A Method of Joint Solution of the Boltzmann and Navier-Stokes Equations with Application to the Problems of Gas Flows over Plane Plates

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Abstract. For numerical simulation of rarefied gas flows at small and moderate Knudsen numbers we developed a method in which thermodynamically highly non equilibrium zones of the flow are computed by the Boltzmann kinetic equation and the near equilibrium zones by the Navier-Stokes equations. In the paper a $2D$ supersonic flow over a plane plate is studied. The fields of hydrodynamic parameters and the aerodynamic reactions on the plate are reported.

Keywords: Numerical Method, Boltzmann Equation, Navier-Stokes Equations, Rarefied Gas Flow, Plane Plate
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INTRODUCTION

At present time, solving of the Boltzmann kinetic equation in a wide range of flow parameters does not present too much difficulty and can be performed on a common PC [1, 2]. However, it requires sufficiently more resources than solution of Euler or Navier-Stokes equations. In case of small Knudsen numbers, complex geometry of the flow, and when one is interested mainly in processes that occur at large distances outside of boundary layers, it is reasonable to apply in the low perturbed area the asymptotic consequence of the Boltzmann equation, namely the Navier-Stokes equations. The method in which thermodynamically highly non equilibrium zones of the flow are computed by the Boltzmann kinetic equation and near equilibrium area by Navier-Stokes equations was proposed in [3, 4]. The equations are solved by a finite-difference method on a fixed coordinate grid. For solving the Boltzmann equation a fixed velocity grid is used. The evaluation of the collision integral is made by a projection method [1] that preserves strict fulfillment of conservation laws and null the integral from a local Maxwellian function. At each time step of the algorithm the kinetic equation provides boundary conditions for Navier-Stokes equations in a form of fluxes of mass, impulse, and energy through interfaces of the domains and the solution of hydrodynamic equations defines boundary conditions for the Boltzmann equation in a form of Chapman-Enskog distribution function. In numerical realization of the method the fluxes of mass, impulse, and energy through an interface are kept continuous. The considered approach was applied to different study of rarefied gas flows [3-6].

MATCHING PROCEDURE

We apply the time-dependant method of solution of the Boltzmann and Navier-Stokes equations. The Boltzmann equation is solved by splitting at a free molecular equation (a) and a relaxation equation (b) on a time interval $\tau << \tau_0$, τ_0 being the mean molecular inter collision time.

$$\partial f / \partial t + \xi \cdot \partial f / \partial x = 0 \qquad \text{(a)}$$

$$\partial f / \partial t = I(f, f) \qquad \text{(b)}$$

Here $I(f, f)$ is the Boltzmann collision integral. The equation (a) is solved by an explicit second order finite-difference scheme of SHASTA type [7]. In the Navier-Stokes equations the convective terms are approximated by the similar explicit scheme, and for the dissipative terms an implicit second order scheme is applied.

The matching procedure on a rectangular grid x_k, y_l is explained in **Figure 1**. The boundary at $x_{i+1/2}$ for a fixed ***k=i*** separates the Navier-Stokes domain at the left side from the kinetic domain at the right side. The matching points are marked by circles. To compute the Boltzmann equation in the layer ***(i+1, l),*** one should have a distribution function $f^{+}_{i+1/2,l}$ for all positive molecular velocities. We choose it in a form of the Chapman-Enskog function, in witch hydrodynamic parameters n, u, v, T are taken in the layer ***(i,l)*** and the derivatives $\partial u / \partial x, \partial u / \partial y, \partial v / \partial x, \partial v / \partial y, \partial T / \partial x, \partial T / \partial y$ are calculated using hydrodynamic parameters in layers with first indices ***i-1, i, i+1/2***. In these calculations the values of $n_{i+1/2,l}, u_{i+1/2,l}, v_{i+1/2,l}, T_{i+1/2,l}$ are taken from the lower time layer. The leading term of the Chapman-Enskog function – the Maxwellian function - is determined by flow parameters in the Navier-Stokes area.

