Problem #1

The strategy is to express the satellite position in the b basis first, and then transform the individual components from the b basis to the ENU basis. I will do this through two successive rotations involving an intermediate c basis.

First, I will rotate the b basis about the b_3 axis in order to obtain basis c.

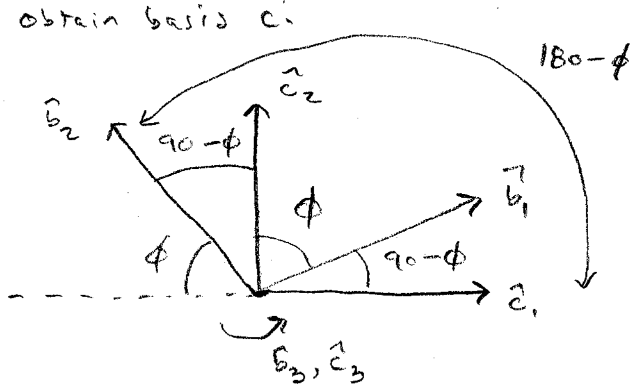


Figure 1.

Assume that we have a vector, $\bar{R}(t)$, whose components are expressed in terms of the b basis. In general, we have:

$$\bar{R}(t)_{|b} = d\hat{b}_1 + e\hat{b}_2 + f\hat{b}_3$$

where d, e, and f are scalar components. Our goal is to express this same vector in terms of the c basis:

$$\bar{R}(t)_{|c} = g\hat{c}_1 + h\hat{c}_2 + m\hat{c}_3$$

So we must compute g, h, and m. In order to compute g, for example, we must find out "how much" of $\bar{R}(t)_{|b}$ is in the direction of g. So, we take the dot product of $\bar{R}(t)_{|b}$ with $\hat{c}_1$, which is the unit vector associated with g.

$$g = \bar{R}(t)_{|b} \cdot \hat{c}_1$$

In general, portions of components d, e, and f could contribute to component g, so all three unit vectors $\hat{b}_1$, $\hat{b}_2$ and $\hat{b}_3$ must be dotted with $\hat{c}_1$ in order to ensure that the appropriate amounts of d, e, and f are "fully represented" in g. So:

$$g = \bar{R}(t)\big|_b \cdot \hat{c}_1 = d(\hat{b}_1 \cdot \hat{c}_1) + e(\hat{b}_2 \cdot \hat{c}_1) + f(\hat{b}_3 \cdot \hat{c}_1)$$

Using a similar analysis, we can compute the component h. Again, we must add up the appropriate contributions of d, e, and f. The "appropriate contributions" are found quantitatively by taking the dot products of $\hat{b}_1$, $\hat{b}_2$ and $\hat{b}_3$ with $\hat{c}_2$ respectively,

$$h = \bar{R}(t)\big|_b \cdot \hat{c}_2 = d(\hat{b}_1 \cdot \hat{c}_2) + e(\hat{b}_2 \cdot \hat{c}_2) + f(\hat{b}_3 \cdot \hat{c}_2)$$

Finally, m can be computed in an analagous fashion

$$m = \bar{R}(t)\big|_b \cdot \hat{c}_3 = d(\hat{b}_1 \cdot \hat{c}_3) + e(\hat{b}_2 \cdot \hat{c}_3) + f(\hat{b}_3 \cdot \hat{c}_3)$$

The three equations written above may be written more compactly in matrix format:

$$\begin{bmatrix} g \\ h \\ m \end{bmatrix} = \begin{bmatrix} (\hat{b}_1 \cdot \hat{c}_1) & (\hat{b}_2 \cdot \hat{c}_1) & (\hat{b}_3 \cdot \hat{c}_1) \\ (\hat{b}_1 \cdot \hat{c}_2) & (\hat{b}_2 \cdot \hat{c}_2) & (\hat{b}_3 \cdot \hat{c}_2) \\ (\hat{b}_1 \cdot \hat{c}_3) & (\hat{b}_2 \cdot \hat{c}_3) & (\hat{b}_3 \cdot \hat{c}_3) \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

