Divergent And Stochastic Combined Forms Of The Boltzmann's Collision Integral In Numerical Simulation

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Abstract. A method for modeling a gas of colliding molecules by a finite system of quasi-particles, which are moving with variable weight along continues trajectories is proposed.

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A description of the rarefied gas dynamics is based on the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\partial}{\partial \boldsymbol{v}} \cdot \frac{\boldsymbol{F}}{m} f = I(f, f), \tag{1}$$

where $f(\boldsymbol{v}, \boldsymbol{r}, t)$ is a distribution function, $\boldsymbol{F}$ is an external force, m is a mass of molecule and $I(f, f)$ is the Boltzmann's collision integral [1]. Recently, the two independent approaches to rewrite identically Boltzmann's collision integral in a divergence form had been developed [2,3]. In this paper we follow the approach [3-5], which is based on the parameterization of a collision by a rotation matrix and utilization of the powerful Lie group constructions. Representation of the Boltzmann's collision integral in the divergence form

$$I(f, \psi) = -\frac{\partial}{\partial \boldsymbol{v}} \cdot \boldsymbol{J}(f, \psi) \tag{2}$$

opens new opportunities to construct kinetic models with finite degrees of freedom. There is a serious problem in numerical simulations of the Boltzmann equation: the finite sets of velocities pairs $(\boldsymbol{v}_i, \boldsymbol{v}_k)$ generally are not invariant under the mapping $(\boldsymbol{v}_i, \boldsymbol{v}_k) \to (\boldsymbol{v}'_i, \boldsymbol{v}'_k)$ induced by two particles collision. To account this problem in accurate numerical methods special attention is paid for providing energy conservation in the system of colliding particles [6, 7].

To explain the suggested method we first consider the space-homogeneous Boltzmann equation:

$$\frac{\partial f}{\partial t} = I(f, f). \tag{3}$$

The idea of our approach is to present the collision integral in a divergent and stochastic combined form. In accordance with eq. (2), we can write:

$$I_{comb}(f, f) = \gamma I(f, f) - (1 - \gamma) \frac{\partial}{\partial \boldsymbol{v}} \cdot \boldsymbol{J}(f, f), \tag{4}$$

where $\gamma = \gamma(\boldsymbol{v}, t)$ is an arbitrary function on velocity and time. In this case, equation (3) takes the following form:

$$\frac{\partial f}{\partial t} = I_{comb}(f, f). \tag{5}$$

Then we present the velocity distribution function $f(\boldsymbol{v},t)$ as a superposition of moving model quasi-particles with time dependent weights and velocities:

$$f(\boldsymbol{v},t) = \sum_{i=1}^{N} n_i(t)\delta(\boldsymbol{v} - \boldsymbol{v}_i(t))\,, \tag{6}$$

Eq.(6) means that the system with enormous number of real molecules $\left(N_A \sim 10^{23}\right)$ is substituted by finite system of quasi-particles. The quasi-particles are moving along smooth trajectories in the phase space with the time dependent weights $n_i(t)$ and velocities $\boldsymbol{v}_i(t)$.

Matrix elements of the flow $\boldsymbol{J}$ are given by [2]:

$$\begin{gathered}\boldsymbol{J}\left(n_i\delta(\boldsymbol{v}-\boldsymbol{v}_i), n_k\delta(\boldsymbol{v}-\boldsymbol{v}_k)\right) \\ = \frac{n_i n_k}{2}\int\frac{d\phi}{\pi}d\Omega_{\boldsymbol{n}}d\alpha\,(1-\cos\phi)\,\phi\, b\left(\mathrm{v}'_\alpha, \mu'_\alpha\right)\left[\boldsymbol{n}\times\mathbf{v}'_\alpha\right]\delta(\boldsymbol{v}-\boldsymbol{v}'_\alpha) \\ = \int\frac{d\phi}{\pi}d\Omega_{\boldsymbol{n}}d\alpha\,\frac{n_i n_k\,(1-\cos\phi)}{2}\phi\, b\,(\mathrm{v},\mu)\left[\cos\alpha\phi + (\sin\alpha\phi)\,\boldsymbol{n}\times\right]\boldsymbol{n}\times\mathbf{v}\,\delta(\boldsymbol{v}-\boldsymbol{v}'_\alpha),\end{gathered} \tag{7}$$

