer (UIO)—powerful and promising tool for detecting and monitoring malicious attacks in networked control systems
- Promising directions—large-scale systems
- Significant industrial applications around the corner
- UIO—Way To Go!