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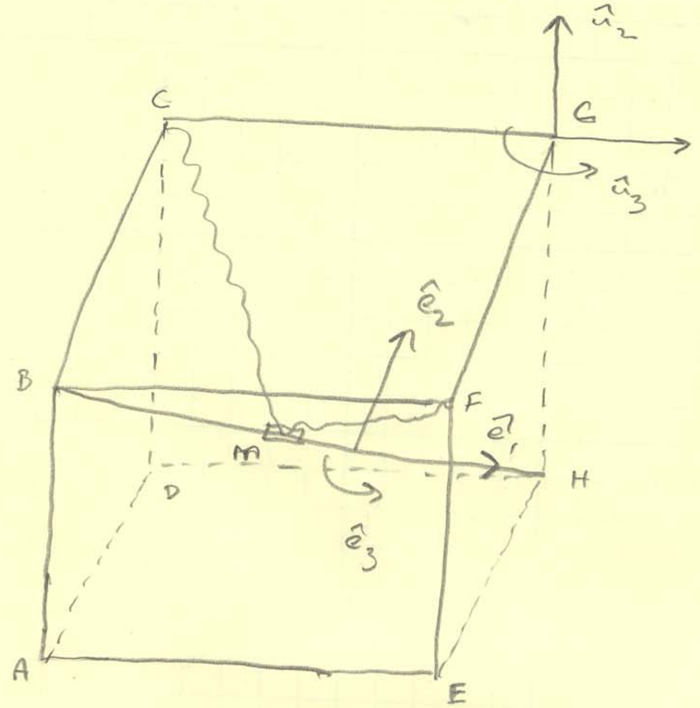
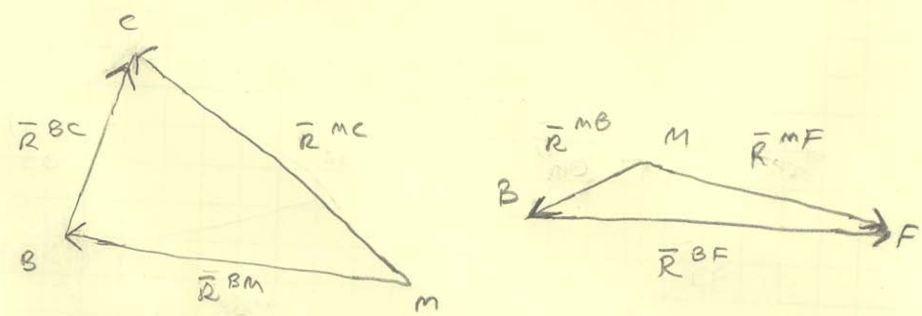
Homework #6 Solution

10/20/04

Define two bases as shown below.



We need expressions for the two unit vectors that point along the springs. Consider triangles BMC and BMF



From vector addition, we know that

$$\bar{R}^{MC} = \bar{R}^{MB} + \bar{R}^{BC} \quad \text{and}$$

$$\bar{R}^{MF} = \bar{R}^{MB} + \bar{R}^{BF}$$

Substituting in appropriate numbers, we have

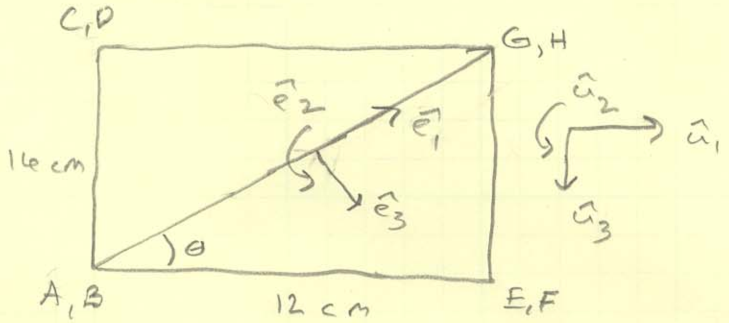
$$\bar{R}^{MC} = -x\,\hat{e}_1 - 16\,\hat{u}_3 \ \text{cm}$$

$$\bar{R}^{MF} = -x\,\hat{e}_1 + 12\,\hat{u}_1 \cdot \text{cm}$$

Convert the $\hat{u}$ basis into the $\hat{e}$ basis. We can define each unit vector in $\hat{e}$ as follows

$$\hat{e}_1 = \frac{\bar{R}^{BH}}{|\bar{R}^{BH}|} = \frac{12\hat{u}_1 - 12\hat{u}_2 - 16\hat{u}_3}{\sqrt{12^2 + (-12)^2 + (-16)^2}} = .514\hat{u}_1 - .514\hat{u}_2 - .686\hat{u}_3$$