Next, the composite distribution function $f_{i+1/2,l} = f^{+}_{i+1/2,l} + f^{-}_{i+1/2,l}$ is constructed in the layer ***(i+1/2, l)*** where $f^{-}_{i+1/2,l}$ presents the solution to the Boltzmann equation at the node ***(i+1, l)*** for negatives molecular velocities. This function is used to calculate the new values $n_{i+1/2,l}, u_{i+1/2,l}, v_{i+1/2,l}, T_{i+1/2,l}$, which determine the boundary conditions for the Navier-Stokes equations. These values are then used to obtain the fluxes of flow variables along the ***x***-axis, which are required to compute the convective terms of Navier-Stokes equations, and for to find the values of partial derivatives and mixed partial derivatives in the dissipative terms near the boundary.

The details of numerical methods for solving the Boltzmann and Navier-Stokes equations and of the matching procedure are described in [3].

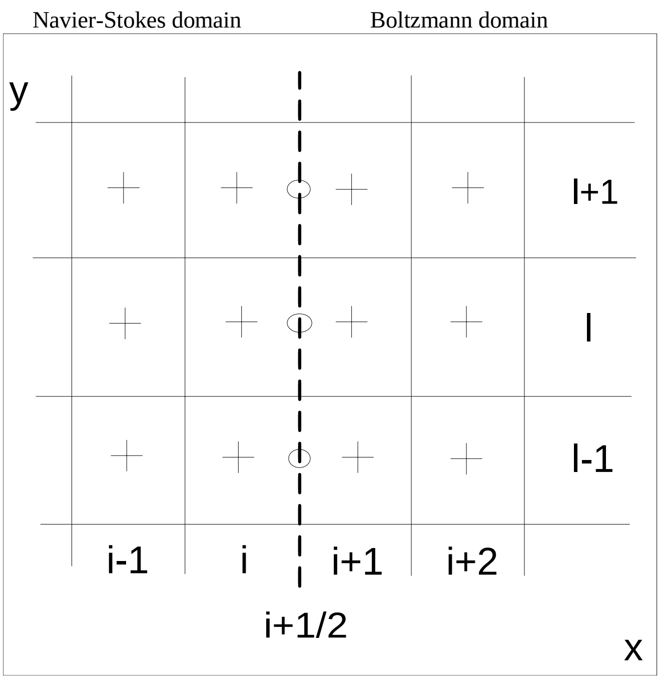

FIGURE 1. Scheme of Matching of Boltzmann and Navier-Stokes Solutions

RESULTS OF COMPUTATIONS

We apply the same combined Boltzmann – Navier-Stokes solver that is described in [4]. As an example consider a $2D$ supersonic flow with $M = 2.8$ and $Kn = 0.1$ over a plane plate at different angles of attack α which vary from 0 to $\pi/2$. The inverse power molecular potential with the exponent $1/12$ is taken. The wall temperature is equal to that of incoming flow, and diffuse molecular reflection with perfect accommodation at the surface is imposed.

In the Figures 2-5 density fields for the angles of attack $\alpha = \{0, 30, 45, 87\}$ are presented. The domain of the Boltzmann equation solution is shown in the Figure 2 by the grey rectangle.

