On the Taylor-Couette problem in the continuum limit

Avshalom Manela* and Itzchak Frankel*

*Faculty of Aerospace Engineering and Asher Space Research Center, Technion-Israel Institute of Technology, Haifa 32000, Israel

Abstract.
The transition to instability in the Taylor-Couette problem at small Knudsen (Kn) numbers and arbitrary Mach numbers is studied via a linear temporal stability analysis of the compressible 'slip-flow' problem. We focus on the case of stationary outer cylinder and equal wall temperatures. The results indicate that occurrence of instability is limited to small Knudsen numbers ($Kn \lesssim 0.016$) owing to the combination of a critical (incompressible) Reynolds number at low Mach numbers and increased dissipation rates at large Mach numbers. Comparison of the linear results with DSMC calculations and existing experimental observations demonstrates that the present analysis correctly predicts the boundaries of the instability domain. We further demonstrate that the neutral curve is well approximated by replacing the incompressible Reynolds number by a Reynolds number based on mean fluid properties. This simple approximation may present a useful alternative in studying the effects of various parameters on the onset of instability in the limit of arbitrarily small Knudsen numbers.

INTRODUCTION

The Taylor-Couette (TC) instability of a fluid between rotating concentric cylinders and the occurrence of secondary vortex flow ('Taylor vortices') is a classical problem in hydrodynamic stability theory [1]. The problem has been investigated extensively for incompressible fluids. Rayleigh [2] formulated a stability criterion for the mean viscous flow according to the inviscid perturbation equations. Later works considered the viscous perturbation equations where transition to instability is governed by some critical value of the Reynolds (Re) number, Re_c, depending on cylinders radii and velocity ratios. Taylor [3] studied the narrow gap approximation assuming axisymmetric perturbations. Others (see [1]) analysed the effect of increasing the gap width, showing that viscosity has a stabilizing effect compared with the inviscid Rayleigh criterion. The onset of instability always occurred via a stationary state (i.e. via 'exchange of stabilities'). Krueger, Gross and Di Prima [4] considered the case of non-axisymmetric perturbations and found that for counter-rotating cylinders the critical mode of transition may be non-axisymmetric and the onset of instability oscillatory (i.e. 'overstability').

Only few works study the compressible TC problem. The nondimensional problem is governed, in addition to the parameters appearing in the incompressible case, by the temperature ratio of the cylinders and the Mach (Ma) number. Kao and Chow [5] assumed axisymmetric perturbations and studied the linear stability problem for a single case of a wide gap. Their results indicate that compressibility (i.e. increasing Ma) has a destabilizing effect in the sense that it decreases Re_c relative to its incompressible value. Hatay *et al.* [6] considered various parameter combinations and non-axisymmetric perturbations. Their results essentially agree with the observations of Kao and Chow [5] indicating the destabilizing effect of compressibility. Similarly to the incompressible results of Ref.[4], the onset of instability in the counter-rotating case was characterized by non-axisymmetric critical modes and oscillatory transition states.

Instability phenomena in the TC problem occur at $Re \lesssim O(10^2)$ [1]. Both Ref. [5] and [6] study the problem for $0 < Ma \leq 4$. The resulting range of Knudsen numbers ($\propto Ma/Re$) is $0 < Kn \lesssim O(10^{-1})$. Thus, use of the continuum approach may be inconsistent. Furthermore, both studies base their Reynolds number definition on the gas density at the inner cylinder instead of its average value. Consequently, the Reynolds numbers obtained at high Mach numbers were much smaller than those based on the average density. The interpretation thus given to the comparison of their parameter with its incompressible counterpart as to the destabilizing effect of compressibility may be incorrect.

Kuhlthau [7] conducted experiments with dry air to study the effects of compressibility on the TC instability. During each set of experiments the rotation speed (Ma) was maintained constant while the average density was increased (i.e. Kn was decreased). The onset of instability was characterized by a sharp increase in the measured torque on the outer (stationary) cylinder. The experiments were carried out for $0.7 \lesssim Ma \lesssim 1.5$ showing an increase in the critical Knudsen number (Kn_c) with increasing Ma (see Fig. 1). The resulting Re_c ($\propto Ma/Kn_c$) was nearly unchanged compared with the corresponding incompressible value.

