Self-similar interpolation in rarefied gas dynamics

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Abstract. The effectivness of the method of self-similar interpolation in its simplest version is demonstrated by solving problems of slow plane Couette and Poiseuille flows of rarefied gas and the problem of the structure of a strong shock wave in a monoatomic gas. Interpolations of the function with respect to its specifield asymptotic representations of a different form at the ends of the interval in which the function is specefield, usually semi-infinite, are obtained. †

1. THE METHOD OF SELF-SIMILAR INTERPOLATION

Suppose we know the following asymptotic forms for the required function $f(x), x=[0,\infty)$

$$f(x)=\begin{cases}\sum\limits_{i=0}^{K} b_i x^i, \ x\to 0\\ \sum\limits_{i=0}^{N} A_i \quad x^{\alpha_i} \ , \ x\to\infty\end{cases} \tag{1.1}$$

Using the method of self-similar interpolation [1], we construct interpolation formula of different orders. The asymptotic form for $x\to 0$ is fixed, and hence we will only give approximations for $f(x\to\infty)$.

The first order. Let us use the main member of expansion

$$f(x)=A_0 x^{\alpha_0}, \ x\to\infty \tag{1.2}$$

The interpolation formula can be written as

$$f^*(x)=\left(b^{1/n}+B \quad x\right)^n \tag{1.3}$$

Symbol "*" marks variables received by the given method

At $x\to 0$ we have $f^*(x)\to f(x)$. Unknown values B and n we take from the equation

$$B^n x^n = A_0 x^{\alpha_0}$$

$$f^*(x)=\left(b^{1/\alpha_0}+{A_0}^{1/\alpha_0} x\right)^{\alpha_0} \tag{1.4}$$

which follows from equations (1.2) and (1.3) at $x\to\infty$. Hence $n=\alpha_0,\ B={A_0}^{1/\alpha_0}$. As a result we have the formula that gives to us true asymptotics at $x\to 0$ and at $x\to\infty$ too.

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The second order. We have

$$f(x) = A_0 \ x^{\alpha_0} + A_1 \ x^{\alpha_1}, \ x \to \infty \tag{1.5}$$

$$f^*(x) = \left[\left(b^{1/n} + C \ x \right)^{n/m} + D \ x^2 \right]^m \tag{1.6}$$

For definition of $n, \ m, \ C, \ D$ asymptotic at $x \to \infty$ is considered . At first the main member in equation (1.6) is equal to the main member in (1.5) and as a result we have equation

$$D^m x^{2m} = A_0 x^{\alpha_0}$$

From here $m = \alpha_0 / 2, \quad D = A_0{}^{2/\alpha_0}$. Then we neglect item $b^{1/n}$ in the equation (1.6) in comparison with Cx and linearize formula. After that we have:

$$n = \frac{\alpha_0}{2} \ \beta, \ C = \left(\frac{2}{\alpha_0} \ A_0^{\gamma} \ A_1 \right)^{1/\beta}, \ \beta = \alpha_1 - \alpha_0 + 2, \ \gamma = \frac{2}{\alpha_0} - 1.$$

The third order. We have

$$f(x) = A_0 \ x^{\alpha_0} + A_1 \ x^{\alpha_1} + A_2 \ x^{\alpha_2}, \ x \to \infty$$

$$f^*(x) = \left\{ \left[\left(b^{1/n} + D \ x \right)^{n/m} + E \ x^2 \right]^{m/\rho} + G \ x^3 \right\}^{\rho}$$

In practical problems, as a rule, some members of asimptotics series are given at the ends of an interval. To receive equations like the formula (1.1), algebraic transformations are used (addition or subtraction of some constants, multiplication or division by some function of x, as, for example, in Section 2), if asimptotic series has a logarithmic member then exponential transformation is used. After that exponential function is represented as power series at $x \to 0$ (section 3). If asimptotic series has an exponential member then logarithmic transformation is used. (section 4). After that interpolatory formula is looked for modified expressions and then the inverse transformation is used.

