HW #8 – Solution

Problem 1

Refer to HW #7 solutions for the figures. The same unit vectors ($\hat{e}$'s and $\hat{b}$'s) and the same observers (I, S, M) will be used!
Basepoints O, C, A as defined previously!

Kinematics method II

$\vec{R}^{CA} = L\,\hat{b}_1 + y\,\hat{b}_2 + z\,\hat{b}_3$ is given. $(y = y(t)\ ,\ z = z(t))$

$$\vec{R}^{OA} = \vec{R}^{OC} + \vec{R}^{CA} = R_o\,\hat{b}_2 + L\,\hat{b}_1 + y\,\hat{b}_2 + z\,\hat{b}_3$$

$$\vec{R}^{OA} = L\,\hat{b}_1 + (R_o + y)\,\hat{b}_2 + z\,\hat{b}_3$$

Differentiate to find ${}^{I}\vec{V}^{OA}$ (note: we need to differentiate the unit vectors since they are <u>not</u> inertially fixed)

$$\frac{d\vec{R}^{OA}}{dt} = L\,\frac{{}^{I}d\hat{b}_1}{dt} + \dot{y}\,\hat{b}_2 + (R_o + y)\,\frac{{}^{I}d\hat{b}_2}{dt} + \dot{z}\,\hat{b}_3 + z\,\frac{{}^{I}d\hat{b}_3}{dt}$$

$$\frac{{}^{I}d\hat{b}_1}{dt} = {}^{I}\vec{\omega}^{S} \times \hat{b}_1 \qquad {}^{I}\vec{\omega}^{S} = -\dot{\theta}\,\hat{b}_3$$

$$= -\dot{\theta}\,\hat{b}_3 \times \hat{b}_1 = -\dot{\theta}\,\hat{b}_2$$

$$\frac{{}^{I}d\hat{b}_2}{dt} = -\dot{\theta}\,\hat{b}_3 \times \hat{b}_2 = \dot{\theta}\,\hat{b}_1$$

$$\frac{{}^{I}d\hat{b}_3}{dt} = -\dot{\theta}\,\hat{b}_3 \times \hat{b}_3 = 0$$

Plug this back into ${}^{I}\vec{V}^{OA}$. $\left(= \frac{{}^{I}d\vec{R}^{OA}}{dt}\right)$

$${}^{I}\vec{V}^{OA} = -L\,\dot{\theta}\,\hat{b}_2 + \dot{y}\,\hat{b}_2 + (R_o + y)\,\dot{\theta}\,\hat{b}_1 + \dot{z}\,\hat{b}_3$$

$$\dot{\theta} = \text{const} = \Omega$$

$${}^I\vec{v}^{OA} = (\dot{y} - L\Omega)\,\hat{b}_2 + (R_0 + y)\Omega\,\hat{b}_1 + \dot{z}\,\hat{b}_3$$

Differentiate again to get ${}^I\vec{a}^{OA} = \dfrac{{}^I d\,{}^I\vec{v}^{OA}}{dt}$

$$
\begin{aligned}
{}^I\vec{a}^{OA} &= \ddot{y}\,\hat{b}_2 + (\dot{y} - L\Omega)\frac{{}^I d\hat{b}_2}{dt} + \dot{y}\Omega\,\hat{b}_1 + (R_0 + y)\Omega\frac{{}^I d\hat{b}_1}{dt} + \ddot{z}\,\hat{b}_3 + \dot{z}\frac{{}^I d\hat{b}_3}{dt} \\
&= \ddot{y}\,\hat{b}_2 + (\dot{y} - L\Omega)\dot{\theta}\,\hat{b}_1 + \dot{y}\Omega\,\hat{b}_1 - (R_0 + y)\Omega\dot{\theta}\,\hat{b}_2 + \ddot{z}\,\hat{b}_3 \\
&= \left[(\dot{y} - L\Omega)\Omega + \dot{y}\Omega\right]\hat{b}_1 + \left[\ddot{y} - (R_0 + y)\Omega^2\right]\hat{b}_2 + \ddot{z}\,\hat{b}_3
\end{aligned}
$$

$$\boxed{{}^I\vec{a}^{OA} = \left[2\Omega\dot{y} - L\Omega^2\right]\hat{b}_1 + \left[\ddot{y} - (R_0 + y)\Omega^2\right]\hat{b}_2 + \ddot{z}\,\hat{b}_3}$$

Problem 2

Refer to HW #7 solution for figures and definitions!