The dot product between vectors $\bar{A}$ & $\bar{B}$ is defined as

$$\bar{A} \cdot \bar{B} = \|\bar{A}\| \, \|\bar{B}\| \cos\beta$$

where the angle β is the angle formed when the two vectors are connected tail to tail while maintaining the same relative orientation to each other. In the special case of two unit vectors, $\|\bar{A}\|$ and $\|\bar{B}\|$ are always one, so the dot product just reduces to the cosine of the angle between them. After examining Figure 1, it should be clear what the matrix elements are in terms of ϕ:

$$\begin{bmatrix} g \\ h \\ m \end{bmatrix} = \begin{bmatrix} \cos(90-\phi) & \cos(180-\phi) & 0 \\ \cos\phi & \cos(90-\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

From trig. identities, we know that $\cos(90-\phi) = \sin\phi$ and $\cos(180-\phi) = -\cos\phi$. Making these substitutions, we have

$$\begin{bmatrix} g \\ h \\ m \end{bmatrix} = \underbrace{\begin{bmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

This is our 1^{st} rotation matrix. We need to rotate the c basis about the $\hat{c}_2$ axis in order to obtain the ENU basis, which is our ultimate goal.

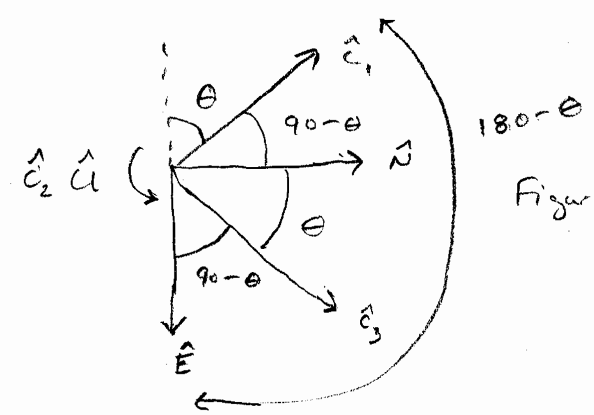


Figure 2

We wish to express $\bar{R}(t)|_c$, currently using components g, h, & m, in terms of components in the ENU basis:

$$\bar{R}(t)|_{ENU} = n\hat{E} + p\hat{N} + q\hat{U}$$

Using the same analysis used previously, we can compute the n, p, and q components in terms of the components of g, h, and m using 3 equations:

$$n = \bar{R}(t)\big|_{c} \cdot \hat{E} = g(\hat{c}_1 \cdot \hat{E}) + h(\hat{c}_2 \cdot \hat{E}) + m(\hat{c}_3 \cdot \hat{E})$$

$$p = \bar{R}(t)\big|_{c} \cdot \hat{N} = g(\hat{c}_1 \cdot \hat{N}) + h(\hat{c}_2 \cdot \hat{N}) + m(\hat{c}_3 \cdot \hat{N})$$

$$q = \bar{R}(t)\big|_{c} \cdot \hat{U} = g(\hat{c}_1 \cdot \hat{U}) + h(\hat{c}_2 \cdot \hat{U}) + m(\hat{c}_3 \cdot \hat{U})$$

In matrix format,

$$\begin{bmatrix} n \\ p \\ q \end{bmatrix} = \begin{bmatrix} (\hat{c}_1 \cdot \hat{E}) & (\hat{c}_2 \cdot \hat{E}) & (\hat{c}_3 \cdot \hat{E}) \\ (\hat{c}_1 \cdot \hat{N}) & (\hat{c}_2 \cdot \hat{N}) & (\hat{c}_3 \cdot \hat{N}) \\ (\hat{c}_1 \cdot \hat{U}) & (\hat{c}_2 \cdot \hat{U}) & (\hat{c}_3 \cdot \hat{U}) \end{bmatrix} \begin{bmatrix} g \\ h \\ m \end{bmatrix}$$

