Influence of the dispersion on the Boltzmann equation for nonhomogeneous case

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Abstract. Influence of the variation the angular moment in an elementary volume is studied. The theory of N.N. Bogolubov has difficulties to write the Boltzmann equation for nonhomogeneous gas. Influence of great gradients density, velocity or temperature on integral of collisions is investigated. The modified conservation laws are suggested.

Keywords. Angular moment, Boltzmann equation, Navie-Stokes equations

INTRODUCTION

The classical Boltzmann equation is written for the one-particle distribution function without influence of the angular moment variation in an elementary volume [1-6]. It is known that for the relaxation problem the Boltzmann equation can be obtained from the Hamilton formalism for the system of the material points. The Navie-Stokes equations can be deduced from the Boltzmann equation. Their justification (Navie-Stokes and Boltzmann) is based on some general conditions. The Boltzmann method for obtaining the one-particle distribution function has the interior contradictions: the mechanism of the binary collision on one hand and effective cross-sections calculated from classic equations of mechanics on the other hand . This method is used also for the quantum statistic with additional supplement about symmetry of some values. However, the Boltzmann method neglects all correlations between dynamic states of molecules and all interactions of three or more particles. So N.N. Bogolubov make an attempt to overcome these difficulties. The principal assumptions in the Bogolubov theory are: infinite volume is considered; potential forces of the interaction of two particles rapidly abate with increasing of distance between two particles; the appearance of a finite volume's effect is investigated when the influence of the boundary is unessential, gas is rarefied but has a finite average density. The theory was developed originally in case of the gas which does not interact with the surface but it is not valid for general case. Additional the assumption is on molecular chaos which is permitted to pass to average description but the description is based on the same ideas near and far from the surface. So there are existence of the four time scales (space scale), by individual class of solution for the s-distribution function, binary interactions, the potential which abates rapidly, and we neglect the influence of the boundary. The Boltzmann equation is for the case when the middle distance between particles is much greater than the mean free path. Additional implicit condition is that the deviation from equilibrium is small in nonhomogeneous case. The integral of collisions does not have the Boltzmann form for potential (i.e. $\Phi \neq \Phi\left|r-r_0\right|$) and depends on the angle. Some consequences from this fact are considered. The method of obtaining the Navie-Stokes equation agrees with assumption for the Boltzmann equation. The principle difference between suggested and the classical Boltzmann equation consists in consideration of the angular moment variation in an elementary volume, as usually disregarded. Often the conservation laws are obtained as balance relations for an elementary volume is located in infinity. A law of angular momentum is postulated in spite of the fact that in general case the movement of points is noninertial. The main attention in our report is given to analysis of deduction of the Navie-Stokes equations and the Boltzmann equation since they are the most general laws conservation for mass, linear momentum, energy and these equations have the common region of definition if the Bogolubov conditions are fulfilled. We deduced the modified Bolzmann and the Navie-Stokes equations taking into account the angular moment law for the gas without structure in infinite region.

SOME SINGULARITIES OF THE KINEMATIC EQUATIONS

We now construct over equation for N-particle distribution function near the surface taking into account the great gradient of the physical values by using the ordinary notations, i.e.[7-9] $\vec{\mathbf{r}}$ – radius-vector; $\vec{\mathbf{x}}$ – coordinate of the point and according to definition function f_N in element of physical volume $d\vec{\mathbf{x}}$ near the point $\vec{\mathbf{x}}$ in moment t probable number of molecules with velocity in element $d\vec{\xi}$ near the $\vec{\xi}$ are equal to $f_N\left(t,\vec{\mathbf{x}},\vec{\xi}\right)d\vec{\xi}d\vec{\mathbf{x}}$. We suggest the next process to construct convective operator for N-particle distribution function. Let us calculate the change of the angular moment near the point $\vec{\mathbf{x}}$ $\delta L = m(\mathbf{r}+\delta\mathbf{r})\times f_N(t,\mathbf{r}+\delta\mathbf{r},\xi)\cdot\xi - m\mathbf{r}\times f_N(t,\mathbf{r},\xi)\cdot\xi$, $f_N(t,\mathbf{r}+\delta\mathbf{r},\xi) = f_N(t,\mathbf{r},\xi)+\delta\mathbf{r}\cdot\dfrac{\partial f_N}{\partial\mathbf{r}}$.