Method II

$$\vec{R}^{OP} = x\,\hat{e}_3 - y\,\hat{e}_2$$

$${}^I\vec{V}^{OP} = \frac{{}^Id\vec{R}^{OP}}{dt} = \dot{x}\,\hat{e}_3 + x\frac{{}^Id\hat{e}_3}{dt} - y\frac{{}^Id\hat{e}_2}{dt}$$

$$\frac{{}^Id\hat{e}_3}{dt} = {}^I\vec{\omega}^B \times \hat{e}_3 \qquad {}^I\vec{\omega}^B = -\dot{\theta}\,\hat{e}_3 \qquad (\dot{\theta} \neq \text{const!})$$

$$= -\dot{\theta}\,\hat{e}_3 \times \hat{e}_3 = 0$$

$$\frac{{}^Id\hat{e}_2}{dt} = -\dot{\theta}\,\hat{e}_3 \times \hat{e}_2 = \dot{\theta}\,\hat{e}_1$$

$${}^I\vec{V}^{OP} = \dot{x}\,\hat{e}_3 - y\dot{\theta}\,\hat{e}_1$$

$${}^I\vec{a}^{OP} = \frac{{}^Id\,{}^I\vec{V}^{OP}}{dt} = \ddot{x}\,\hat{e}_3 + \dot{x}\frac{{}^Id\hat{e}_3}{dt} - y\ddot{\theta}\,\hat{e}_1 - y\dot{\theta}\frac{d\hat{e}_1}{dt}$$

$$\frac{{}^Id\hat{e}_1}{dt} = -\dot{\theta}\,\hat{e}_3 \times \hat{e}_1 = -\dot{\theta}\,\hat{e}_2$$

$$\boxed{{}^I\vec{a}^{OP} = \ddot{x}\,\hat{e}_3 - y\ddot{\theta}\,\hat{e}_1 + y\dot{\theta}^2\,\hat{e}_2}$$

$$\ddot{x} = -3{,}35\ \frac{cm}{sec^2}$$

$$\dot{\theta} = 100\ \frac{rev}{min} = 10{,}47\ \frac{rad}{sec}$$

$$\ddot{\theta} = -3000\ \frac{rev}{min} = -5{,}24\ \frac{rad}{s^2}$$

$${}^I\vec{a}^{OP} = -3{,}35\ \frac{cm}{sec^2}\,\hat{e}_3 - 0{,}75\cdot(-5{,}24)\ \frac{cm}{sec^2}\,\hat{e}_1 + 0{,}75\,(10{,}47)^2\ \frac{cm}{sec^2}\,\hat{e}_2$$

$$\boxed{{}^I\vec{a}^{OP} = 3{,}93\ \hat{e}_1 + 82{,}2\ \hat{e}_2 - 3{,}35\ \hat{e}_3 \quad \left(\frac{cm}{sec^2}\right)}$$

Problem 3

Refer to HW #7 solution for figures!

$$\vec{R}^{OP} = (L+x)\,\hat{u}_2$$

$$^I\vec{V}^{OP} = \frac{^I d\vec{R}^{OP}}{dt} = \dot{x}\,\hat{u}_2 + (L+x)\frac{^I d\hat{u}_2}{dt}$$

$$\frac{^I d\hat{u}_2}{dt} = {^I\vec{\omega}^S} \times \hat{u}_2 \qquad {^I\vec{\omega}^S} = -\omega_1\hat{u}_1 - \omega_2\hat{e}_3$$

$$\hat{e}_3 = -s\theta_1\,\hat{u}_2 + c\theta_1\,\hat{u}_3 \qquad \text{(see transformation matrix HW \#7)}$$

$$^I\vec{\omega}^S = -\omega_1\hat{u}_1 + \omega_2 s\theta_1\hat{u}_2 - \omega_2 c\theta_1\hat{u}_3$$

$$\frac{^I d\hat{u}_2}{dt} = \left[-\omega_1\hat{u}_1 + \omega_2 s\theta_1\hat{u}_2 - \omega_2 c\theta_1\hat{u}_3\right]\times\hat{u}_2 =$$

$$= -\omega_1\hat{u}_3 + \omega_2 c\theta_1\hat{u}_1$$

$$^I\vec{V}^{OP} = \dot{x}\,\hat{u}_2 + (L+x)\left(\omega_2 c\theta_1\hat{u}_1 - \omega_1\hat{u}_3\right)$$

$$^I\vec{a}^{OP} = \frac{^I d\,{^I\vec{V}^{OP}}}{dt} = \dot{x}\frac{^I d\hat{u}_2}{dt} + \dot{x}\left[\omega_2 c\theta_1\hat{u}_1 - \omega_1\hat{u}_3\right] + (L+x)\Big[-\omega_2\omega_1 s\theta_1\hat{u}_1$$
$$+ \omega_2 c\theta_1\frac{^I d\hat{u}_1}{dt} - \omega_1\frac{^I d\hat{u}_3}{dt}\Big]$$

$$\frac{^I d\hat{u}_1}{dt} = -\omega_2 s\theta_1\hat{u}_3 - \omega_2 c\theta_1\hat{u}_2$$

$$\frac{^I d\hat{u}_3}{dt} = \omega_1\hat{u}_2 + \omega_2 s\theta_1\hat{u}_1$$

$$^I\vec{a}^{OP} = \dot{x}\left[-\omega_1\hat{u}_3 + \omega_2 c\theta_1\hat{u}_1\right] + \dot{x}\left[\omega_2 c\theta_1\hat{u}_1 - \omega_1\hat{u}_3\right] +$$
$$+ (L+x)\left[-\omega_2\dot{\theta}_1 s\theta_1\hat{u}_1 - \omega_2^2 c\theta_1\left(s\theta_1\hat{u}_3 + c\theta_1\hat{u}_2\right) - \omega_1\left(\omega_1\hat{u}_2 + \omega_2 s\theta_1\hat{u}_1\right)\right]$$