where

$$\boldsymbol{v}'_\alpha = \boldsymbol{v}_i + \frac{\left[(1-\cos\alpha\phi)\,\hat{n}^2 + (\sin\alpha\phi)\,\hat{n}\right](\boldsymbol{v}_i - \boldsymbol{v}_k)}{2}, \tag{8}$$

$$\boldsymbol{u}'_\alpha = \boldsymbol{v}_k - \frac{\left[(1-\cos\alpha\phi)\,\hat{n}^2 + (\sin\alpha\phi)\,\hat{n}\right](\boldsymbol{v}_i - \boldsymbol{v}_k)}{2}. \tag{9}$$

$$\mathbf{v}'_\alpha = \boldsymbol{v}'_\alpha - \boldsymbol{u}'_\alpha, \quad \mathbf{v} = \boldsymbol{v}_i - \boldsymbol{u}_k, \quad \hat{n}\mathbf{v} \overset{df}{=} \boldsymbol{n}\times\mathbf{v}, \tag{10}$$

$$\begin{gathered}\mu'_\alpha = \frac{\mathbf{v}'_\alpha\cdot\hat{R}\mathbf{v}'_\alpha}{\mathrm{v}'^{\,2}_\alpha} = \mu_\alpha = \frac{\mathbf{v}\cdot\hat{R}\mathbf{v}}{\mathrm{v}^2} \\ = 1 - \frac{(1-\cos\alpha\phi)\left[\boldsymbol{n}\times\mathbf{v}'_\alpha\right]^2}{\mathrm{v}'^{\,2}_\alpha} = 1 - \frac{(1-\cos\alpha\phi)\left[\boldsymbol{n}\times\mathbf{v}\right]^2}{\mathrm{v}^2}.\end{gathered} \tag{11}$$

The unit vector $\boldsymbol{n}$ is a direction of the rotation axis, ϕ is an angle of rotation, α is a parameter of collision weakening. The other notations are standard. A collision is defined by the following set of parameters:

$$\boldsymbol{n} \in \left\{\boldsymbol{n}^2 = 1\right\};\quad 0 \le \phi \le \pi;\quad 0 \le \alpha \le 1. \tag{12}$$

In the similar equation for matrix elements of the stochastic form of the Boltzmann's collision integral $I(f,f)$ [2] the post-collisional velocities do not depend on weakening parameter α:

$$\begin{gathered}I\left(n_i\delta(\boldsymbol{v}-\boldsymbol{v}_i), n_k\delta(\boldsymbol{v}-\boldsymbol{v}_k)\right) \\ = n_i n_k\int\frac{d\phi}{\pi}d\Omega_{\boldsymbol{n}}(1-\cos\phi)b\left(\mathrm{v}',\mu'\right)\left[\delta\left(\boldsymbol{v}-\boldsymbol{v}'\right) - \delta(\boldsymbol{v}-\boldsymbol{v}_i)\right] \\ = n_i n_k\int\frac{d\phi}{\pi}d\Omega_{\boldsymbol{n}}(1-\cos\phi)b\,(\mathrm{v},\mu)\left[\delta\left(\boldsymbol{v}-\boldsymbol{v}'\right) - \delta(\boldsymbol{v}-\boldsymbol{v}_i)\right],\end{gathered} \tag{13}$$

where

$$\boldsymbol{v}' = \boldsymbol{v}_i + \frac{\left[(1-\cos\phi)\,\hat{n}^2 + (\sin\phi)\,\hat{n}\right](\boldsymbol{v}_i - \boldsymbol{v}_k)}{2}, \tag{14}$$

$$\boldsymbol{u}' = \boldsymbol{v}_k - \frac{\left[(1-\cos\phi)\,\hat{n}^2 + (\sin\phi)\,\hat{n}\right](\boldsymbol{v}_i - \boldsymbol{v}_k)}{2}. \tag{15}$$

