Numerical study of the spiral cylindrical Couette flow of rarefied gas

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Abstract. We study the spiral Couette flow between two cylinders, the inner of which not only rotates but also slides along its axis. The analysis is based on the numerical solution of the model kinetic equation by a second-order accurate finite-volume scheme conservative with respect to the model collision integral. The influence of ratio of cylinder radiuses, velocities of the inner cylices as well as the Knudsen number on shear stresses, mass flow rates as well as macroparameters are investigated in the broad range of Knudsen numbers.

Keywords: Shakhov model kinetic equation, Couette flow, conservative method, implicit TVD scheme
PACS: 47.45.-n

INTRODUCTION

The plane Couette flow of a rarefied gas between two plates is a classical problem of fluid dynamics. Its axisymmetric extension is given by the flow between moving cylinders and is of not only theoretical but also practical interest. The current literature is devoted to the study of Knudsen-number dependence of the solution for the pure rotation case, e.g. [1, 2, 3, 4]. The influence of two other important parameters of the problem, the sliding velocity of the cylinder and the ratio of their radiuses, has so far been studied only in the pure longitudinal Couette flow [5]. However, these parameters can have a significant influence on the flow pattern in the general case as well in which the inner cylinder both rotates and slides along its axis.

From the numerical point of view the cylindrical Couette flow is a difficult problem due to the different behavior of the velocity distribution function for large and low Knudsen numbers. The method of steady-state iterations [6, 1] is most commonly used to construct the solution. The shortcomings of this method are its poor convergence at low Knudsen numbers and the complexity of its software implementation; also the second order version [4] is non-monotone around steep gradients due to the use of linear differentiation operators.

The basic goal of this study is two fold. Firstly, we develop a universal numerical method for analyzing axisymmetric Couette flows on the basis of kinetic equation with the S-model collision integral [7, 8]. The method can be viewed as an extension of [5] and is second-order accurate, implicit and conservative with respect to the collision integral. Secondly, we apply this numerical method to the Couette problem in the generalized formulation in which the inner cylinder not only rotates but also slides along its axis. This effectively gives three possible flow patterns: longitudinal, rotational and spiral cylindrical Couette flows. For each pattern we study the dependence of macroparameters, stresses and mass flow rates on Knudsen number and ration of cylinder radiuses.

FORMULATION OF THE PROBLEM

We consider a steady-state rarefied monatomic gas flow between two infinite coaxial cylinders with radii r_1 and r_2 ($r_1 < r_2$). The outer cylinder is at rest. The gas flows due to the motion of the inner cylinder which travels along the symmetry axis at a constant velocity U_z and rotates with a constant angle velocity Ω. On the surfaces of the cylinders a constant temperature T_w (the same for both cylinders) is maintained. The momenta and energies of the incident molecules are totally accommodated by the cylinder surfaces and diffusely reflected with the equilibrium distribution functions at the given temperature T_w. The radial gas velocity component is equal to zero.

We introduce the cylindrical coordinate system (r, φ, z), where the z axis coincides with the axis of the cylinders, r is the distance from this axis, and φ is the azimuthal coordinate. The axisymmetric state of the rarefied gas, which is independent of z, is determined by the molecular velocity distribution function $f(t, r, \xi_r, \xi_\varphi, \xi_z)$, where ξ_r, ξ_φ and ξ_z are

the orthogonal components of the velocity vector in the radial, azimuthal, and axial directions, respectively. In velocity space we also introduce a cylindrical coordinate system with the axis parallel to the z axis. We will denote by ζ the velocity component in a plane perpendicular to the axis of the cylinders and by ω the angle between this component and the radial direction outward from the symmetry axis. The relationship between the orthogonal components of the velocity, ξ_r and ξ_φ, and its polar coordinates is given by $\xi_r = \zeta\cos\omega$, $\xi_\varphi = \zeta\sin\omega$.