$\dot{x} = v$

$\dot{\theta}_1 = \omega_1$

Plug in and simplify:

$$\boxed{\begin{aligned} {}^{I}\vec{a}^{\,OP} = {} & \left[2 v \omega_2 c\theta_1 - 2(L+x)\omega_1 \omega_2 s\theta_1\right]\hat{u}_1 - \left[(L+x)\left(\omega_1^2 + \omega_2^2 c\theta_1^2\right)\right]\hat{u}_2 \\ & - \left[2 v \omega_1 + (L+x)\omega_2^2 s\theta_1 c\theta_1\right]\hat{u}_3 \end{aligned}}$$

Problem 4

Figure in HW # 3 Solution.

Basepoint O on the gimbal (inertially fixed), basepoint S is on the satellite.
Observer A on the antenna.

$$\vec{R}^{os} = R\,\hat{b}_1$$

$$^I\vec{V}^{os} = \frac{^I d\vec{R}^{os}}{dt} = \dot{R}\,\hat{b}_1 + R\,\frac{^I d\hat{b}_1}{dt}$$

$$\frac{^I d\hat{b}_1}{dt} = {^I\vec{\omega}^A} \times \hat{b}_1 \qquad {^I\vec{\omega}^A} = -\dot{\phi}\,\hat{b}_3 - \dot{\theta}\,\hat{u} = -\Gamma\,\hat{b}_3 - \Omega\,\hat{u}$$

$$\hat{u} = c\phi\,\hat{b}_1 + s\phi\,\hat{b}_2$$

$$^I\vec{\omega}^A = -\Omega\,c\phi\,\hat{b}_1 - \Omega\,s\phi\,\hat{b}_2 - \Gamma\,\hat{b}_3$$

$$\frac{^I d\hat{b}_1}{dt} = \left[-\Omega\,c\phi\,\hat{b}_1 - \Omega\,s\phi\,\hat{b}_2 - \Gamma\,\hat{b}_3\right] \times \hat{b}_1 = \Omega\,s\phi\,\hat{b}_3 - \Gamma\,\hat{b}_2$$

$$^I\vec{V}^{os} = \dot{R}\,\hat{b}_1 + R\left[\Omega\,s\phi\,\hat{b}_3 - \Gamma\,\hat{b}_2\right]$$

$$^I\vec{a}^{os} = \frac{^I d\,^I\vec{V}^{os}}{dt} = \ddot{R}\,\hat{b}_1 + \dot{R}\,\frac{^I d\hat{b}_1}{dt} + \Omega\left(\dot{R}\,s\phi + R\Gamma\,c\phi\right)\hat{b}_3 + R\Omega\,s\phi\,\frac{^I d\hat{b}_3}{dt} - \dot{R}\Gamma\,\hat{b}_2 - R\Gamma\,\frac{^I d\hat{b}_2}{dt}$$

$$\frac{^I d\hat{b}_2}{dt} = -\Omega\,c\phi\,\hat{b}_3 + \Gamma\,\hat{b}_1$$

$$\frac{^I d\hat{b}_3}{dt} = \Omega\,c\phi\,\hat{b}_2 - \Omega\,s\phi\,\hat{b}_1$$

$$^I\vec{a}^{os} = \ddot{R}\,\hat{b}_1 + \dot{R}\left[\Omega\,s\phi\,\hat{b}_3 - \Gamma\,\hat{b}_2\right] + \Omega\dot{R}\,s\phi\,\hat{b}_3 + \Omega R\Gamma\,c\phi\,\hat{b}_3 + R\Omega\,s\phi\left[\Omega\,c\phi\,\hat{b}_2 - \Omega\,s\phi\,\hat{b}_1\right] - \dot{R}\Gamma\,\hat{b}_2 - R\Gamma\left[-\Omega\,c\phi\,\hat{b}_3 + \Gamma\,\hat{b}_1\right]$$

Rearrange and combine some terms :

$$
{}^{I}\vec{a}^{\,OS} = \left[\ddot{R} - R\Omega^2 s^2\phi - R\Gamma^2 \right] \hat{b}_1 + \left[R\Omega^2 s\phi\, c\phi - 2\dot{R}\Gamma \right] \hat{b}_2 +
$$

$$
+ \left[2\dot{R}\Omega s\phi + 2R\Omega\Gamma c\phi \right] \hat{b}_3
$$