$\hat{e}_3$ is $\perp$ to the plane ABGH. If we look "down" on the figure, we see



$$\hat{e}_3 = \sin\theta\,\hat{u}_1 + \cos\theta\,\hat{u}_3$$

$$\sin\theta = \frac{16}{\sqrt{12^2 + 16^2}} = .8$$

$$\cos\theta = \frac{12}{\sqrt{12^2 + 16^2}} = .6$$

$$\hat{e}_3 = .8\hat{u}_1 + .6\hat{u}_3$$

$\hat{e}_2$ completes a right-handed set, so we have

$$\hat{e}_2 = \hat{e}_3 \times \hat{e}_1 = (.8\hat{u}_1 + .6\hat{u}_3) \times (.514\hat{u}_1 - .514\hat{u}_2 - .686\hat{u}_3)$$

$$= \cancel{(.8\hat{u}_1 \times .514\hat{u}_1)}^{0} + (.8\hat{u}_1 \times -.514\hat{u}_2) + (.8\hat{u}_1 \times -.686\hat{u}_3)$$

$$+ (.6\hat{u}_3 \times .514\hat{u}_1) + (.6\hat{u}_3 \times -.514\hat{u}_2) + \cancel{(.6\hat{u}_3 \times -.686\hat{u}_3)}^{0}$$

$$= -.4112\hat{u}_3 + .5488\hat{u}_2 + .3084\hat{u}_2 + .3084\hat{u}_1$$

$$= .3084\hat{u}_1 + .8572\hat{u}_2 - .4112\hat{u}_3$$

I can convert the $\hat{u}$ components to the $\hat{e}$ components now:

$$\bar{R}^{MC} = -x\hat{e}_1 - 16(-.686)\hat{e}_1 - 16(-.4112)\hat{e}_2 - 16(.6)\hat{e}_3$$
$$= (10.976 - x)\hat{e}_1 + 6.5792\hat{e}_2 - 9.6\hat{e}_3$$
$$\bar{R}^{MF} = -x\hat{e}_1 + 12(.514)\hat{e}_1 + 12(.3084)\hat{e}_2 + 12(.8)\hat{e}_3$$
$$= (6.168 - x)\hat{e}_1 + 3.7008\hat{e}_2 + 9.6\hat{e}_3$$

These spring vectors can be converted into unit vectors:

$$\hat{e}^{MC} = \frac{(10.976 - x)\hat{e}_1 + 6.5792\hat{e}_2 - 9.6\hat{e}_3}{\sqrt{(10.976-x)^2 + (6.5792)^2 + (-9.6)^2}}$$

$$\hat{e}^{MF} = \frac{(6.168 - x)\hat{e}_1 + 3.7008\hat{e}_2 + 9.6\hat{e}_3}{\sqrt{(6.168-x)^2 + (3.7008)^2 + (-9.6)^2}}$$

The unit vector for the weight force is

$\hat{e}_w = -\hat{u}_2$ OR after converting to the $\hat{e}$ basis

$= .514\hat{e}_1 - .8572\hat{e}_2$

The normal force that the rod exerts on the mass will have two orthogonal components. Note that the rod does not exert a force parallel to it since it is frictionless.

$\hat{e}_{N1} = \hat{e}_2$

$\hat{e}_{N2} = \hat{e}_3$

We need expressions for the spring forces. Since both springs are linear, they obey Hooke's Law:

$$\bar{F}_{spring} = K(y - l)\hat{u}_{spring}$$

where "l" is the unloaded length of the spring, "y" is the loaded length of the spring, and k is the spring constant, $\hat{u}_{spring}$ is the unit vector pointed in the direction of spring extension, and $\bar{F}_{spring}$ is the force that the spring exerts. The usual negative sign has been omitted since we are assuming positive displacements to be in the direction of spring extension. The spring constant k is given for both springs, as is the equilibrium length "l." $\hat{u}_{spring}$ for each spring was computed on page 3. The loaded length, "y," is just the magnitude of the position vectors along the springs, $\|\bar{R}^{MC}\|$ and $\|\bar{R}^{MF}\|$. These magnitudes appear in the denominators of the unit vectors $\hat{e}^{MC}$ and $\hat{e}^{MF}$, computed on page 3. We can substitute those expressions into Hooke's Law for both springs and also write down the vector force equations for the other forces.