We now introduce non-dimensional variables, taking as the scales for the spatial coordinate r, velocity ξ, density n, temperature T, viscosity μ, and distribution function f, respectively,

$$r_1, \quad \sqrt{2RT_w}, \quad n_0, \quad T_w, \quad \frac{5}{16}mn_0\sqrt{2\pi RT_w}, \quad \lambda_0, \quad n_0(2RT_w)^{-3/2}, \quad n_0 = \frac{2}{r_2^2-r_1^2}\int\limits_{r_1}^{r_2} rn(r)dr \tag{1}$$

Here, n_0 is the mean density of the gas between the cylinders, m is the molecular mass, R is the gas constant, λ_0 is the free path in the gas at rest with density n_0 and temperature T_w. Although in the non-dimensional variables the radius of the inner cylinder is equal to unity, below the notation r_1 is retained for convenience of presentation. In what follows, we will denote the non-dimensional quantities by the same letters as the corresponding dimensional ones.

We will assume that the distribution function $f(t,r,\xi_r,\xi_\varphi,\xi_z)$ satisfies the Boltzmann equation with an S-model collision operator [7, 8]. Following [9], we reduce the problem dimension by going over from the distribution function to its integral $\boldsymbol{g} = (g_1,g_2,g_3,g_4)^T$ by means of the formula

$$\boldsymbol{g} = \int\limits_{-\infty}^{+\infty} \left(1, \xi_z, \xi_z^2, \xi_z^3\right)^T f\, d\xi_z$$

The kinetic equation for $\boldsymbol{g}$ in the cylindrical coordinate system can be written in the following conservative form [10]:

$$\frac{\partial}{\partial t}(r\boldsymbol{g}) = -\frac{\partial}{\partial r}(\boldsymbol{g}r\zeta\cos\omega) + \frac{\partial}{\partial\omega}(\boldsymbol{g}\zeta\sin\omega) + r\boldsymbol{H}, \quad \boldsymbol{H} = \nu(\mathbf{G}-\boldsymbol{g}) \tag{2}$$

where the components of G can be found from the gas macroparameters using the formulas

$$G_1 = g_M\left(1+B(S_rc_r+S_\varphi c_\varphi)(c_r^2+c_\varphi^2-2)\right), \quad G_3 = \frac{1}{2}Tg_M\left(1+B(S_rc_r+S_\varphi c_\varphi)(c_r^2+c_\varphi^2-1)\right)+2u_zG_2-u_z^2G_1,$$

$$G_2 = \frac{1}{2}\sqrt{T}g_MBS_z(c_r^2+c_\varphi^2-1)+u_zG_1, \quad G_4 = \frac{3}{4}T^{3/2}BS_z(c_r^2+c_\varphi^2)g_M+3u_z^3G_3-3u_z^2G_2+u_z^3G_1,$$

$$g_M = \frac{n}{\pi T}\exp(-c_r^2-c_\varphi^2), \quad (S_r,S_\phi,S_z) = \frac{2(q_r,q_\phi,q_r)}{nT^{3/2}}, \quad (c_r,c_\varphi) = \frac{1}{\sqrt{T}}(\xi_r,\xi_\varphi-u_\varphi), \quad B = \frac{4}{5}(1-\mathrm{Pr})$$

Here, $\mathrm{Kn} = \lambda_0/r_1$ is the Knudsen number, which determines the degree of gas rarefaction; $\mathrm{Pr} = 2/3$ is the Prandtl number. The gas density n, velocity u_φ, u_z, temperature T, heat fluxes q_r, q_φ, q_z and the shear stresses $P_{\varphi z}$, P_{rz} can be expressed in terms of the distribution function f in the form of integrals with respect to the molecular velocity

$$\left(n, nu_\varphi, nu_z, \frac{3}{2}nT+nu_\varphi^2+nu_z^2\right) = \int\limits_{-\pi}^{\pi}\int\limits_{0}^{\infty}\left(g_1, \xi_\phi g_1, g_2, \zeta^2g_1+g_3\right)\zeta\, d\zeta\, d\omega,$$

$$(q_r,q_\varphi) = \frac{1}{2}\int\limits_{-\pi}^{\pi}\int\limits_{0}^{\infty}(\xi_r,v_\varphi)\left[(\xi_r^2+v_\varphi^2)g_1-2u_zg_2+g_3\right]\zeta\, d\zeta\, d\omega, \quad P_{rz} = \int \xi_rg_2\zeta\, d\zeta\, d\omega, \tag{3}$$

$$q_z = nu_z^3+\frac{1}{2}\int\limits_{-\pi}^{\pi}\int\limits_{0}^{\infty}\left((\xi_r^2+v_\varphi^2)(g_2-u_zg_1)+(g_4-3u_zg_3)\right)\zeta\, d\zeta\, d\omega, \quad P_{r\varphi} = \int \xi_r\xi_\varphi g_1\zeta\, d\zeta\, d\omega$$

Note that, if $|U_z| > 0$, the longitudinal heat flux q_z is also nonzero [11]; therefore, we have to use g_4.