$$\mathbf{v}' = \boldsymbol{v}' - \boldsymbol{u}', \qquad \mathbf{v} = \boldsymbol{v}_i - \boldsymbol{u}_k, \quad \hat{n}\mathbf{v} \overset{df}{=} \boldsymbol{n} \times \mathbf{v}, \tag{16}$$

$$\begin{aligned} \mu' &= \frac{\mathbf{v}' \cdot \hat{R}\mathbf{v}'}{\mathrm{v}'^2} = \mu = \frac{\mathbf{v} \cdot \hat{R}\mathbf{v}}{\mathrm{v}^2} \\ &= 1 - \frac{(1-\cos\phi)\left[\boldsymbol{n} \times \mathbf{v}'\right]^2}{\mathrm{v}'^2} = 1 - \frac{(1-\cos\phi)\left[\boldsymbol{n} \times \mathbf{v}\right]^2}{\mathrm{v}^2}. \end{aligned} \tag{17}$$

For discretization of eq. (5), we substitute the integrals in eq. (7) and eq.(13) by the Monte Carlo sums according to the following rules:

$$\sum_{ik} \int d\alpha \frac{d\phi}{\pi} d\Omega_{\boldsymbol{n}} \{\ldots\} \to \left(\sum_{ik} \int d\alpha \frac{d\phi}{\pi} d\Omega_{\boldsymbol{n}} \right) \frac{1}{N_{coll}} \sum_{coll} \{\ldots\} = \frac{4\pi N^2}{N_{coll}} \sum_{coll} \{\ldots\}, \tag{18}$$

$$\sum_{ik} \int \frac{d\phi}{\pi} d\Omega_{\boldsymbol{n}} \{\ldots\} \to \left(\sum_{ik} \int \frac{d\phi}{\pi} d\Omega_{\boldsymbol{n}} \right) \frac{1}{N_{coll}} \sum_{coll} \{\ldots\} = \frac{4\pi N^2}{N_{coll}} \sum_{coll} \{\ldots\}, \tag{19}$$

where $\sum\limits_{coll} \{\ldots\}$ is the sum on N_{coll} collisions. A collision in eqs. (18)-(19) is specified by setting the following parameters: $\left\{(i,k),\, 1 \le i,k \le N\right\};\ \boldsymbol{n} \in S_2;\ 0 \le \phi \le \pi; 0 \le \alpha \le 1$.

It should be noted, that in the standard construction of the two-particles collision integral the velocity change is considered for the "bombarding" particle only. So that, each molecule is accounted twice, the first time as "bombarding" and the second time as a "target" one. Therefore, we characterize the collision in eqs.(18)-(19) by ordered pair of indexes (i,k), where the "bombarding" particle number is on the first place and the "target" particle number is on the second. That is why, N_{coll} is the number of ordered collisions and is equal to doubled number of the acts of the real physical collisions.

To proceed further, we divide the velocity space into the Wigner-Seitz Cells $\Delta(\boldsymbol{v}_j),\, 1 \le j \le N$. The Wigner-Seitz Cell $\Delta(\boldsymbol{v}_j)$ around a velocity knot $\boldsymbol{v}_j$ is defined as the locus of points in the velocity space, which are closer to that knot than any of the other knots. In the sum for the "Gain" term of stochastic collision integral (20), we group the collisions in accordance with falling the velocities $\boldsymbol{v}'$ into the Wigner-Seitz Cells $\Delta(\boldsymbol{v}_j)$. That is, we present the sum on collisions in the form $\sum\limits_{coll} = \sum\limits_{j} \sum\limits_{\boldsymbol{v}' \in \Delta(\boldsymbol{v}_j)}$, where $\sum\limits_{\boldsymbol{v}' \in \Delta(\boldsymbol{v}_j)}$ includes only those collisions, due to which the post-collisional velocities of "bombarding" particles fall into the cell $\Delta(\boldsymbol{v}_j)$.