$$\bar{F}_{MC} = k\left[\sqrt{(10.976-x)^2+(6.5792)^2+(-9.6)^2}-l\right]\hat{e}_{MC}$$

$$\bar{F}_{MF} = k\left[\sqrt{(6.168-x)^2+(3.7008)^2+(-9.6)^2}-l\right]\hat{e}_{MF}$$

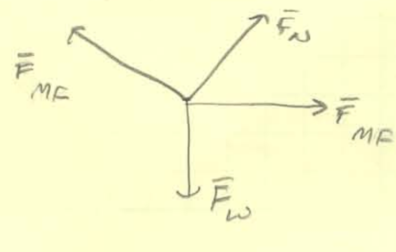
where $\hat{e}_{MC}$ and $\hat{e}_{MF}$ are given on page 3.

$$\bar{F}_N = F_{N1}\,\hat{e}_2 + F_{N2}\,\hat{e}_3$$

$$\bar{F}_w = (.5\text{ kg})\left(9.806\ \tfrac{m}{s^2}\right)\hat{e}_w$$

where $\hat{e}_w$ is given on page 3.

A free-body diagram for the block appears below



From static equilibrium, we have

$$\Sigma F = \bar{F}_{MC} + \bar{F}_{MF} + \bar{F}_N + \bar{F}_W = 0$$

This is a vector equation, so it can be split up into 3 scalar equations. I am substituting in $k = 1.22 \frac{N}{cm}$, $l = 6\,cm$, and the expressions for $\hat{e}_{MC}$, $\hat{e}_{MF}$, and $\hat{e}_W$ as well.

$$(1.22)\left[\sqrt{(10.976-x)^2+(6.5792)^2+(-9.6)^2}-6\right]\left[\frac{10.976-x}{\sqrt{(10.976-x)^2+(6.5792)^2+(-9.6)^2}}\right]$$

$$+(1.22)\left[\sqrt{(6.168-x)^2+(3.7008)^2+(-9.6)^2}-6\right]\left[\frac{6.168-x}{\sqrt{(6.168-x)^2+(3.7008)^2+(-9.6)^2}}\right]+\overbrace{2.5201}^{mg(.514)}=0$$

$$(1.22)\left[\sqrt{(10.976-x)^2+(6.5792)^2+(-9.6)^2}-6\right]\left[\frac{6.5792}{\sqrt{(10.976-x)^2+(6.5792)^2+(-9.6)^2}}\right]$$

$$+(1.22)\left[\sqrt{(6.168-x)^2+(3.7008)^2+(-9.6)^2}\right]\left[\frac{3.7008}{\sqrt{(6.168-x)^2+(3.7008)^2+(-9.6)^2}}\right]+F_{N1}-\overbrace{4.203}^{mg(-.8572)}=0$$

$$
\begin{aligned}
&(1.22)\left[\sqrt{(10.976-x)^2+(6.5792)^2+(-9.6)^2}-6\right]\left[\frac{-9.6}{\sqrt{(10.976-x)^2+(6.5792)^2+(-9.6)^2}}\right] \\
&+(1.22)\left[\sqrt{(6.168-x)^2+(3.7008)^2+(-9.6)^2}-6\right]\left[\frac{9.6}{\sqrt{(6.168-x)^2+(3.7008)^2+(-9.6)^2}}\right]+F_{N2}=0
\end{aligned}
$$

You will notice that the first equation contains only one variable, x, the variable that we are solving for. Because we transformed all of the vectors to a basis that was "lined up" with the rod, the two normal force components appeared in only two equations. If we had chosen a different basis, then at least one of the normal force components would have appeared in all 3 equations, complicating the solution process.

The first equation was solved numerically in MATLAB. See the attached M files for the details. If you run HW6, you will get $\boxed{x = 10.78\text{ cm}}$ for an answer.

We were not asked to solve for the normal force components, but you could certainly do so using the second and third equations.

```
function F = spr3(x)

F = 1.22.*(sqrt((10.976 - x).^2 + (6.5856).^2 + (-9.6).^2) - 6).*(10.976 - x)./(sqrt((10.
976 - x).^2 + (6.5856).^2 + (-9.6).^2)) + 1.22.*(sqrt((6.168 - x).^2 + (3.7044).^2 + (-9.
6).^2) - 6).*(6.168 - x)./(sqrt((6.168 - x).^2 + (3.7044).^2 + (-9.6).^2)) + 2.5201;
return
```

```
clear
format long

x = fsolve(@spr3, 10)
```