On the surface of the cylinders we assign the conditions of diffuse scattering of the molecules with total thermal accommodation to the surface temperature

$$\begin{aligned} r = r_1,\ \ \xi_r > 0: \quad \boldsymbol{g}_{w1} &= \frac{1}{\pi}n_{w1}e^{-\xi_r^2-(\xi_\varphi-U_\varphi)^2}\left(1, U_z, \tfrac{1}{2}+U_z^2, \tfrac{3}{2}U_z+2U_z^3\right)^T \\ r = r_2,\ \ \xi_r < 0: \quad \boldsymbol{g}_{w2} &= \frac{1}{\pi}n_{w2}e^{-\zeta^2}\left(1, 0, \frac{1}{2}, 0\right)^T \end{aligned} \tag{4}$$

where $U_\varphi = \Omega r_1$ and is taken into account that in the non-dimensional variables $T_w \equiv 1$. The densities of the reflected particles n_{w1} and n_{w2} can be found from the conventional impermeability condition.

Interesting integral characteristics of the problem considered are the gas flow-rates per unit time in longitudinal and azimuthal direction which with account for the normalization condition (1) are given by the integrals

$$Q_\varphi = 2\pi \int_{r_1}^{r_2} n u_\varphi r\, dr, \quad Q_z = 2\pi \int_{r_1}^{r_2} n u_z r\, dr$$

We will conclude this section with the following integral relations. which are the consequence of the kinetic equation:

$$r^2 P_{r\varphi} = \text{const} = \text{A}, \quad r P_{rz} = \text{const} = \text{B}, \quad r E_r = \text{const} = \text{C} \tag{5}$$

where E_r is the energy flux in the radial direction, The constants A, B and C depend on the degree of gas rarefaction, the ratio of the radii $\varepsilon = r_1/r_2$ and the dimensionless velocities U_z, U_φ. We used conditions (5) for checking the accuracy of the computations.

METHOD OF THE SOLUTION

The steady-state solution of the problem is found by marching in time to the steady state. Let us define $\Delta t = t^{n+1} - t^n$, $\boldsymbol{g}^n = \boldsymbol{g}(t^n, r, \omega, \zeta)$, $\boldsymbol{\delta} = r(\boldsymbol{g}^{n+1} - \boldsymbol{g}^n)$. A fully implicit difference scheme for kinetic equation (2) can be written in the following form [12, 13]:

$$\left(\frac{1}{\Delta t} + \zeta \frac{\partial}{\partial r} \cos\omega - \left(\frac{\zeta}{r} \right) \frac{\partial}{\partial \omega} \sin\omega + \nu \right) \boldsymbol{\delta} = \boldsymbol{K}^n, \quad \boldsymbol{K}^n = -\frac{\partial}{\partial r} \left(r\zeta \cos\omega\, \mathbf{g} \right)^n + \frac{\partial}{\partial \omega} \left(\zeta \sin\omega\, \mathbf{g} \right)^n + r\boldsymbol{H}^n \tag{6}$$

Next in variables r, ω we introduce a finite-volume mesh with cell centers r_i, ω_j that is refined toward the surfaces of the cylinders. With respect to ζ we use a uniform mesh with nodes ζ_k and cell size $\Delta\zeta$. The range of ω is divided into four subdomains depending on the sign of the molecular velocity components ξ_r, ξ_φ: $-\pi + m\pi/2 \le \omega \le -\pi/2 + m\pi/2$, where $m = 0, \ldots 3$. For each subdomain, the derivatives with respect to r and ω on the left-hand side of (6) are approximated by first-order accurate upwind differences. As a result, the distribution function at a new time level is determined according to a running scheme. For example, for $m = 1$, the computation is performed from left to right with respect to r by the formula