It is necessary be do because flow tube of an elementary volume must be of older of free path and must contain many molecules, i.e. our consideration the distribution function can change. Finally, we obtain the variation of angular moment in any volume and this results in $\delta\mathbf{L} = m(\delta\mathbf{r}\times f_N + \mathbf{r}\times\delta f_N)\cdot\xi$.

Hence it follows that

$$\delta\mathbf{L} = m\delta\mathbf{r}\times\left(f_N + \mathbf{r}\cdot\frac{\delta f_N}{\delta\mathbf{r}}\right)\cdot\xi .$$

The convective operator is modified as in [10], i.e.

$$\frac{df_N}{dt} = \frac{\partial f_N}{\partial t} + \xi_i\cdot\left[\frac{\partial f_N}{\partial x_i}\right] + \xi_i\cdot\frac{\partial}{\partial x_i}\left[x_j\frac{\partial f_N}{\partial x_j}\right] - \frac{X_i}{m}\frac{\partial f_N}{\partial \xi_i}.$$

There are variations of angular moment connected both with variation of the distribution function inside the volume and increase of radius vector of moment after passage in another point. It is essential that the fourth term in formula (1) [2] does not vanish when $|r_j|\to\infty$ for nonhomogeneous gas and for gas is not in equilibrium.

$$\begin{aligned}
\frac{1}{V^s}\frac{\partial F_s}{\partial t} &= -\frac{1}{V^s}\sum_{i=1}^{s}\frac{p_i}{m}\frac{\partial F_s}{\partial r_i} + \frac{1}{2V^s}\sum_{\substack{i,j=1\\(j\neq i)}}^{s}\frac{\partial\Phi(|r_i-r_j|)}{\partial r_i}\frac{\partial F_s}{\partial p_i} + \\
&+\frac{1}{V^{s+1}}\sum_{i=1}^{s}\int\sum_{j=s+1}^{N}\frac{\partial\Phi(|r_i-r_j|)}{\partial r_i}\frac{\partial F_{s+1}(t,r_1,\dots,r_s,r_j,p_1,\dots,p_s,p_j)}{\partial p_i}dr_j dp_j - \\
&-\frac{1}{V^{s+1}}\sum_{j=s+1}^{N}\int\frac{p_j}{m}\frac{\partial F_{s+1}(t,r_1,\dots,r_s,r_j,p_1,\dots,p_s,p_j)}{\partial r_j}dr_j dp_j + \\
&+\frac{1}{V^{s+1}}\sum_{i=1}^{s}\int\sum_{j=s+1}^{N}\frac{\partial\Phi(|r_i-r_j|)}{\partial r_j}\frac{\partial F_{s+1}(t,r_1,\dots,r_s,r_j,p_1,\dots,p_s,p_j)}{\partial p_j}dr_j dp_j + \\
&+\frac{1}{2V^{s+2}}\sum_{\substack{i,j=s+1\\(j\neq i)}}^{s}\int\frac{\partial\Phi(|r_i-r_j|)}{\partial r_i}\frac{\partial F_{s+2}(t,r_1,\dots,r_s,r_i,r_j,p_1,\dots,p_s,p_i,p_j}{\partial p_i}dr_i dr_j dp_i dp_j
\end{aligned} \qquad (1)$$

There are: t – time, r – coordinate, p – moment, ρ – density, Φ – potential of molecules interaction. Generally accepted

$$\frac{\partial}{\partial r_j}F_n \to 0 \text{ if } |r_j|\to\infty , \quad |p_j|\to\infty \qquad \frac{\partial}{\partial p_j}F_n\to 0 .$$

For the enumerated cases particles are not pairwise permutable and we do not arrive at

$$\int\sum_{j=s+1}^{N}\frac{\partial\Phi_{ij}}{\partial r_i}\frac{\partial F_{s+1}(t,r_1,\dots,r_s,r_j,p_1,\dots,p_s,p_j)}{\partial p_i}dr_j dp_j \approx N\int\frac{\partial\Phi_{is+1}}{\partial r_i}\frac{\partial F_{s+1}}{\partial p_i}dr_{s+1}dp_{s+1}$$