In the "Loss" term we separate collisions on the groups $\sum\limits_{coll} = \sum\limits_{j} \sum\limits_{coll(\boldsymbol{v}_i = \boldsymbol{v}_j)}$, in which the "bombarding" particle velocity is equal to $\boldsymbol{v}_j$. Therefore, the sum $\sum\limits_{coll(\boldsymbol{v}_i = \boldsymbol{v}_j)}$ includes those collisions, in which the velocity of "bombarding" particle is equal to $\boldsymbol{v}_j$. As a result, we arrive to the expression:

$$\begin{aligned} I(f,f) &= \sum_{ik} I\left(n_i \delta(\boldsymbol{v} - \boldsymbol{v}_i), n_k \delta(\boldsymbol{v} - \boldsymbol{v}_k)\right) = \sum_{coll} \left[\nu' n_i \delta(\boldsymbol{v} - \boldsymbol{v}') - \nu n_i \delta(\boldsymbol{v} - \boldsymbol{v}_i)\right] \\ &= -\sum_j \left(\sum_{coll(\boldsymbol{v}_i = \boldsymbol{v}_j)} \nu \right) n_i \delta(\boldsymbol{v} - \boldsymbol{v}_i) + \sum_j \sum_{\boldsymbol{v}' \in \Delta(\boldsymbol{v}_j)} \nu' n_i \delta(\boldsymbol{v} - \boldsymbol{v}') \\ &= -\sum_j \overline{\nu}_j n_j \delta(\boldsymbol{v} - \boldsymbol{v}_j) + \sum_j \sum_{\boldsymbol{v}' \in \Delta(\boldsymbol{v}_j)} \nu' n_i \delta(\boldsymbol{v} - \boldsymbol{v}'), \end{aligned} \tag{20}$$

where $\overline{\nu}_j$ - is a total collision frequency of particle (j) with the other ones:

$$\overline{\nu}_j = \sum_{coll(\boldsymbol{v}_i=\boldsymbol{v}_j)} \nu\,, \tag{21}$$

$$\nu = \nu' = \frac{4\pi N^2 n_k}{N_{coll}}(1-\cos\phi) b(\mathrm{v}', \mu') = \frac{4\pi N^2 n_k}{N_{coll}}(1-\cos\phi) b(\mathrm{v}, \mu)\,. \tag{22}$$

Similarly we construct the expression for the flow in the velocity space:

$$\begin{aligned}\boldsymbol{J}(f,f) &= \sum_{ik} \boldsymbol{J}\left(n_i\delta(\boldsymbol{v}-\boldsymbol{v}_i), n_k\delta(\boldsymbol{v}-\boldsymbol{v}_k)\right) = \sum_{coll} \boldsymbol{a}' n_i \delta(\boldsymbol{v}-\boldsymbol{v}'_\alpha) \\ &= \sum_j \sum_{\boldsymbol{v}'_\alpha \in \Delta(\boldsymbol{v}_j)} \boldsymbol{a}' n_i \delta(\boldsymbol{v}-\boldsymbol{v}'_\alpha),\end{aligned} \tag{23}$$

where

$$\begin{aligned}\boldsymbol{a}' &= \frac{4\pi N^2 n_k}{N_{coll}} \frac{(1-\cos\phi)}{2} \phi\, b(\mathrm{v}'_\alpha, \mu')[\boldsymbol{n}\times \mathbf{v}'_\alpha] = \hat{R}\boldsymbol{a} \\ &= \frac{4\pi N^2 n_k}{N_{coll}} \frac{(1-\cos\phi)}{2} \phi\, b(\mathrm{v}, \mu)[\cos\alpha\phi + (\sin\alpha\phi)\boldsymbol{n}\times]\boldsymbol{n}\times\mathbf{v}.\end{aligned} \tag{24}$$

We choose an arbitrary function $\gamma(\boldsymbol{v},t)$ to be constant and equal to γ_i inside each Wigner-Seitz Cell:

$$\gamma(\boldsymbol{v},t) = \gamma_j;\ \ \boldsymbol{v}\in\Delta(\boldsymbol{v}_j) \tag{25}$$