Numerical studies of the TC problem in rarefied gases have attracted considerable interest in recent years. The axisymmetric problem has principally been studied by means of the DSMC method. The numerical simulations follow the evolution of the system through its terminal state which, in turn, serves to classify the system response as stable or unstable. Reichelmann and Nanbu [8] studied the TC instability for a Maxwell gas. They justified the use of the DSMC method for this problem by showing a close agreement with the results of Kuhlthau [7]. Stefanov and Cercignani [9] considered the problem for hard-sphere gas and found, contrary to the above-mentioned results, that compressibility has a stabilizing effect. The authors related this apparent disagreement to the effects of rarefaction which were not taken into account in the above-mentioned studies. They also pointed out the need for a rigorous study of the stability problem based on the 'slip-flow' model. Golshtein and Elperin [10] and Usami [11] carried out DSMC calculations for parameter combinations within the instability domain and studied the form of vortices obtained. Bird [12] investigated the time evolution of the vortices and presented, for the only time so far, results of three-dimensional DSMC calculations. Aoki, Sone and Yoshimoto [13] and Yoshida and Aoki [14] used the DSMC method to study the effects of varying both temperature and velocity ratio on the boundary of instability domain.

All of the above-mentioned DSMC studies demonstrate that TC instability is a small $O(10^{-2})$ Knudsen phenomenon. However, the artificial 'noise' inherent in these simulations makes it difficult to clearly identify and characterize the final states, particularly for parameters combinations in the vicinity of the transition to instability. Furthermore, these simulations become extremely time consuming in the continuum limit, which obstruct an accurate delineation of the domain of instability. Consequently, explicit results in the literature have been presented only for a limited number of parameter combinations. The present contribution is thus intended to avoid these difficulties and complement the above studies by considering the corresponding linear hydrodynamic temporal-stability problem based on the 'slip-flow' near continuum model valid in the limit of (arbitrarily) small Knudsen numbers[1].

FORMULATION OF THE PROBLEM

We consider a perfect monatomic Maxwell gas confined between infinite concentric cylinders which are maintained at the same uniform temperatureT_i. The inner cylinder is rotating with an angular rate Ω_i while the outer is kept stationary. To render the problem dimensionless the position vector is scaled by the gap width $\Delta R = R_o - R_i$ (where R_i and R_o are the radii of the inner and outer cylinders, respectively), the gas density by its mean value $\bar{\rho}$ (associated with the normalization condition (6)), the velocity vector by $U_i = \Omega_i R_i$ and the temperature by T_i. The transport coefficients of shear viscosity and heat conductivity are normalized by their values μ_i and κ_i at T_i and the pressure is normalized by $\bar{\rho} R T_i$. The dimensionless problem is governed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \, , \tag{1}$$

together with the Navier-Stokes

$$\rho \frac{D\mathbf{u}}{Dt} = -\frac{1}{\gamma Ma^2} \nabla p - \frac{1}{Re} \nabla \cdot \mathsf{p} \tag{2}$$

and energy

$$\rho \frac{DT}{Dt} = -\frac{\gamma}{PrRe} \nabla \cdot \mathbf{q} - (\gamma - 1) p \nabla \cdot \mathbf{u} + \gamma(\gamma - 1) \frac{Ma^2}{Re} \Phi \tag{3}$$

equations as well as the perfect gas equation of state

$$p = \rho T \, . \tag{4}$$

In the above, p is the Newtonian deviatoric stress, $\mathbf{q}$ is the heat flux density satisfying the Fourier law and $\Phi = -\mathsf{p} : \nabla \mathbf{u}$ is the rate of dissipation. Appearing in (2) and (3) are the Reynolds number, $Re = U_i \Delta R / \nu_i$ (with $\nu_i = \mu_i / \bar{\rho}$ denoting