2. THE COUETTE PROBLEM

Let us consider slow (Mack number $M << 1$) rarefied gas flow between parallel plates that moves one relative the other with equal bat opposite directed velocities. Integral equation for velocity distribution is given by [2]

$$g(x) \ = f^-(x) + \frac{\alpha}{\sqrt{\pi}} \int_{-1/2}^{1/2} T_{-1}\left(|x - s| \right) g(s) ds \tag{2.1}$$

$$f^{\pm}(x) = \frac{1}{\sqrt{\pi}} \ \left\{ T_0\left(\alpha\left(\frac{1}{2} - x \right) \right) \ \pm T_0\left(\alpha\left(\frac{1}{2} + x \right) \right) \right\} \tag{2.2}$$

$$T_i(x) \ = \ \int_0^{\infty} z^i \exp\left(-z^2 - \frac{x}{z} \right) dz, \ \ i = 0, \ -1, \quad \alpha = 1/Kn$$

We mast find friction stress P_{xz}. At $\alpha \to 0$ we have [2]

$$P = P_{xz} / P_{xz}^0 \ = 1 - \sqrt{\pi}\alpha / 2 \tag{2.3}$$

At $\alpha \to \infty$ (taking slip into account)

$$P = \ \sqrt{\pi} / \alpha - 2\sqrt{\pi} / \alpha^2 \tag{2.4}$$

where P_{xz}^0 - friction stress at free-moleculte case.

Self-similar interpolations of different orders can be represented as:

the first order

$$P=\begin{cases}1\,,\ \alpha\to 0\\ \sqrt{\pi}/\alpha,\ \alpha\to\infty\end{cases};\qquad P^{*}=\frac{\sqrt{\pi}}{\sqrt{\pi}+\alpha} \tag{2.5}$$

the second order

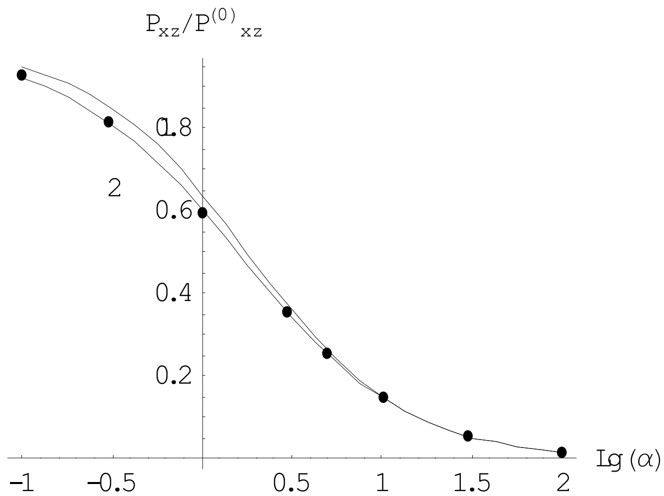
$$P=\begin{cases}1-\alpha\sqrt{\pi}\;\;\alpha/2,\ \alpha\to 0\\ \sqrt{\pi}/\alpha,\ \alpha\to\infty\end{cases};\qquad P^{*}=\frac{\sqrt{\pi}\;\;\alpha}{2}\left[1+\frac{\pi\;\;\alpha}{4}\left(\alpha+2\;\;\sqrt{\pi}\right)\right]^{-1/2} \tag{2.6}$$

the third order

$$P=\begin{cases}1-\sqrt{\pi}\;\;\alpha\;/2,\ \alpha\to 0\\ \sqrt{\pi}/\alpha\;-\;2\;\;\sqrt{\pi}/\alpha^{2},\ \alpha\to\infty\end{cases}$$

$$P^{*}=\frac{\sqrt{\pi}\;\;\alpha}{2}\left[1+\frac{\pi\;\;\alpha}{8}\left(6\;\;\pi\;\;\left(\sqrt{\pi}-1\right)+3\pi\;\;\alpha+\sqrt{\pi}\;\;\alpha^{2}\right)\right]^{-1/3} \tag{2.7}$$

Equation (2.1) was solved numerically for some value α of variation least-squares method. Sampling function is a polynomial of degree 20.