By adding (20) to (23) we have an appraisal for combined Boltzmann's collision integral in the form:

$$\gamma I(f,\psi) - (1-\gamma)\frac{\partial}{\partial \boldsymbol{v}}\cdot \boldsymbol{J}(f,\psi) \tag{26}$$

$$\begin{aligned}= \sum_j \Bigg\{ &-\gamma_j \overline{\nu}_j n_j \delta(\boldsymbol{v}-\boldsymbol{v}_j) + \gamma_j \sum_{\boldsymbol{v}'\in\Delta(\boldsymbol{v}_j)} \nu' n_i \delta(\boldsymbol{v}-\boldsymbol{v}') \\ &- (1-\gamma_j) \sum_{\boldsymbol{v}'_\alpha\in\Delta(\boldsymbol{v}_j)} \frac{\partial}{\partial \boldsymbol{v}}\cdot \boldsymbol{a}'\, n_i\delta(\boldsymbol{v}-\boldsymbol{v}'_\alpha)\Bigg\}.\end{aligned}$$

Then we substitute the sum of delta-functions located in the cell $\Delta(\boldsymbol{v}_j)$ by the delta-function concentrated in the cell center $\boldsymbol{v}_j$ by introducing indefinite coefficients ν_j and $\boldsymbol{a}_j$:

$$\begin{aligned}&-\gamma_j \overline{\nu}_j n_j \delta(\boldsymbol{v}-\boldsymbol{v}_j) + \gamma_j \sum_{\boldsymbol{v}'\in\Delta(\boldsymbol{v}_j)} \nu' n_i \delta(\boldsymbol{v}-\boldsymbol{v}') \\ &-(1-\gamma_j) \sum_{\boldsymbol{v}'_\alpha\in\Delta(\boldsymbol{v}_j)} \frac{\partial}{\partial \boldsymbol{v}}\cdot \boldsymbol{a}'\, n_i\delta(\boldsymbol{v}-\boldsymbol{v}'_\alpha) \\ &\approx \nu_j n_j \delta(\boldsymbol{v}-\boldsymbol{v}_j) - \frac{\partial}{\partial \boldsymbol{v}}\cdot \boldsymbol{a}_j n_j \delta(\boldsymbol{v}-\boldsymbol{v}_j).\end{aligned} \tag{27}$$

We demand the number of particles, momentum and energy conservation in the each cell $\Delta(\boldsymbol{v}_j)$ so that, we multiply the both parts of eq.(27) by 1, $\boldsymbol{v}$ and $\boldsymbol{v}^2$ and integrate on $d\boldsymbol{v}$. As a result, we obtain the following three equations for $\nu_j, \boldsymbol{a}_j, \gamma_j$:

$$\gamma_j \left(-\overline{\nu}_j + \sum_{\boldsymbol{v}'\in\Delta(\boldsymbol{v}_j)} \frac{n_i}{n_j}\nu' \right) = \nu_j\,, \tag{28}$$

$$\gamma_j \left(-\overline{\nu}_j \boldsymbol{v}_j + \sum_{\boldsymbol{v}'\in\Delta(\boldsymbol{v}_j)} \frac{n_i}{n_j}\nu' \boldsymbol{v}' \right) + (1-\gamma_j) \sum_{\boldsymbol{v}'_\alpha\in\Delta(\boldsymbol{v}_j)} \frac{n_i}{n_j} \boldsymbol{a}' = \nu_j \boldsymbol{v}_j + \boldsymbol{a}_j\,, \tag{29}$$

$$\gamma_j \left(-\bar{\nu}_j \boldsymbol{v}_j^{\,2} + \sum_{\boldsymbol{v}' \in \Delta(\boldsymbol{v}_j)} \frac{n_i}{n_j} \nu' \boldsymbol{v}'^2 \right) + 2(1-\gamma_j) \sum_{\boldsymbol{v}'_\alpha \in \Delta(\boldsymbol{v}_j)} \frac{n_i}{n_j} \boldsymbol{v}'_\alpha \cdot \boldsymbol{a}' = \nu_j \boldsymbol{v}_j^{\,2} + 2\boldsymbol{v}_j \cdot \boldsymbol{a}_j \tag{30}$$