[1] Recently, Yoshida and Aoki [15] have considered the corresponding linear stability problem based on the Bhatnagar-Gross-Krook model of the Boltzmann equation for a case of a wide gap.

the kinematic viscosity); the Mach number, $Ma = U_i/\sqrt{\gamma R T_i}$ (with γ denoting the ratio of specific heats at constant pressure, c_p, and volume, c_v, respectively); and the Prandtl number, $Pr = \mu_i c_p/\kappa_i$. For a Maxwell gas $\gamma = 5/3$, $Pr = 2/3$ and

$$\mu(T) = \kappa(T) = T \tag{5}$$

(see [16]).

We write the above equations in a cylindrical coordinate system (r, θ, z) whose origin lies on the axis of symmetry of the cylinders. The problem is supplemented by the normalization condition

$$\int \rho \, \mathrm{d}V = 1, \tag{6}$$

specifying the total amount of gas between the cylinders, and by the boundary conditions

$$u_r = 0,\ u_{\theta,z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm \zeta \frac{\partial u_{\theta,z}}{\partial r},\ T = 1 \pm \tau \frac{\partial T}{\partial r} \ \text{ at } r = (R_R - 1)^{-1} \begin{bmatrix} 1 \\ R_R \end{bmatrix}, \tag{7}$$

respectively imposing the vanishing of the normal velocity component and specifying the magnitudes of velocity slip and temperature jump at the inner and outer cylinders. In (7), $R_R = R_o/R_i$, $\zeta = 1.1466\overline{Kn}$ and $\tau = 2.1904\overline{Kn}$ [17]. $\overline{Kn}$ denotes the mean Knudsen number, the ratio of the mean-free-path $\bar{l}$ (based on *mean* density *and* temperature, see (9)) and the macroscopic scale ΔR. To relate the parameters Re, Ma and $\overline{Kn}$ appearing in the above, we make use of the prevailing Variable Hard Sphere (VHS) model [18] to obtain for a Maxwell gas

$$\overline{Kn} = \left(\frac{10}{3\pi}\right)^{1/2} \frac{Ma}{Re}\ \bar{T}^{1/2}\,. \tag{8}$$

In (8) $\bar{T}$ denotes the mean gas temperature (see (9)).

The viscous-compressible cylindrical Couette flow is calculated numerically assuming a steady-state solution of (1)-(7) depending only on the radial coordinate with a nonzero velocity component in the azimuthal direction. The calculation yields the mean fields $U_\theta^{(0)}(r)$, $T^{(0)}(r)$, $\rho^{(0)}(r)$ and $p^{(0)}(r)$. $\bar{T}$ in (8) is then obtained as

$$\bar{T} = \frac{2(R_R - 1)}{R_R + 1} \int_{1/(R_R-1)}^{R_R/(R_R-1)} T^{(0)} r \, \mathrm{d}r\,. \tag{9}$$

It is worthwhile to note that, although the fluid is considered as compressible, the reference velocity field trivially satisfies $\nabla \cdot \mathbf{u}^{(\mathbf{0})} = 0$ for all Ma.

The linear temporal stability of this reference state is analyzed by assuming that it is perturbed by small spatially harmonic perturbations. Accordingly, each of the above-mentioned fields is generically represented by the sum

$$F = F^{(0)}(r) + \phi^{(1)}(r) \exp[i(k_z z + k_\theta \theta) + \omega t] \tag{10}$$

wherein $F^{(0)}$ denotes the steady reference state and k_z and k_θ are, respectively, the wave-numbers in the axial and tangential directions. Substituting (10) into (1)-(7) and neglecting non-linear terms, we obtain the perturbation problem which is not explicitly presented here for brevity. The dispersion relation $\omega = \omega(k_z, k_\theta; Kn, Ma, R_R,\ R_U, R_T, s)$ is calculated by means of the Chebyshev collocation method [19].