It is well known that for rarefied gas the Boltzmann equation is basic. So conclusion of the Boltzmann equation is given without influence of the boundary that irregular asymptotic near the surface is postulated. Influence of the boundary is taken into account via the boundary condition for distribution function. Influence of the gradient is discovered under large Reynolds number if a variation of the angular moment is considered and an additional term will appear in the Boltzmannn equation. It should be remarked that additional effect of the twisting should be taken into account at large velocity of the medium even if the surface is absent. Thus the modified Boltzmann equation for one-particle distribution function is

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\xi_i\cdot\left[\frac{\partial f}{\partial x_i}\right]+\xi_i\cdot\frac{\partial}{\partial x_i}\left[x_j\frac{\partial f}{\partial x_j}\right]-\frac{X_i}{m}\frac{\partial f}{\partial \xi_i}=I,$$

and we obtain the first term traditionally $\xi_i\cdot\dfrac{\partial f}{\partial x_i}$ and the second one $\xi_i\cdot\dfrac{\partial}{\partial x_j}\left(x_j\dfrac{\partial f}{\partial x_j}\right)$. Directly near the solid wall we shall exploit the method of the balance. It is possible since f is "a macrofunction" [8].

Then the gas dynamics values can be written by the distribution function

$$n(t,\vec{\mathbf{x}})=\int f(t,\vec{\mathbf{x}},\vec{\xi})d\vec{\xi},\qquad \mathbf{u}(t,\vec{\mathbf{x}})=\frac{1}{n}\int\vec{\xi}f(t,\vec{\mathbf{x}},\vec{\xi})d\vec{\xi},$$

$$P_{ij}=m\int c_jc_if(t,\vec{\mathbf{x}},\vec{\xi})d\vec{\xi},\qquad q_i=\frac{m}{2}\int c^2c_if(t,\vec{\mathbf{x}},\vec{\xi})d\vec{\xi}.$$

There is thermal or characteristic velocity of molecules $\mathbf{c}=\vec{\xi}-\mathbf{u}$. We obtained the modified Navie-Stokes equations in [8], where the angular moment was taking into account. Then we have conservation equations

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u_i}{\partial x_i}+\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\rho u_i}{\partial x_i}\right)=0.$$

$$\frac{\partial\rho u_i}{\partial t}+\frac{\partial}{\partial x_i}\left(\rho u_iu_j+P_{ij}+x_i\frac{\partial P_{ij}}{\partial x_i}\right)-\frac{X_i}{m}\rho=0.$$

$$\frac{\partial}{\partial t}\rho\left(\frac{3}{2}RT+\frac{1}{2}u^2\right)+\frac{\partial}{\partial x_j}\left[\rho u_j\left(\frac{3}{2}RT+\frac{1}{2}u^2\right)+u_kP_{kj}+q_j\right]+$$
$$+\frac{\partial}{\partial x_i}x_i\frac{\partial}{\partial x_j}\left[\rho u_j\left(\frac{3}{2}RT+\frac{1}{2}u^2\right)+u_kP_{kj}+q_j\right]=0$$

It is essential that we can not obtain equation for S-particle distribution function under great gradients of physical values by using ordinary notation even for the relaxation case.

Hence it is not possible to construct the Boltzman equation from the Hamilton formalism but we can use only the laws of conservation for mass, moment and energy and angle moment. It is necessary to take into account the difference between concentration of right and back collisions.

The number of molecules are defined by the one-particle distribution function $f\,(r+\vec{\xi}\,\xi\cdot\Delta t)$. Here Δt is spacing time of molecules are, $\vec{\xi}$ is the phase velocity. The general number of collisions of ξ — molecules in element $dxd\,\xi$ is

$$\Delta^-=dt\,d\vec{x}\,d\vec{\xi}\,f\left(t,\vec{x},\vec{\xi}\right)\int[f_1\left(t,\vec{x},\vec{\xi}\right)+O\left(\Delta t\vec{\xi}\frac{\partial f_1}{\partial x}\right)]gbdbd\varepsilon d\vec{\xi}_1$$

$$\Delta^+=dt\,d\vec{x}\,d\vec{\xi}'\int\left[f\left(t,\vec{x},\vec{\xi}'\right)f\left(t,\vec{x},\vec{\xi}_1'\right)+O\left(\Delta t\vec{\xi}\frac{\partial f}{\partial x}\right)\right]g'b'db'd\varepsilon'd\vec{\xi}_1$$