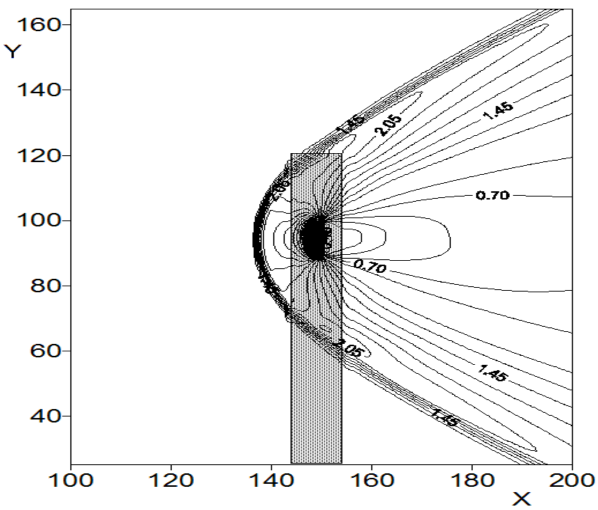

FIGURE 2. Density field for $\alpha = 0$

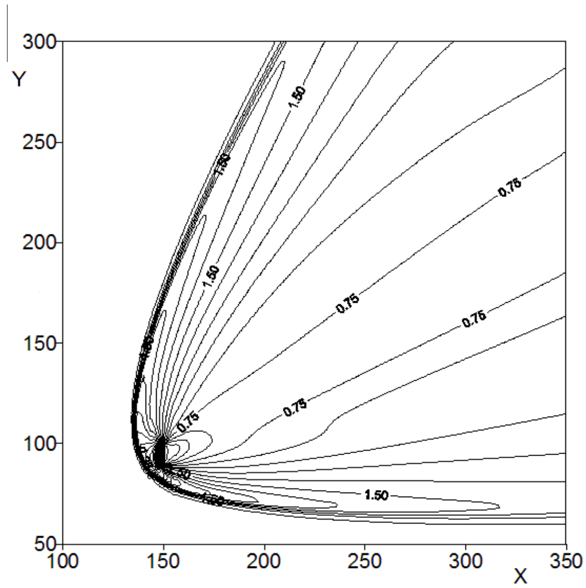

FIGURE 3. Density field for $\alpha = 30^{\circ}$

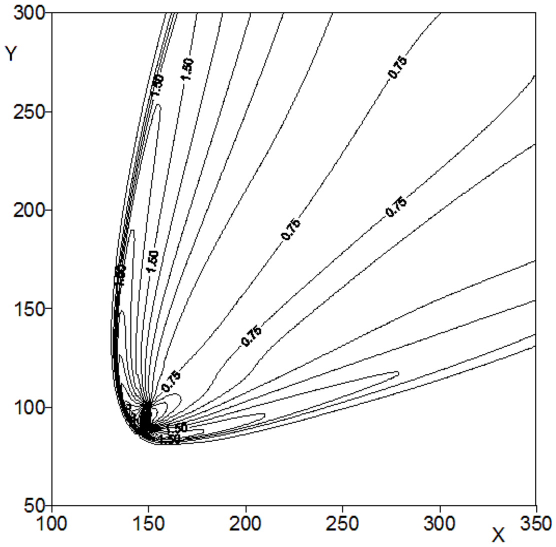

FIGURE 4. Density field for $\alpha = 45^{\circ}$

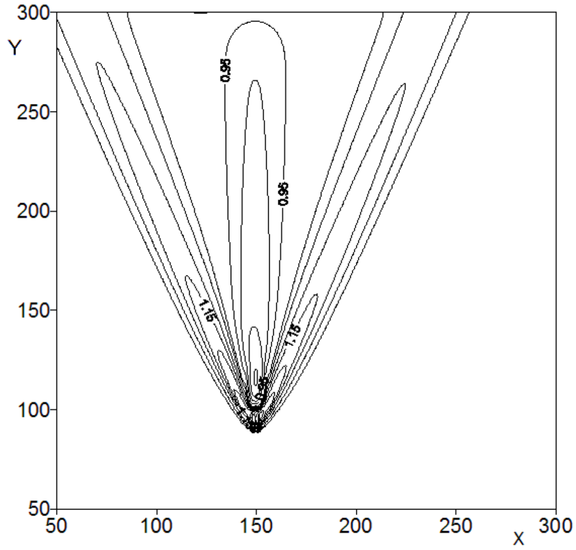

FIGURE 5. Density field for $\alpha = 87^{\circ}$

The Boltzmann equation is solved near the plate inside a rectangular area at a Cartesian grid having 10×120 nodes. To check the influence of the domain decomposition this area is located not symmetrically to the plate. The number of the grid points in velocity space is approximately equal to $(\pi/3) \cdot 16 \times 16 \times 8$ nodes. In the outer area the Navier-Stokes equation is solved at a grid with 350×350 nodes. The steps in the configuration space for both kinetic and Navier-Stokes domains are $\Delta x = \Delta y = \lambda_\infty$, λ_∞ being the molecular mean free path at the incoming flow. The matching of solutions is made with time step $dt = 0.1\tau_\infty$, τ_∞ being the mean time of molecular free path. The steady solution inside all computation area is obtained after $(300 \div 400)\tau_\infty$.