RESULTS AND DISCUSSION

For a stationary outer cylinder our calculations invariably yield real-valued ω. Accordingly, the onset of convection takes place via ‘exchange of stabilities’ [1], i.e. $\omega = 0$. Furthermore, in all cases examined we found that the critical mode of instability is axisymmetric, i.e. $k_\theta = 0$. In the following we focus on a case of $R_R = 1.12$ so as to facilitate comparison with the results of the experiments of Kuhlthau [7].

The solid line in Figure 1 separates the plane of parameters (Ma, Kn) into respective domains of unstable (U), $\omega > 0$, and stable (S), $\omega < 0$, response. Also presented are our corresponding DSMC (circles) and experimental (Ref. [7], crosses) results together with the curves of constant $Re = 127$ (dashed line) and $\overline{Re} = 127$ (dash-dotted line, see (11)). We note that Kn appearing in the figure is based on the *wall* temperature T_i (cf. the above definition of $\overline{Kn}$).

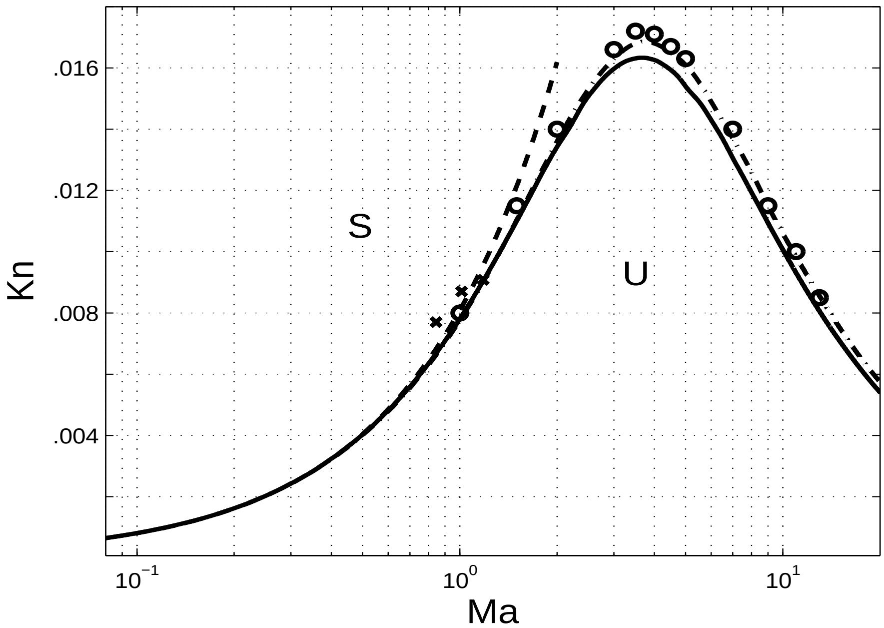

FIGURE 1. Division of the plane of parameters (Ma, Kn) into respective domains of unstable (U), $\omega > 0$, and stable (S), $\omega < 0$, response for $R_R = 1.12$. Also presented are the corresponding DSMC (circles) and experimental (Kuhlthau 1960, crosses) results together with the curves of $Re = 127$ (dashed) and $\overline{Re} = 127$ (dash-dotted).

For all $Kn > 0$ the instability domain is confined to a finite interval of Mach numbers. The extent of this interval is rapidly diminishing with increasing Kn, vanishing entirely for $Kn = Kn_{max} \gtrsim 0.0165$. When $Kn \lesssim 0.008$ (and $Ma \lesssim 1$), the left branch of the neutral curve coincides with the dashed line. The latter corresponds to a constant Reynolds number, $Re = 127$. This is in accordance with the Re_c obtained in the incompressible case (see [1]) at the present R_R. Indeed, our mean flow calculations show that at low Mach numbers ($Ma \lesssim 0.5$) the density and temperature become nearly uniform (see Fig. 2) and thus the energy equation (3) is decoupled from the dynamical problem (1)-(2). However, for $Ma \gtrsim 1$ compressibility effects become increasingly important causing the neutral curve to deviate from the dashed asymptote to larger values of Re. Thus, contrary to the above-mentioned results of Refs. [5] and [6], compressibility has a stabilizing effect. It is worthwhile to note that the critical wave-number obtained was nearly unaffected by Ma, equal to its incompressible value $k_{z_{cr}} \approx 3.13$.