Comparison of value ratio $P=P_{xz}/P_{xz}^{0}$ from inverse Knudsen number α is given on figure (1- the first order, 2- the second order, points is numerical computation) and in the table (N–results of the numerical computation, I_1, I_2, I_3 – interpolation of the first, the second and the third orders (formulas (2.5)-(2.7)).

α	0.1	0.3	1	3	10	30	100
N	0.927	0.819	0.600	0.360	0.147	0.0559	0.0176
I_1	0.947	0.855	0.639	0.371	0.150	0.0558	0.0174
I_2	0.922	0.807	0.585	0.344	0.145	0.0549	0.0173
I_3	0.924	0.817	0.603	0.356	0.147	0.0554	0.0174

The maximum error of the first order interpolation is 6% (α =1), of the second order interpolation is 2.7% (α =3), of the third order interpolation is 0.6% (α=3).

3. POISEUILLE FLOW

The slow rarefied gas flow between parallel plates under the influence of the slow pressure gradient $\partial p / \partial x = -Kp_0$ ($K = const$, p_0 – the pressure value for $x = 0$, the coordinate x direct along up the plate and the coordinate value divided to distance between it) is consider. Integral equation for velocity distribution $u_x = g(y)\ V$, $V^2 = 2kT/m$ is given by [2, 3]

$$g(x) = \frac{K}{2\alpha}\left[1 - f^+(x)\right] + \frac{\alpha}{\sqrt{\pi}} \int_{-1/2}^{1/2} T_{-1}\left(\alpha\ |x-s|\right)\ g(s)\ ds \tag{3.1}$$

The functions f^+ и T_{-1} and the value of α is given by formula (2.2).

We mast find the nondimensional volume gas flow

$$Q(\alpha) = \int_{-1/2}^{1/2} g(x)\ dx$$

The asymptotic presentation of $Q(\alpha)$ is given [2, 3] (γ -Euler constant)

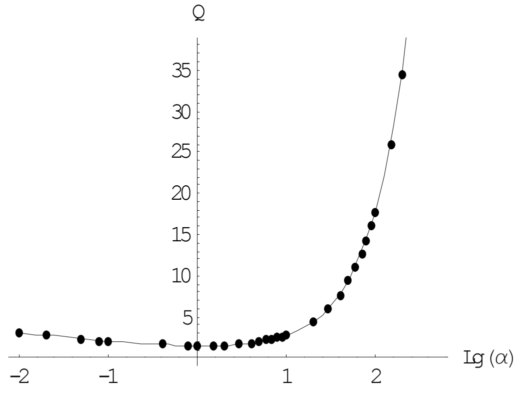
$$Q(\alpha) = \begin{cases} -\ln\alpha/\sqrt{\pi} + (1-\gamma/2)/\sqrt{\pi} & \alpha \to 0 \\ \alpha/6 + 1.0162 & \alpha \to \infty \end{cases} \tag{3.2}$$

Self-similar interpolation is given by formula

$$Q^*(\alpha) = A + B\ \ \alpha + C\ \ \ln\left(D + E\ \ \alpha^{-1}\right),\ A = 1.0162,$$
$$B = 1/6,\ C = 1/\sqrt{\pi},\ D = 3/2,\ E = 0.3363 \tag{3.3}$$

Function $Q(\alpha)$ have minimum (known Knudsen paradox [2]). At $\alpha \to 0$ function $Q(\alpha)$ is contained the logarithmical item. This singularities are complicated the interpolation determination (in analyzed comparison with section 2). Nevertheless, calculation result by the formula (3.3) low differ from the numerical calculations [3] (see table, N – numerical calculation result $Q(\alpha)$ [3], I– formula use result (3.3)).