Looking a Figure 2, the dot products can be replaced:

$$\begin{bmatrix} n \\ p \\ q \end{bmatrix} = \begin{bmatrix} \cos(180-\theta) & 0 & \cos(90-\theta) \\ \cos(90-\theta) & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} g \\ h \\ m \end{bmatrix}$$

Using trig identities, the matrix can be written as:

$$\begin{bmatrix} n \\ p \\ q \end{bmatrix} = \underbrace{\begin{bmatrix} -\cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix}}_{} \begin{bmatrix} g \\ h \\ m \end{bmatrix}$$

This is our 2^{nd} rotation matrix. Remember that our ultimate goal was to transform the components in the b basis, or d, e, and f in this case, to the components in the ENU basis, or n, p, and q in this case. So, we eliminate the 3×1 matrix containing g, h, and m by substituting the matrix equation containing the 1^{st} rotation matrix into the matrix equation containing the 2^{nd} rotation matrix. The resulting matrix equation looks like this:

$$\begin{bmatrix} n \\ p \\ q \end{bmatrix} = \underbrace{\begin{bmatrix} -\cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix}}_{2^{nd} \text{ rotation matrix}} \underbrace{\begin{bmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{1^{st} \text{ rotation matrix}} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

If the two rotation matrices are multiplied together, the result is

$$\begin{bmatrix} n \\ p \\ q \end{bmatrix} = \underbrace{\begin{bmatrix} -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \\ \cos\phi & \sin\phi & 0 \end{bmatrix}}_{\text{Transformation matrix from b basis to ENU basis.}} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

So, we have achieved our goal. θ and ϕ are functions of t which are given in the notes. I have not yet explicitly stated what d, e, and f are. Remember that the b basis rotates such that the $\hat{b}_1$ unit vector maintains a perfect lock on the satellite as it moves around. Thus, the range of the satellite has only a $\hat{b}_1$ component; the $\hat{b}_2$ and $\hat{b}_3$ components are always 0 for the direction of the tracking. In mathematical terms,

$$\bar{R}(t)_{|b} = d\,\hat{b}_1 + \cancelto{0}{e\hat{b}_2} + \cancelto{0}{f\hat{b}_3} = d\,\hat{b}_1 \quad \text{where}$$

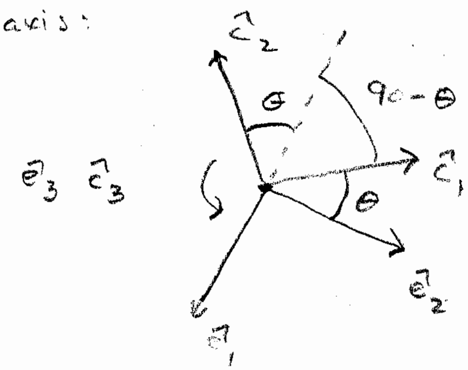
$d = 8000 + 10.0t$ km, as given in the problem.

Sample MATLAB code and the plot are included.

Problem #2

Part 1:

Define an intermediate basis c. Rotate the e basis about the $\vec{e}_3$ or $\hat{c}_3$ axis:

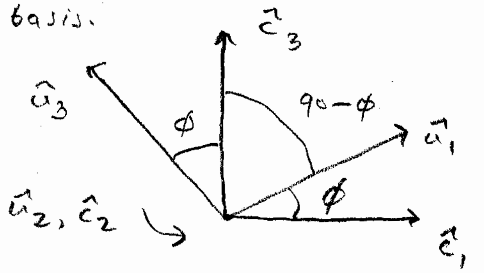


The first rotation matrix is

$$\begin{bmatrix} \hat{c}_1\cdot\hat{e}_1 & \hat{c}_1\cdot\hat{e}_2 & \hat{c}_1\cdot\hat{e}_3 \\ \hat{c}_2\cdot\hat{e}_1 & \hat{c}_2\cdot\hat{e}_2 & \hat{c}_2\cdot\hat{e}_3 \\ \hat{c}_3\cdot\hat{e}_1 & \hat{c}_3\cdot\hat{e}_2 & \hat{c}_3\cdot\hat{e}_3 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $\theta = 25^\circ$ rather than 115° because I have chosen to measure from the $\hat{e}_2$ axis rather than the $\vec{e}_1$ axis.