The agreement between the linear analysis and experiments is satisfying. The minor differences appearing may be attributed to the different fluids considered (we apply our analysis to homogeneous monatomic gas whereas Ref. [7] used dry air for his experiments). However, the limited experimental data are insufficient for a comprehensive comparison, in particular regarding the occurrence of the right branch of the neutral curve. Thus, for further confirmation of our analysis we carry out DSMC calculations. We use the DS2V version by Bird [20]. The axisymmetric flow is studied in a rectangular computational domain $D\{(r,z) \in D(\Delta R \times \Delta Z)\}$ with an aspect ratio $R_L = \Delta Z / \Delta R = 2$. The domain is confined in the r-direction by two diffusively reflecting cylindrical walls while a periodic boundary condition is imposed on the bounding walls in the z-direction[2]. The gas is initially in thermodynamic equilibrium with the uniform density $\bar{\rho}$ and temperature T_i. At $t = 0$ the inner cylinder starts rotating impulsively. The simulation follows the evolution of the flow field through its terminal state which, in turn, serves to classify the system response as stable or unstable. Each circle appearing in Fig. 1 denotes the maximal value of Kn where instability occurs at the corresponding Ma. As mentioned above, the artificial 'noise' inherent in the simulations makes it difficult to clearly identify

---

[2] This combination of boundary conditions and R_L is equivalent to the choice of $k_z = n\pi$, $n = 1, 2, \ldots$ as the discrete spectra in the linear analysis (Manela and Frankel [21]). The value $k_z = \pi$ is approximately equal to the critical wave-number obtained by our calculations across the neutral curve.

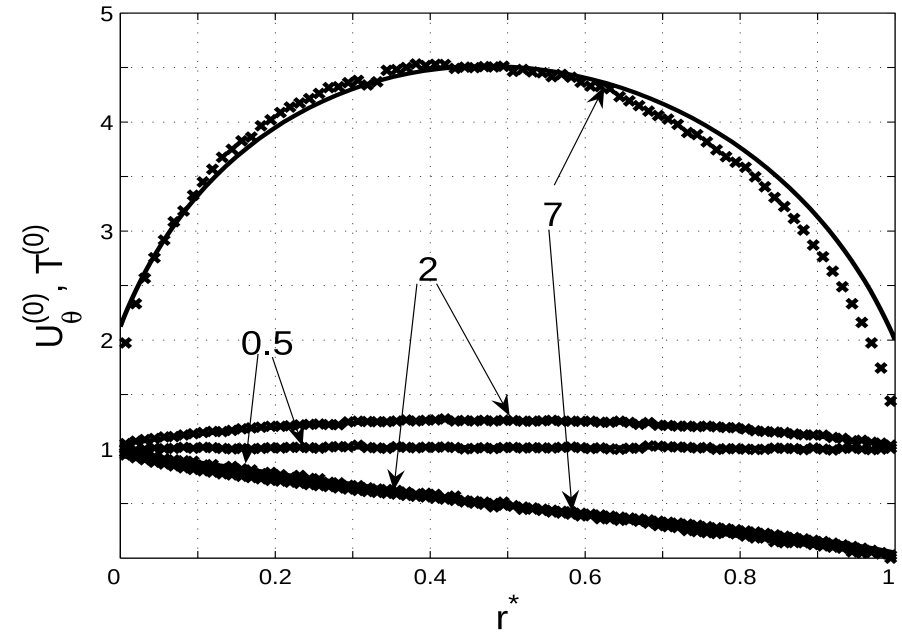

FIGURE 2. High-Mach effects on the reference state: $U_\theta^{(0)}$ (lower part) and $T^{(0)}$ (upper part) at the indicated values of Ma for Maxwell gas at $Kn = 0.014$ with $R_R = 1.12$. The solid lines correspond to the numerical mean-flow solution and the crosses are obtained from DSMC simulations.

and characterize the final states obtained in the vicinity of the neutral curve. Thus, each of the circles may be better referred to as the upper bound of an 'intermediate range' of Knudsen numbers in which the neutral curve passes.