α	0.01	0.1	1	10	100
N	3.050	2.033	1.539	2.768	17.70
I	3.044	2.066	1.470	2.721	17.69



Self-similar interpolation results (curve) and numerical calculation result [3] (points) is given on figure. Maximal difference equal to 4.5% at $\alpha = 1$, nearby of minimum volume gas flow.

4. SHOCK WAVE STRUCTURE IN MONOATOMIC GAS

Onedimensional stationary flow of the monatomic gas in the line of axis x is describe by means of the kinetic Boltzmann equation

$$c_x \partial f(x,\mathbf{c})/\partial x = \int \left[f(x,\mathbf{c}') f(x,\mathbf{c}_1') - f(x,\mathbf{c}) f(x,\mathbf{c}_1) \right] gbdbd\varepsilon d\mathbf{c}_1 = J_1 - fJ_2 \tag{4.1}$$

$\mathbf{c} = (c_x, c_y, c_z)$ - molecular velocity, $g = |\mathbf{c} - \mathbf{c}_1|$ - module of the relative molecular velocity under molecular collision, b– target interval, ε– horizontal angle, $\mathbf{c}'$, $\mathbf{c}_1'$ - relative molecular after molecular collision.

Numerical gas density evaluated express as distribution function $f(x,\mathbf{c})$ by the formula

$$n(x) = \int f(x,\mathbf{c}) d\mathbf{c} \tag{4.2}$$

At $x \to \mp\infty$ distribution function tend to Maxwell functions

$$f_j^M = n_j \left(\frac{\theta_j}{\pi}\right)^{3/2} \exp\left\{-\theta_j \left[(c_x - u_j)^2 + c_y^2 + c_z^2 \right]\right\},\ \theta_j = \frac{m}{2kT_j},\ j = 1,2 \tag{4.3}$$

Values of gas density, rate and temperature front $x = -\infty$ and behind $x = \infty$ shock wave are equal n_1, u_1, T_1 и n_2, u_2, T_2 accordingly. This values are bind of known equation [4].

Integral form of the equation (4.1) for shock wave structure problem is given [2] ($\hat{A}$ is an operator)

$$f(x,\ \mathbf{c}) = \hat{A}f \equiv \left\{ \int_{-\infty}^{x} W(\tau,\mathbf{c}) d\tau,\ c_x > 0;\ \int_{x}^{\infty} W(\tau,\mathbf{c}) d\tau,\ c_x < 0 \right\},$$

$$W(\tau,\mathbf{c}) = \frac{J_1(\tau,\ \mathbf{c})}{c_x} \exp\left[-\int_{\tau}^{x} J_2(\tau_1,\ \mathbf{c}) \frac{d\tau_1}{c_x} \right] \tag{4.4}$$

We will use "pseudo-Maxwellian" model of the molecules [5], introducing value σ_0 by equality $\pi d^2 = \sigma_0 g^{-1}$. Then $J_2(x,c) = \sigma_0 n$.

We are solve the integral equation (4.4) by method of successive approximations $f^{(n)} = \hat{A}f^{(n-1)}$ taking into account boundary condition (4.3) by null approximation [6]:

$$f^{(0)}(x,\mathbf{c}) = \begin{cases} f_1^M,\ \text{when}\ -\infty < x < 0,\ \text{or}\ x = 0,\ c_x > 0 \\ f_2^M,\ \text{when}\ 0 < x < \infty,\ \text{or}\ x = 0,\ c_x < 0 \end{cases}$$

The quantities f_1^M и f_2^M are calculated for (n_1, u_1, T_1) и (n_2, u_2, T_2) respectively. We obtain in the first approximation

$$f^{(1)}(x,\mathbf{c}) = \begin{cases} f_1^M + \xi\left[f_2^M - f_1^M \right]\ B_1(x,\ c_x)\ \text{when}\ \ x < 0 \\ f_2^M + (1-\xi)\left[f_1^M - f_2^M \right]\ B_2(x,\ c_x)\ \text{when}\ \ x > 0 \end{cases} \tag{4.5}$$

$$\xi = \begin{cases} 1\ \text{when}\ c_x < 0 \\ 0\ \text{when}\ c_x < 0 \end{cases},\quad B_j(x,\ c_x) = \exp(-\sigma_0\ \ n_j\ \ x / c_x),\ j = 1,\ 2 \tag{4.6}$$