The distances are expressed in molecular mean path λ_∞, and the density is reported to its value at the incoming flow. It is seen the shock wave in front of the plate and rarefaction zone behind the plate. The shock wave is mostly located inside the Navier-Stokes zone. The interface between the kinetic and the hydrodynamic domains is practically not visible, except the area where it is crossed by the shock wave where small disturbances of the flow field take place. In Figure 6 the fields of density, temperature, longitudinal flow velocity **u**, and transversal velocity **v** are shown for the case $\alpha = 60^\circ$.

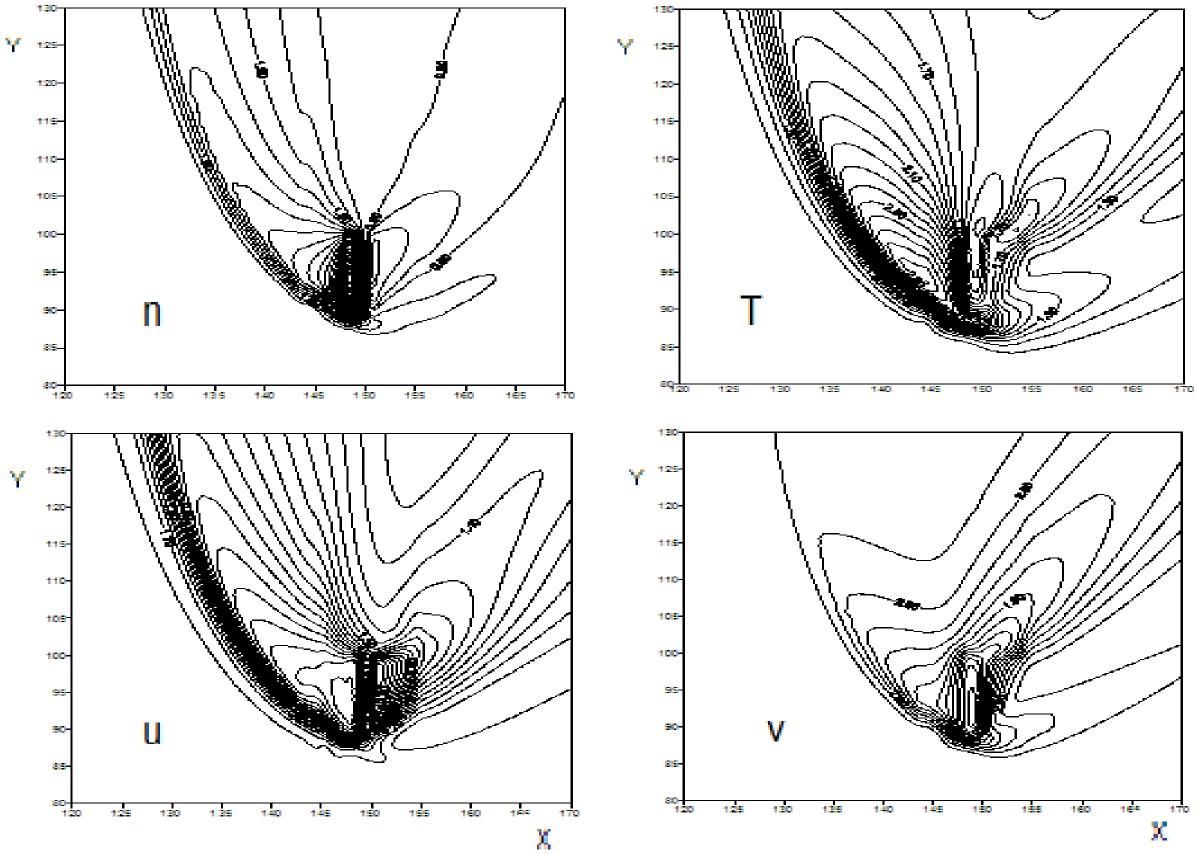

FIGURE 6. Detailed flow field for $\alpha = 60^\circ$

In Figure 7 the aerodynamic reactions on the plate: the drag coefficient C_x, the friction coefficient C_f , and the heat flux coefficient C_q are presented. They are defined as follows:

$$C_x = F_x / (n_\infty m u_\infty^2 L)\,,\quad C_f = F_y / (n_\infty m u_\infty^2 L)\,,\quad C_q = 2Q / (n_\infty m u_\infty^3 L)$$