In view of the vastly different methods of calculation (a linearized eigenvalue problem as opposed to a non-linear initial value problem), the close agreement between the linear and DSMC results is gratifying. The relative differences between the respective results increase with increasing Ma. Indeed, at high Mach numbers the density in the vicinity of the rotating cylinder decreases considerably while sharp gradients appear in part of the macroscopic fields (see Fig. 2). Thus, the local Knudsen number (based on the local mean free path and macroscopic scale) increases. This, in turn, impairs the accuracy of the 'slip-flow' model. Nevertheless, the occurrence of Kn_{max} and the high-Mach right branch of the U-domain boundary is evident.

To gain some insight into the effects of compressibility, we study the reference flow at increasing Mach numbers. Figure 2 presents the reference velocity and temperature distributions at $Kn = 0.014$ and the indicated values of Ma against the shifted $r^* = r - 1/(R_R - 1)$ coordinate. The solid lines correspond to the numerical mean-flow solution and the crosses are obtained from the DSMC calculations. The agreement between the calculations is excellent for the low Mach numbers. Differences are distinguishable only for the temperature distribution at $Ma = 7$.

We first observe that the velocity field is nearly unaffected by Ma. This is attributed to the relation trivially satisfied, $\nabla \cdot \mathbf{u}^{(\mathbf{0})} = 0$, irrespective of the Mach number. However, the temperature field changes considerably with Ma. At $Ma = 0.5$ it is nearly uniform in accordance with the incompressible limit. With increasing Ma a maximum appears approximately in the middle of the gap. Its value grows substantially to $T_{max}^{(0)} \approx 4.5$ at $Ma = 7$ along with large temperature jumps at the walls. As can be seen from the energy balance governing the mean flow, this increase results from large dissipation rates (proportional to Ma^2) occuring within the fluid. We find, therefore, that at high Mach numbers the kinetic energy of the rotating cylinders is irreversibly transformed into internal energy of the fluid. This results in large values of viscosity and heat conductivity ($\propto T$ in the present case of Maxwell gas). Moreover, the local Mach numbers obtained within the fluid are considerably lower than the defined Ma (based on inner-wall properties) owing to the increase in the local speed of sound ($\propto T^{1/2}$).

The above discussion seems to suggest that, similarly to the corresponding incompressible problem, stability is essentially determined through a balance between destabilizing inertial- and retarding viscous- effects. Such a balance is expressible in a critical Reynolds number. However, one needs to consider that, owing to the elevated temperatures at supersonic speeds, bulk-fluid density and viscosity greatly differ from their corresponding values at the inner cylinder. To roughly account for these we introduce

$$\overline{Re} = \frac{U_i \Delta R}{\bar{\nu}} \tag{11}$$

based on $\bar{\nu}$, the mean kinematic viscosity in the reference state. Comparison with (8) yields

$$\overline{Re} = \left(\frac{10}{3\pi}\right)^{1/2} \frac{Ma}{Kn} \overline{\left(\frac{T^{(0)}}{\rho^{(0)}}\right)}^{-1} . \tag{12}$$

As mentioned above, the dash-dotted curve in the figure represents $\overline{Re} \approx 127$. At subsonic speeds $Ma \lesssim 1$, $T^{(0)}/\rho^{(0)}$ is nearly uniform throughout the gap and this curve nearly coincides with the dashed line $Re \approx 127$. At supersonic speeds $T^{(0)}/\rho^{(0)}$ increases faster than linearly with Ma. Consequently, in order to maintain a constant value of $\overline{Re}$, Kn needs to diminish with increasing Ma. Thus, the appearance of the descending branch of the neutral curve is related to increased dissipation rates at supersonic speeds.