Now I will do a second rotation to transform from the c basis to the u basis.



$\phi = 40^\circ$

$$\begin{bmatrix} \hat{u}_1\cdot\hat{c}_1 & \hat{u}_1\cdot\hat{c}_2 & \hat{u}_1\cdot\hat{c}_3 \\ \hat{u}_2\cdot\hat{c}_1 & \hat{u}_2\cdot\hat{c}_2 & \hat{u}_2\cdot\hat{c}_3 \\ \hat{u}_3\cdot\hat{c}_1 & \hat{u}_3\cdot\hat{c}_2 & \hat{u}_3\cdot\hat{c}_3 \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

So, the final transformation matrix from the e basis to the u basis is:

$$\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

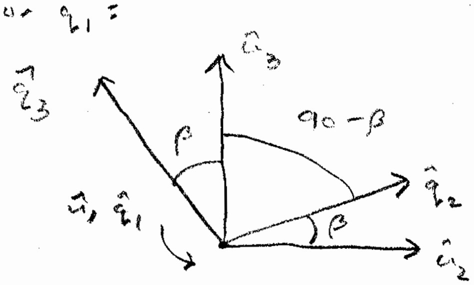
$$\begin{bmatrix} -\cos\phi\sin\theta & \cos\phi\cos\theta & \sin\phi \\ -\cos\theta & -\sin\theta & 0 \\ \sin\phi\sin\theta & -\sin\phi\cos\theta & \cos\phi \end{bmatrix}$$

Now substitute in $\theta = 25°$ and $\phi = 40°$

$$\boxed{\begin{bmatrix} -.32 & .69 & .64 \\ -.91 & -.42 & 0 \\ .27 & -.58 & .77 \end{bmatrix}}_{e \to u}$$

Part #2:

First find the matrix from u to q. This requires only one rotation about $\vec{u}_1$ or $\vec{q}_1$:



$\beta = 30°$

The rotation matrix is:

$$\begin{bmatrix} \hat{u}_1\cdot\hat{q}_1 & \hat{u}_2\cdot\hat{q}_1 & \hat{u}_3\cdot\hat{q}_1 \\ \hat{u}_1\cdot\hat{q}_2 & \hat{u}_2\cdot\hat{q}_2 & \hat{u}_3\cdot\hat{q}_2 \\ \hat{u}_1\cdot\hat{q}_3 & \hat{u}_2\cdot\hat{q}_3 & \hat{u}_3\cdot\hat{q}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix}$$

Now we can go from $e \to u \to q$. We already know the transformation matrix for $e \to u$ since it was calculated in part 1.

Now I simply need to multiply the matrix that I just computed by the one obtained into part 1 to obtain the transformation matrix from $e \to q$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix}}_{\beta=30^\circ} \begin{bmatrix} -.32 & .69 & .64 \\ -.91 & -.42 & 0 \\ .27 & -.58 & .77 \end{bmatrix} = \underbrace{\begin{bmatrix} -.32 & .69 & .64 \\ -.65 & -.65 & .38 \\ .69 & -.29 & .67 \end{bmatrix}}_{e \to q}$$

Part #3:

a) Since we are transforming from the e basis to the q basis, use the matrix computed in part 2.