$I=\Delta^- - \Delta^+$

Here ε — angle, b — sighting distance, $g=\xi_1-\xi$, (the parameters of the molecules after collisions are supplied upper by prime). Now we shall obtain the equation for angular moment from the modified Boltzmann equation. The usual Boltzmann equation follows from the above if macrofunction is varied more slowly at the mean free path.

$$\int\left(\vec{r}\times m\vec{\xi}_i\right)\frac{\partial f}{\partial t}d\vec{\xi}+\int\left(\vec{r}\times m\vec{\xi}_j\right)\xi_i\frac{\partial f}{\partial x_i}d\xi_i+\int\left(\vec{r}\times\vec{\xi}_j m\right)\xi_i\frac{\partial f}{\partial x_i}x_j\frac{\partial f}{\partial x_j}d\xi=I_M,$$

$$\frac{\partial\vec{r}}{\partial x}\times\vec{p}_x+\frac{\partial\vec{r}}{\partial y}\times\vec{p}_y+\frac{\partial\vec{r}}{\partial z}\times\vec{p}_z+x_j\frac{\partial}{\partial x_j}\left(\vec{P}_j\right)=M_I.$$

For the mentioned conditions the angular moment of $M_I\approx 0$.

Influence of the dispersion in the fluid mechanics

As an example the infinite plate is investigated in case of influence of angular moment.
This profile can be theoretically defined from Prandtle formula, but not from Navier-Stokes equations or from boundary-layer equations. In our case the modified Navier-Stokes equation is

$$\frac{d}{dy}\left(\mu\frac{du}{dy}\right)+\frac{d}{dy}\left(\mu y\frac{d^2u}{dy^2}\right)=0.$$

The boundary conditions are

$$u=0,\ \ \mu\frac{du}{dy}=\tau_w,\quad y=0,\ \ u=U_\infty,\ \ y\to\infty.$$

Its solution is

$$u=C*\ln y+\tau_w/\mu*y+Const.$$

We suppose that for some point inside the boundary layer near the surface that u = 0 is possible at both boundaries, a stationary flow is formed in a sublayer, so that the inverse direction flow is not possible. Then the longitudinal gradient begins to work. A model of intermitted laminar and turbulent layers is suggested.
The Fokner-Sken problem.
New equations of uniform translational movement of cylinder are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_e\frac{\partial U_e}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\mu y\frac{\partial^2u}{\partial y^2}\right),$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

boundary conditions are

$$u=0,\qquad v=0,\qquad \mu\frac{\partial u}{\partial y}=\tau_w,\qquad y=0;$$

$$u=U_e,\qquad y\to\infty,\qquad x>0;\qquad u=U_e,\qquad x=0.$$

Here U_e is velocity at upper boundary, τ_ω is friction.

$$u=cx^m\phi(\eta),\qquad \eta=\sqrt{\frac{c}{\mu}}yx^{(m-1)/2},\qquad v=\sqrt{\mu c}x^{(m-1)/2}V(\eta),\qquad \nu=\mu y$$

In this case we have

$$m\Phi^2+\frac{m-1}{2}\eta\Phi\frac{d\Phi}{d\eta}+V\frac{d\Phi}{d\eta}=m+\mu\frac{d^2\Phi}{d\eta^2}+\frac{d}{d\eta}\left(\mu\eta\frac{d^2\Phi}{d\eta^2}\right);$$

$$m\Phi+\frac{m-1}{2}\eta\frac{d\Phi}{d\eta}+\frac{dV}{d\eta}=0.$$

If $\Phi=\psi'$ we have

$$(\eta\psi''')'+\psi'''+\frac{m-1}{2}\psi\psi''=m(\psi'^2-1)$$

Boundary conditions are

$$\psi(0)=0, \qquad \psi'(0)=0, \qquad \psi''(0)=\alpha, \qquad \psi'_\infty=1.$$

It is possible that the boundary condition may be defined more exactly. For example it can be $\psi''(\infty)=0$. We do not have new constants. The velocity u, u_2/ u_3, v profiles for the boundary condition (it was computed by A.I. Voronkova) for m=0 is at fig. d = 0.9 is multiplied by Blasius function at infinity.