Expressions for the coefficients are given by

$$\gamma_j = \frac{2 \sum\limits_{\boldsymbol{v}'_\alpha \in \Delta(\boldsymbol{v}_j)} n_i (\boldsymbol{v}'_\alpha - \boldsymbol{v}_j) \cdot \boldsymbol{a}'}{2 \sum\limits_{\boldsymbol{v}'_\alpha \in \Delta(\boldsymbol{v}_j)} n_i (\boldsymbol{v}'_\alpha - \boldsymbol{v}_j) \cdot \boldsymbol{a}' - \sum\limits_{\boldsymbol{v}' \in \Delta(\boldsymbol{v}_j)} n_i \nu' (\boldsymbol{v}' - \boldsymbol{v}_j)^2}, \tag{31}$$

$$\nu_j = \gamma_j \left(-\bar{\nu}_j + \sum_{\boldsymbol{v}' \in \Delta(\boldsymbol{v}_j)} \frac{n_i}{n_j} \nu' \right), \tag{32}$$

$$\boldsymbol{a}_j = \gamma_j \sum_{\boldsymbol{v}' \in \Delta(\boldsymbol{v}_j)} \frac{n_i}{n_j} \nu' (\boldsymbol{v}' - \boldsymbol{v}_j) + (1-\gamma_j) \sum_{\boldsymbol{v}'_\alpha \in \Delta(\boldsymbol{v}_j)} \frac{n_i}{n_j} \boldsymbol{a}'. \tag{33}$$

For convenience, we repeat the main notations used in eqs. (31) – (33):

$$\nu' = \nu = \frac{4\pi N^2 n_k}{N_{coll}} (1 - \cos\phi) b(\mathrm{v}, \mu), \tag{34}$$

$$\bar{\nu}_j = \sum_{coll(\boldsymbol{v}_i = \boldsymbol{v}_j)} \nu, \tag{35}$$

$$\boldsymbol{a}' = \hat{R}\boldsymbol{a} = \frac{4\pi N^2 n_k}{N_{coll}} \frac{(1-\cos\phi)}{2} \phi\, b(\mathrm{v}, \mu) [\cos\alpha\phi + (\sin\alpha\phi)\, \boldsymbol{n} \times] \boldsymbol{n} \times \mathbf{v}, \tag{36}$$

$$\mu = \mu' = 1 - \frac{(1-\cos\phi)[\boldsymbol{n} \times \mathbf{v}]^2}{\mathrm{v}^2}, \tag{37}$$

$$\mathbf{v} = \boldsymbol{v}_i - \boldsymbol{u}_k, \qquad \mathrm{v} = \mathrm{v}' = |\boldsymbol{v}_i - \boldsymbol{u}_k|, \tag{38}$$

Eqs. (31)-(38) provide the final expression for discretization of the collision integral in the combined form:

$$\begin{aligned} I_{comb} &= \gamma I(f,f) - (1-\gamma) \frac{\partial}{\partial \boldsymbol{v}} \cdot \boldsymbol{J}(f,f) \\ &= \sum_j \left[\nu_j n_j \delta(\boldsymbol{v} - \boldsymbol{v}_j) - \frac{\partial}{\partial \boldsymbol{v}} \cdot \boldsymbol{a}_j n_j \delta(\boldsymbol{v} - \boldsymbol{v}_j) \right]. \end{aligned} \tag{39}$$

The time derivative of distribution function (6) is given by equation:

$$\frac{\partial f}{\partial t} = \sum_j \left[\dot{n}_j(t) \delta(\boldsymbol{v} - \boldsymbol{v}_j(t)) - \dot{\boldsymbol{v}}_j(t) \cdot \frac{\partial}{\partial \boldsymbol{v}} n_j(t) \delta(\boldsymbol{v} - \boldsymbol{v}_j(t)) \right]. \tag{40}$$