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} -.32 & .69 & .64 \\ -.65 & -.65 & .38 \\ .69 & -.29 & .67 \end{bmatrix} \begin{bmatrix} 0.5 \\ -1.3 \\ -0.5 \end{bmatrix}_e = \begin{bmatrix} -1.38 \\ .33 \\ .39 \end{bmatrix}$$

Therefore, $\bar{R} = -1.38 q_1 + .33 q_2 + .39 q_3$ cm

b) Here we are transforming from $u \to q$, so use the second rotation matrix computed in part 2:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .87 & .50 \\ 0 & -.50 & .87 \end{bmatrix} \begin{bmatrix} 13.1 \\ 5.5 \\ -2.4 \end{bmatrix} = \begin{bmatrix} 13.1 \\ 3.56 \\ -4.83 \end{bmatrix}$$

Therefore, $\bar{\omega} = 13.1 q_1 + 3.56 q_2 - 4.83 q_3$ sec^{-1}

Part #4:

a) The goal is to go from $u \to e$, so we use the <u>transpose</u> of the matrix that was computed in part 1:

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -.32 & -.91 & -.27 \\ .69 & -.42 & -.58 \\ .64 & 0 & .77 \end{bmatrix} \begin{bmatrix} 2.5 \times 10^3 \\ 10.4 \times 10^3 \\ -1.1 \times 10^3 \end{bmatrix} = \begin{bmatrix} -1.05 \times 10^4 \\ -.20 \times 10^4 \\ .08 \times 10^4 \end{bmatrix}$$

Therefore, $\bar{R} = (-1.05 \times 10^4)\hat{e}_1 - (2.0 \times 10^4)\hat{e}_2 + (.08 \times 10^4)\hat{e}_3$ km

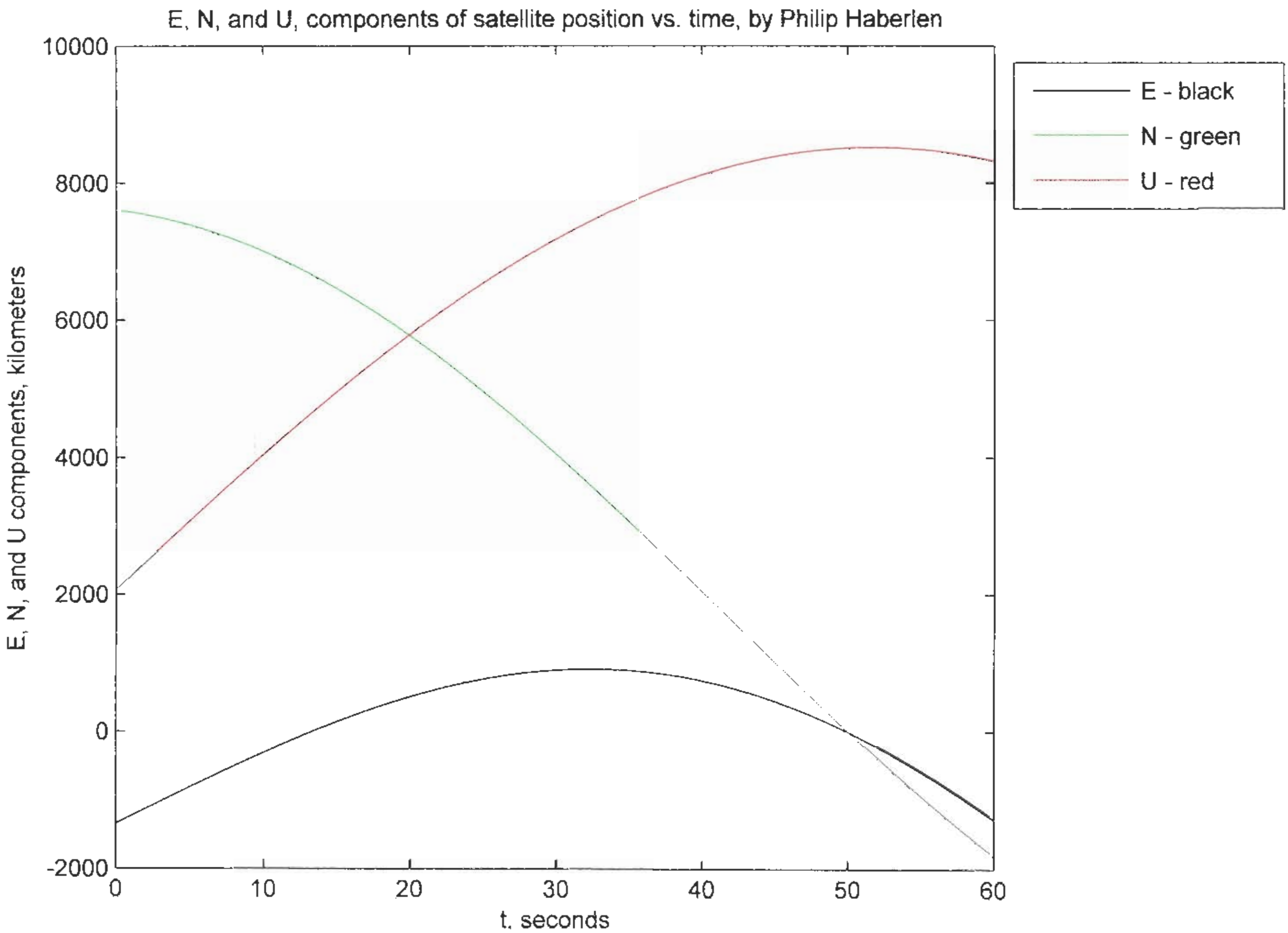
b)