Using relations (4.2) and (4.6), we obtain an expression for density in the first approximation

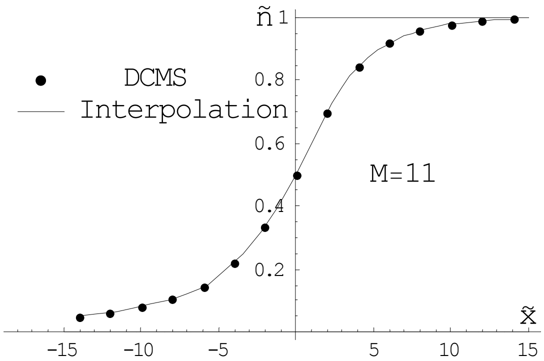
$$n = \begin{cases} n_1 \quad \left(1 - n_{11} + n_{12}\right) \quad x < 0 \\ n_2 \quad \left(1 - n_{22} + n_{21}\right) \quad x > 0 \end{cases} \tag{4.7}$$

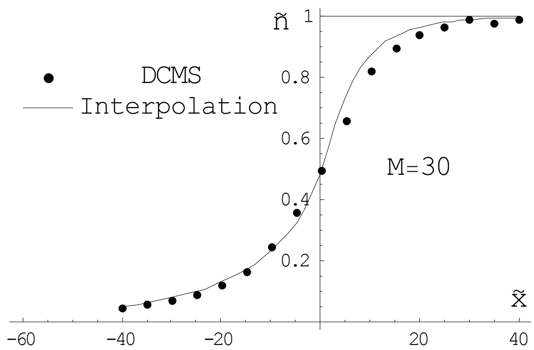
$$n_{jm} = \frac{n_m}{n_j}\left(\frac{\theta_m}{\pi}\right)^{1/2} N_{jm},\ N_{1m} = \int_{-\infty}^{0} A_m B_1 dc_x,\ N_{2m} = \int_{0}^{\infty} A_m B_2 dc_x,\ A_m = \exp\left[-\theta_m \left(c_x - u_m\right)^2\right],\ m = 1,\ 2$$

We obtain the following asymptotic expression as $x \to \infty$ [4]:

$$n_{ij} = 3^{-1/2}\frac{n_j}{n_i} \exp\left[-3 \quad \theta_j^{1/3}\left(\frac{\sigma_0 \quad n_i \quad |x|}{2}\right)^{2/3}\right],\ i,\ j = 1,\ 2 \tag{4.8}$$

The method of constructing the interpolation formula differs from those used in Sections 2 and 3 in specifying the asymptotic representations at the ends of the infinite interval. The following method is used. It is assumed that the function $n(x)$ has continuous derivatives up to the third order inclusive, and the function is split into left- and right-hand parts at the point $x = 0$. Interpolations are found for the left- and right-hand parts, containing the unknown values of function and its first derivative at the point $x = 0$. To determine these values, it is assumed that the second and third derivatives of the left- and right-hand parts are equal at $x = 0$.





The results of calculations of density profile,

$$\tilde{n} = \left(n\left(\tilde{x}\right) - n\left(-\infty\right)\right) / \left(n\left(+\infty\right) - n\left(-\infty\right)\right), \quad \tilde{x} = x/\ell\left(-\infty\right),\ \ell\text{-mean of free path,}$$

for $M_1 = 10 - 30$ (running line) were compared with the results of calculations by the DCMS method (points). The relative difference between these results does not exceed 10%, which is close to the error of calculations by the DCMS method.

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