$$\left(\frac{1}{\Delta t} + B_{ijk} + C_{ij-1,k} + \nu_i \right) \boldsymbol{\delta}_{ijk} = B_{ijk} \boldsymbol{\delta}_{i-1,jk} + C_{ij-1,k} \boldsymbol{\delta}_{ij-1,k} + \boldsymbol{K}^n_{ijk}, \quad B_{ijk} = \frac{\zeta_k \cos\omega_j}{r_i - r_{i-1}}, \quad C_{ijk} = -\frac{\zeta_k \sin\omega_j}{r_i \Delta\omega} \tag{7}$$

where $\boldsymbol{\delta}_{ijk} = r_i(\mathbf{g}^{n+1}_{ijk} - \mathbf{g}^n_{ijk})$; $\mathbf{K}^n_{ijk}$ approximates the (time-independent) right-hand side of kinetic equation (2):

$$\boldsymbol{K}^n_{ijk} = r_i \boldsymbol{H}^n_{ijk} - \frac{\mathbf{X}^n_{i+1/2,jk} - \mathbf{X}^n_{i-1/2,jk}}{\Delta r_i} - \frac{\mathbf{Y}^n_{ij+1/2,k} - \mathbf{Y}^n_{ij-1/2,k}}{2\sin(\Delta\omega/2)} \tag{8}$$

where the numerical fluxes through the cell boundaries are determined by the formulas

$$\boldsymbol{X}^n_{i+1/2,jk} = \zeta_k \cos\omega_j r_{i+1/2}\, \mathbf{g}^n_{i+1/2,jk}, \quad \mathbf{Y}^n_{ij+1/2,k} = -\zeta_k \sin\omega_{j+1/2}\, \mathbf{g}^n_{ij+1/2,k}$$

On the surfaces of the cylinders, we set $\boldsymbol{\delta} \equiv 0$. The distribution function values on the boundaries of the cells $\mathbf{g}^n_{i+1/2,jk}$, $\mathbf{g}^n_{ij+1/2,k}$ are determined by monotone piecewise-linear interpolation [14], which ensures second-order accuracy on smooth solutions and guarantees that no spurious oscillations occur on the discontinuities. The macroscopic gas parameters, involved in $\mathbf{G}_{ijk}$, are determined in such a way that the mass, momentum, and energy conservation laws are satisfied. In order to achieve this, we use a direct approximation of the conditions used in deriving the model equation [15, 5]. For a fixed spatial position r this results in the following system for macroparameters (index i is omitted for simplicity):

$$\begin{aligned}
\sum_{jk} \left[H_1, \zeta \sin\omega H_1, H_2, \zeta^2 H_1 + H_3 \right]_{jk} \zeta_k A_{jk} &= \mathbf{0}, \\
\sum_{jk} (v_\varphi, v_r)_{jk} \left[(v_r^2 + v_\phi^2 + u_z^2) H_1 - 2u_z H_2 + H_3 \right]_{jk} \zeta_k A_{jk} &= -(4/3)\nu(q_r, q_\varphi), \\
\sum_{jk} \left[(v_r^2 + v_\phi^2 + u_z^2)(H_2 - u_z H_1) + H_4 - 3u_z H_3 + 2u_z^2 H_2 - u_z^3 H_1 \right]_{jk} \zeta_k A_{jk} &= -(4/3)\nu q_z
\end{aligned} \tag{9}$$