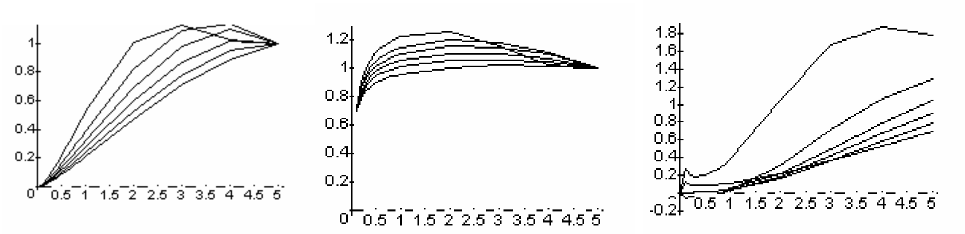

Fig.1 Blasius problem and our results.

Nonstationary problem. Small times. Sudden movement. Another example.
New equations of uniform translational movement of cylinder are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_e\frac{\partial U_e}{\partial x}+\tilde{\nu}\frac{\partial^2 u}{\partial y^2}+\tilde{\nu}\frac{\partial}{\partial y}\left(y\frac{\partial^2 u}{\partial y^2}\right), \quad \tilde{\nu}=\frac{\mu}{\rho},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Boundary and initial conditions are

$$u=U_e(x), \quad v=0, \quad y=0, \quad t=0;$$

$$u=0, \quad v=0, \quad \mu\frac{\partial u}{\partial y}=\tau_w, \quad y=0;$$

$$u=U_e(x), \quad y\to\infty, \quad x>0; \quad u=U_e(0), \quad x=0.$$

Here t is time variable, x, y are the coordinates, u_v are the velocities, $\tilde{\nu}$ is the viscosity, U_e is the velocity on external boundary. For small t boundary layer is very thin, velocity u is close to U ; v is close to 0. We use the theory of small perturbations. Here we have well known solution which is the Gauss function of mistakes instead of solution of the following problem

$$\tilde{\nu}\frac{\partial^2 u_1}{\partial y^2}+\tilde{\nu}\frac{\partial}{\partial y}y\left(\frac{\partial^2 u_1}{\partial y^2}\right)-\frac{\partial u_1}{\partial t}=0.$$

After introduction of a new function and a new variable

$$u_1=U_e(x)f_1(\eta), \quad \eta=y/2\sqrt{\tilde{\nu}t}$$

the equation takes the form

$$f_1''+\frac{d}{d\eta}\left(\eta f_1''\right)+2\eta f_1'=0.$$

To estimate influence of dispersion term we decompose equation into series near the surface.
The first term of the series is

$$f_{10}=\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-y^2}dy.$$

Then for classical solution we have the exact value of velocity for $\eta < 0.5$.

In the general case the nonstationary operator for movement equation is

$\frac{\partial}{\partial x_j} x_j \frac{\partial}{\partial t}$. From the fact that $\frac{\partial u}{\partial t}$ has the dimension of force follows the equations

$$\frac{\partial}{\partial x_j} x_j \frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_e \frac{\partial U_e}{\partial x} + \tilde{v}\frac{\partial^2 u}{\partial y^2} + \tilde{v}\frac{\partial}{\partial y}\left(y\frac{\partial^2 u}{\partial y^2}\right), \quad \tilde{v} = \frac{\mu}{\rho},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Boundary and initial conditions as before are

$$u = U_e(x), \quad v = 0, \quad y = 0, \quad t = 0;$$

$$u = 0, \quad v = 0, \quad \mu\frac{\partial u}{\partial y} = \tau_w, \quad y = 0;$$

$$u = U_e(x), \quad y \to \infty, \quad x > 0; \quad u = U_e(0), \quad x = 0.$$

Then we have

$$-\frac{\partial}{\partial y} y \frac{\partial u_1}{\partial t} + \tilde{v}\frac{\partial^2 u_1}{\partial y^2} + \tilde{v}\frac{\partial}{\partial y} y\left(\frac{\partial^2 u_1}{\partial y^2}\right) - \frac{\partial u_1}{\partial t} = 0.$$

After introduction of a new function and a new variable

$$u_1 = U_e(x) f_1(\eta), \quad \eta = y/2\sqrt{\tilde{v}t}$$

the equation takes the form

$$f_1'' + \frac{d}{d\eta}(\eta f_1'') + 2\eta f_1' + 2\frac{d}{d\eta}\left(\eta^2 f_1'\right) = 0$$

CONCLUSION

Our investigations show that it is necessary to include the low of angular moment conservation in equation of continuous mechanics and the pressure tensor is not symmetric for large gradients of physical values.

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