We are going from $q \rightarrow e$, so I will use the transpose of the matrix that was computed in problem #2:

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -.32 & -.65 & .69 \\ .69 & -.65 & -.29 \\ .64 & .38 & .67 \end{bmatrix} \begin{bmatrix} 3.1 \\ -9.9 \\ 3.1 \end{bmatrix} = \begin{bmatrix} 7.61 \\ 7.70 \\ .24 \end{bmatrix}$$

Therefore $\bar{\omega} = 7.61\hat{e}_1 + 7.71\hat{e}_2 + .24\hat{e}_3$ sec^{-1}

```
clear
format long
step = .01;
t = 0: step: 60;
num_t = length(t);
deg_rad = pi./180;
a = 1;
for a = 1:num_t,
    phi(1, a) = (75 - 1.5.*t(1, a)).*deg_rad;
    a = a + 1;
end
b = 1;
for b = 1:num_t,
    theta(1, b) = (80 + 0.75.*t(1, b)).*deg_rad;
    b = b + 1;
end
c = 1;
for c = 1:num_t,
    range(c, 1) = 8000 + 10.*t(1, c);
    c = c + 1;
end
d = 1;
for d = 1:num_t,
    trans_b_enu = [-cos(theta(1, d)).*sin(phi(1, d)), cos(theta(1, d)).*cos(phi(1, d)),
sin(theta(1, d));
        sin(theta(1, d)).*sin(phi(1, d)), -sin(theta(1, d)).*cos(phi(1, d)), cos(theta(1,
d));
        cos(phi(1, d)), sin(phi(1, d)), 0];
    e_n_u = trans_b_enu*[range(d, 1); 0; 0];
    enu_E(1, d) = e_n_u(1, 1);
    enu_N(1, d) = e_n_u(2, 1);
    enu_U(1, d) = e_n_u(3, 1);
    d = d + 1;
end
plot(t, enu_E, 'k', t, enu_N, 'g', t, enu_U, 'r')
title('E, N, and U, components of satellite position vs. time, by Philip Haberlen')
xlabel('t, seconds')
ylabel('E, N, and U components, kilometers')
legend('E - black', 'N - green', 'U - red', 'Location', 'NorthEastOutside')
```



E, N, and U, components of satellite position vs. time, by Philip Haberlen
10000
8000
6000
4000
2000
0
-2000
0
10
20
30
40
50
60
t. seconds
E, N, and U components, kilometers
E - black
N - green
U - red