Figure 1 demonstrates that $\overline{Re} \approx 127$ is indeed a surprisingly close approximation of the neutral curve. The close agreement obtained suggests the mean-Reynolds criterion as a tractable means for easily examining, among other things, the effects of other values of parameters and models of molecular interaction on the boundary of instability domain.

REFERENCES

1. Chandrasekhar, S., *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford, 1961.
2. Rayleigh, L., *"On the dynamics of revolving fluids", Sci. Pap.*, **6**, 447 (1916).
3. Rayleigh, G., *"Stability of a viscous liquid contained between two rotating cylinders", Phil. Trans. Roy. Soc. A*, **223**, 289 (1923).
4. Krueger, E., A.Gross, and Prima, R. D., *"On the relative importance of Taylor-vortex and non-axisymmetric modes in flow between rotating cylinders", J Fluid Mech*, **24**, 521 (1966).
5. Kao, K., and Chow, C., *"Linear stability of compressible Taylor-Couette flow", Phys. Fluids A*, **4**, 984 (1992).
6. Hatay, F., Biringen, S., Erlebacher, G., and Zorumski, W., *"Stability of high-speed compressible Couette flow", Phys. Fluids A*, **5**, 393 (1993).
7. Kuhlthau, A., *"Recent low-density experiments using cylinder techniques", Rarefied Gas Dynamics (ed. Devienne, F.M.), Pergamon, London*, p. 192 (1960).
8. Riechelmann, D., and Nanbu, K., *"Monte Carlo simulation of the Taylor instability in rarefied gas", Phys. Fluids A*, **5**, 2585 (1993).
9. Stefanov, S., and Cercignani, C., *"Monte Carlo simulation of the Taylor-Couette flow of a rarefied gas", J. Fluid Mech.*, **256**, 199 (1992).
10. Golshtein, E., and Elperin, T., *"Investigation of Bénard, Taylor and Thermal stresses instabilities in rarefied gases by direct simulation Monte Carlo method", AIAA paper 95-2054* (1995).
11. Usami, M., *"Direct simulation Monte Carlo on Taylor vortex flow", Rarefiied Gas Dynamics (eds. J. Harvey and G. Lord), Oxford University Press, Oxford, vol. 1 pages =* (1995).
12. Bird, G., *"Recent advances and current challenges for DSMC", Computers Math. Applic.*, **35**, 1 (1998).
13. Aoki, K., Sone, Y., and Yoshimoto, M., *"Numerical analysis of the Taylor-Couette problem for a rarefied gas by the direct simulation Monte Carlo method", Rarefied Gas Dynamics (ed. Brun, R., Camprague, R., Gatignol, R. and Lengrand J.-C.), CÉPADUES, Toulouse, vol. 2*, p. 109 (1999).
14. Yoshida, H., and Aoki, K., *"A numerical study of Taylor-Couette problem for a rarefied gas: Effect of rotation of the outer cylinder", Rarefied Gas Dynamics (ed. Capitelli, M.), American Institute of Physics, New York*, p. 467 (2005).
15. Yoshida, H., and Aoki, K., *"Linear stability of the cylindrical Couette flow of a rarefied gas", Phys. Rev. E*, **73** (2006).
16. Chapman, S., and Cowling, T., *The Mathematical Theory of Non-Uniform Gases*, 3rd edn. Cambridge University Press, Cambridge, 1970.
17. Albertoni, S., Cercignani, C., and Gotusso, L., *"Numerical evaluation of the slip coefficient", Phys. Fluids*, **6**, 993 (1963).
18. Bird, G., *Molecular Gas Dynamics and the Direct Simulations of Gas Flows*, Clarendon Press, Oxford, 1994.
19. Peyret, R., *Spectral methods for incompressible viscous flow*, Springer-Verlag, New York, 2002.
20. Bird, G., *"The DS2V general program for the computation of two-dimensional and axially-symmetric flows by the DSMC method", Available at http://www.gab.com.au/DS2V2.html* (2005).
21. Manela, A., and Frankel, I., *"On the Rayleigh-Bénard problem in the continuum limit", Phys. Fluids*, **17**, 036101 (2005).