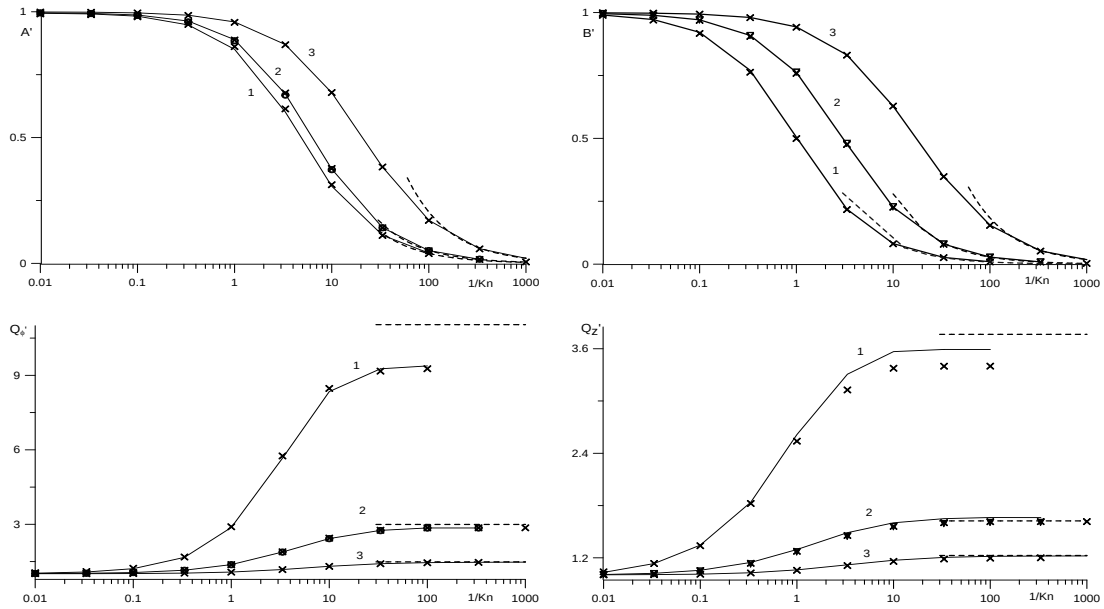

FIGURE 1. Results for $U_\varphi = U_z = 1/2$. Solid lines – spiral movement, circles – pure rotation, triangles – pure longitudinal movement, crosses – linearized solution, dashed lines – Navier-Stokes solution. Curves 1,2,3 correspond to $\varepsilon = 0.1$, 0.5, 0.9.

Here A_{jk} are the weights used in the quadrature formula. As a rule, the integrals with respect to ζ are evaluated by using the composite Simpson rule, while the integrals with respect to ω are evaluated by applying the second-order accurate midpoint formula involving the distribution function values at the cell centers ω_j. System (9) for the macroscopic gas parameters n, u_ϕ, u_z, T, $\mathbf{q}$ is easily solved by Newton's method.

The resulting numerical method has no time-step restrictions associated with the Courant condition or the approximation of the collision integral and as $t \to \infty$, it approximates the steady kinetic equation to second-order accuracy. The converged steady-state solution satisfies the discrete conservation laws for the density, the z-component of momentum, and energy. This property follows from the conservativeness of the approximation to the right-hand side of the scheme and from the midpoint formula used for evaluating the integrals with respect to ω.

RESULTS

All the results presented below were obtained for $\mu = \sqrt{T}$, which corresponds to molecular interaction in accordance with the hard-sphere law. The calculations were performed using nonlinear kinetic equation for the radius ratios $\varepsilon = r_1/r_2 = 0.1$, 0.5 and 0.9, which correspond to large, moderate, and small widths of the gap between the cylinders, respectively. We have also used the linearized kinetic equation and the incompressible Navier-Stokes equations for comparisons. The description of these equations is omitted here to save space. We note that for $\varepsilon \approx 1$ it is more natural to define the effective Knudsen number as $\mathrm{Kn}_1 = \lambda_0/(r_2 - r_1) = \varepsilon\,\mathrm{Kn}\,/(1-\varepsilon)$ whereas in the opposite case $\varepsilon << 1$ we can use $\mathrm{Kn}_2 = \lambda_0/r_2 = \varepsilon\,\mathrm{Kn}$ so that $\mathrm{Kn}_2 << \mathrm{Kn} << \mathrm{Kn}_1$.

The solution of the problem depends on four parameters: Kn, U_φ, U_z and ε. For the continuum solution as well as the linearized equations the gas flow is obtained by superposition of longitudinal and rotational motions which depend linearly on the cylinder velocity. In the case of the nonlinear kinetic equation the influences of U_φ and U_z cannot in general be separated except the free-molecular case, in which the longitudinal motion of the cylinder does not affect the flow characteristics corresponding to rotation: u_φ, $P_{r\varphi}$. The opposite is not true: u_z and P_{rz} depend in a nonlinear way on U_φ. One of the aims of the calculations was to study if the above-mentioned properties of the free-molecular solution hold for the finite Knudsen numbers as well.