The forces and the heat flux are defined as the integrals along the plate length by both sides of the plate

$$F_x = \int_0^L (p_{xx}^{(1)} - p_{xx}^{(2)})dy\,,\; F_y = \int_0^L (p_{xy}^{(1)} - p_{xy}^{(2)})dy\,,\; Q = \int_0^L (q^{(1)} - q^{(2)})dy$$

Here m is the molecular mass, n_∞, u_∞ are the density and the velocity of the incoming flow, L is the length of the plate, and p_{xx}, p_{xy}, q are components of the stress tensor and the projection of the heat flux at the normal to the plate. The upper indices mark the front and the rear sides of the plate. The values p_{xx}, p_{xy}, q are defined by the computed velocity distribution function at the plate.

The drag force approximately follows the law $C_x \sim \cos\alpha$, when the other reactions depend on the angle of attack nearly linearly. The heat flux for normally posed plate is about twice higher then for the longitudinal one.

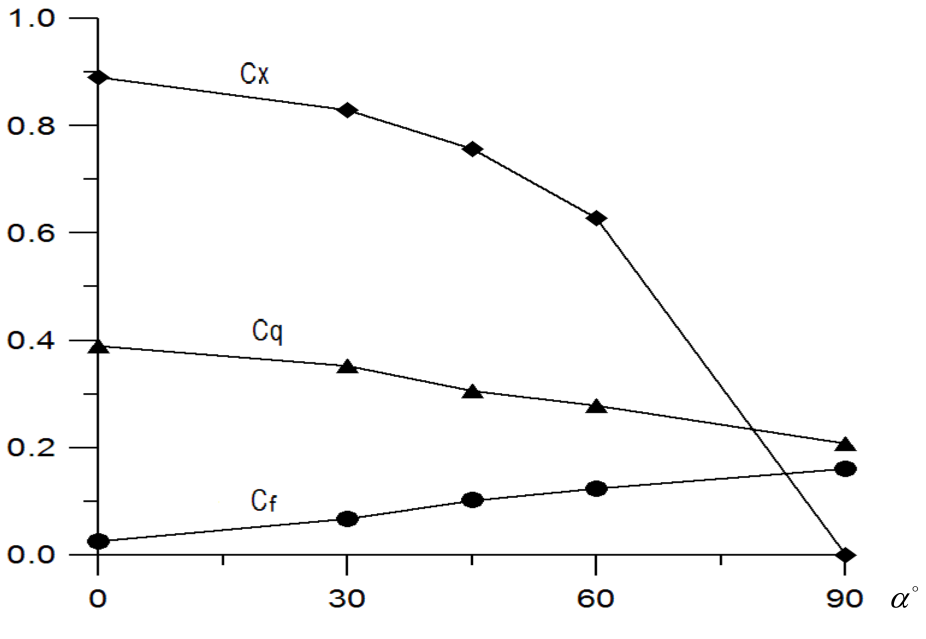

FIGURE 7. Aerodynamic reactions at the plate

The CPU time at PC Pentium 4, 3.06 Ghz was about 2h per each presented case, the used memory was about 50 Mb. The kinetic domain occupied less then 10% of the total computational area, but the time of solution of the Boltzmann equation took nearly 95% of the total CPU time.

CONCLUSION

We considered the supersonic rarefied gas flow about an inclined plate for different angles of attack at moderately small Knudsen number. For all the inclinations of the plate the flow is characterized by formation of a narrow highly non equilibrium zone near the plate and a large area where the flow can be simulated by the hydrodynamic equations. The solution of the Boltzmann kinetic equation in the first zone gives the boundary conditions for the outer area and provides aerodynamic reactions at the plate. It was shown that the application of the combined Boltzmann – Navier-Stokes solver developed by the authors permits to compute the flow field without visible discontinuities at the matching interfaces between the kinetic and the continuum flow domains. The use of the proposed approach considerably reduces the required computational resources. The gain in the efficiency of computations shall increase with the diminishing of the Knudsen number.

ACKNOWLEDGMENTS

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