Fig. 1 illustrates the dependence of shear stresses and mass flow rates, normalized by their free-molecular values, e.g. $A' = A/A_\infty$, on Kn for $U_\varphi = 1/2$, $U_z = 1/2$. Solid lines correspond to the spiral movement, circles show A', Q'_φ in the case of pure rotation ($U_z = 0$), triangles show B' Q'_z in the case of pure longitudinal movement ($U_\varphi = 0$), crosses

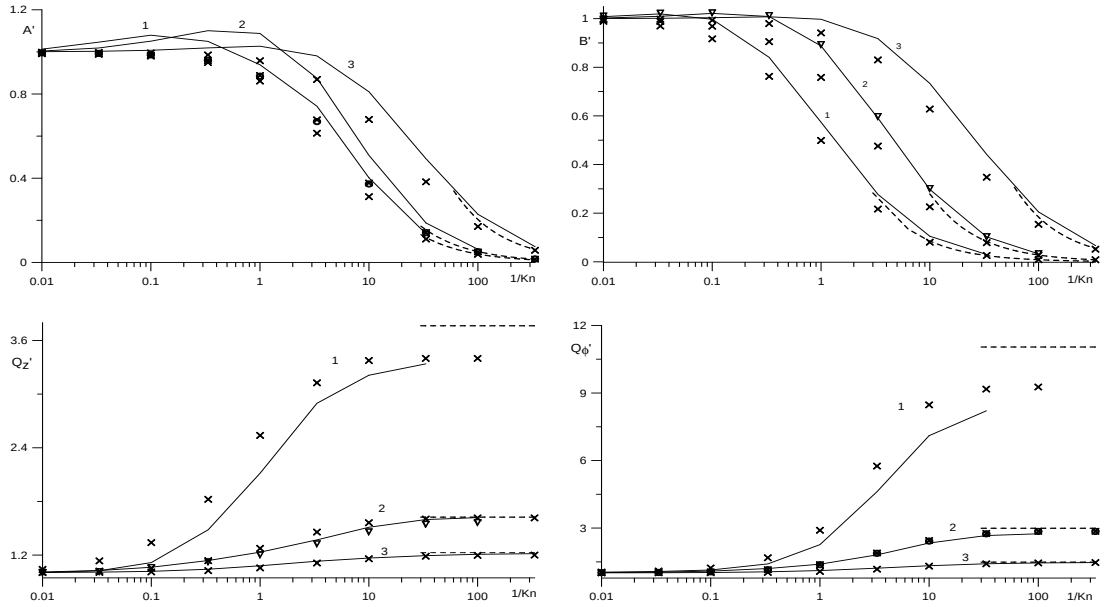

FIGURE 2. Results for $U_\varphi = 1/2$, $U_z = 4$. Notation as in Fig. 1.

depict solution of the linearized kinetic equation and dashed lines correspond to the incompressible Navier-Stokes solution. Curves 1,2,3 correspond to $\varepsilon = 0.1$, 0.5, 0.9. We see that overall the solution of the problem depends strongly on ε. Interestingly, the normalized shear stresses have the following properties:

$$A'(\mathrm{Kn}, \varepsilon, U_\varphi, U_z) = A'(\mathrm{Kn}, \varepsilon, U_\varphi, 0), \quad B'(\mathrm{Kn}, \varepsilon, U_\varphi, U_z) = B'(\mathrm{Kn}, \varepsilon, 0, U_z)$$

Moreover, there is a good agreement with the linearized solution in the whole range of the Knudsen numbers. The continuum solution is close to the kinetic one for small effective Knudsen numbers, defined above.

Dependence of mass flow rate on the cylinder movement is more complex. Similar to shear stresses, Q'_φ is not influenced by the value of U_z; nonlinear and linearized solutions agree reasonably well. However, the values of Q'_z, are affected by rotation of the cylinder, especially for small ε and Kn. Since in the linearized problem rotation and longitudinal movement of the cylinder do not influence each other, the linearized solution for Q'_z coincides with the nonlinear solution in the case $U_\varphi = 0$ but differs from the nonlinear solution for the spiral movement. The continuum solution for mass flow rates does not depend on Kn and for small ε does not agree with the kinetic one.

The further increase of U_z leads to strong nonlinearity of the solution. Fig. 2 illustrates the same quantities as in Fig. 1 but for $U_z = 4$, $U_\varphi = 1/2$. The most significant difference is the appearance of the stress maximum effect in both longitudinal and azimuthal directions: with decrease in the Knudsen number, A', B' first increase and then begin to fall. Surprisingly, the maximum in A' is much more pronounced than in B' and is achieved for smaller Knudsen numbers. We remark that for the plane Couette flow the stress maximum effect was first observed in [16]. We also not that the spiral nonlinear solution for shear stresses differs from both linearized and or pure rotational/longitudinal movement for all Knudsen numbers. On the other hand, the dependencies of the mass flow rates on Kn and ε, shown in Fig. 2 are qualitatively similar to those plotted in Fig. 1 an for $\varepsilon \geq 1/2$ agree well with the linearized solution.

Finally we investigate the dependence of gas macroparameters on ε and movement of the cylinder. Fig. 3 illustrate the distributions of density and temperature as functions of reduced spatial coordinate $x = (r - r_1)/(r_2 - r_1)$ for $\mathrm{Kn} = 0.01$, $U_z = 1/2$, $U_\varphi = 1/2$ and different ε. Solid lines correspond to the solution of the nonlinear kinetic equation, circles show the case of pure rotation, triangles – pure longitudinal flow. Curves 1,2,3 correspond to $\varepsilon = 0.1$, 0.5, 0.9. We can see a strong dependence of the results on ε as well as on the type of the cylinder movement. The velocity profiles (not shown here) are not sensitive to the latter factor; the linearized solution for velocity agrees well with the nonlinear one. However, the incompressible Navier-Stokes solution agrees with the kinetic one only for $\varepsilon = 0.1$ and 0.5, when the effective Knudsen number is sufficiently small.

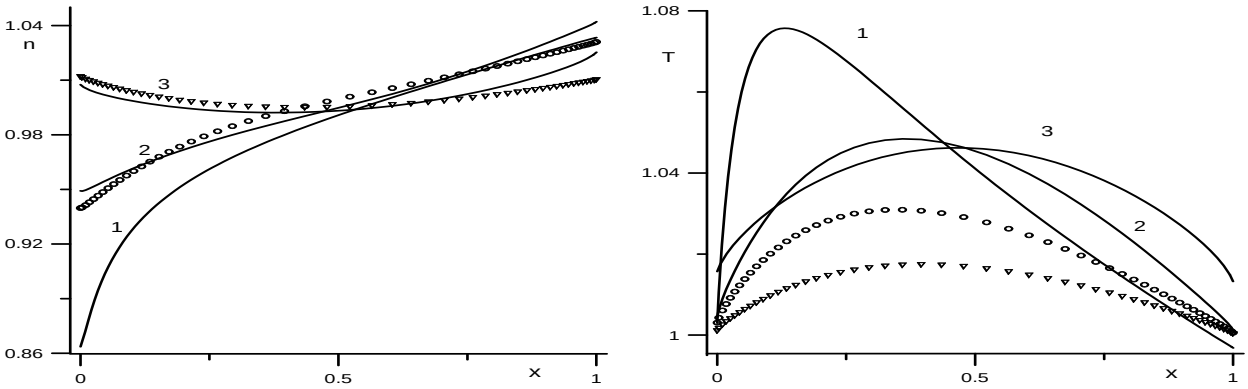

FIGURE 3. Gas macroparameters for $U_\varphi = U_z = 1/2$ and $\mathrm{Kn} = 0.01$.

CONCLUSIONS

We have presented a second order accurate non-oscillatory method for solving the S-model kinetic equation for the cylindrical Couette flow. The method is easy to code and is conservative with respect to the collision integral. We have applied the method to the Couette problem in the generalized spiral formulation in which the inner cylinder not only rotates but also slides along its axis. Our study also included the linearized kinetic equation as well as incompressible Navier-Stokes solution.

We have analyzed the influence of four main problem parameters: Knudsen number, two cylinder velocities as well as ratio of radiuses on shear stresses, mass flow rates and distribution of macroparameters. For large sliding velocities a stress maximum effect is observed in both azimuthal and longitudinal directions: with decrease in the Knudsen number the normalized shear stresses first increase and then begin to fall.

ACKNOWLEDGMENTS

The first author acknowledges the partial support of the Russian Foundation for Basic Research (project No. 04-01-00347). The second author acknowledges the financial support provided by the PRIN programme (2004-2006) of Italian Ministry of Education and Research (MIUR). He would also like to thank Professor E.F. Toro OBE for numerous discussions on the construction of the Godunov-type methods.

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