Comparing eq.(39) and eq.(40), we obtain evolution equations for weights and knots of distribution function (6):

$$\frac{\dot{n}_j}{n_j} = \frac{\partial}{\partial t} \ln n_j = \nu_j, \qquad \frac{\partial}{\partial t} \boldsymbol{v}_j = \boldsymbol{a}_j, \tag{41}$$

where expressions for ν_j and $\boldsymbol{a}_j$ are given by eqs.(31)-(33). Keeping eq.(41) in mind, we obtain formulae for weight and velocity calculation in the time step:

$$\ln n_j(t + \Delta t) = \ln n_j(t) + \nu_j \Delta t, \qquad \boldsymbol{v}_j(t + \Delta t) = \boldsymbol{v}_j(t) + \boldsymbol{a}_j \Delta t. \tag{42}$$

Note, that Eq.(42) guarantees positiveness of the weights $n_j(t)$.

In the case of the non-homogeneous Boltzmann equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} f + \frac{\partial}{\partial \boldsymbol{v}} \cdot \frac{\boldsymbol{F}}{m} f = I_{comb}(f,f), \tag{43}$$

we present a distribution function in the following form:

$$f(\boldsymbol{v},\boldsymbol{r},t)=\sum_j n_j(\boldsymbol{r},t)\delta(\boldsymbol{v}-\boldsymbol{v}_j(\boldsymbol{r},t))\,. \tag{44}$$

Taking into account that

$$\begin{aligned}
&\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{r}}n_j(\boldsymbol{r},t)\delta(\boldsymbol{v}-\boldsymbol{v}_j(\boldsymbol{r},t))\\
&=\left(\boldsymbol{v}_j\cdot\frac{\partial}{\partial\boldsymbol{r}}n_j\right)\delta(\boldsymbol{v}-\boldsymbol{v}_j)-n_j\boldsymbol{v}\cdot\left(\frac{\partial}{\partial\boldsymbol{r}}\boldsymbol{v}_j\right)\cdot\frac{\partial}{\partial\boldsymbol{v}}\delta(\boldsymbol{v}-\boldsymbol{v}_j)\\
&=\left[\left(\boldsymbol{v}_j\cdot\frac{\partial}{\partial\boldsymbol{r}}n_j\right)+n_j\left(\frac{\partial}{\partial\boldsymbol{r}}\cdot\boldsymbol{v}_j\right)\right]\delta(\boldsymbol{v}-\boldsymbol{v}_j)-n_j\boldsymbol{v}_j\cdot\left(\frac{\partial}{\partial\boldsymbol{r}}\boldsymbol{v}_j\right)\cdot\frac{\partial}{\partial\boldsymbol{v}}\delta(\boldsymbol{v}-\boldsymbol{v}_j)
\end{aligned} \tag{45}$$

and according to eq.(40), we obtain the left part of equation (43) in the form:

$$\begin{aligned}
\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{r}}f+\frac{\partial}{\partial\boldsymbol{v}}\cdot\frac{\boldsymbol{F}}{m}f=\sum_j\Bigg\{&\left[\dot{n}_j+\frac{\partial}{\partial\boldsymbol{r}}\cdot n_j\boldsymbol{v}_j\right]\delta(\boldsymbol{v}-\boldsymbol{v}_j)\\
&-n_j\left[\dot{\boldsymbol{v}}_j(t)+\boldsymbol{v}_j\cdot\frac{\partial}{\partial\boldsymbol{r}}\boldsymbol{v}_j-\frac{\boldsymbol{F}}{m}\right]\cdot\frac{\partial}{\partial\boldsymbol{v}}\delta(\boldsymbol{v}-\boldsymbol{v}_j)\Bigg\}
\end{aligned} \tag{46}$$

Comparing eq.(39) and eq.(46), we obtain the equation governing rarefied gas dynamics of quasi-particles:

$$\begin{aligned}
&\frac{\partial}{\partial t}n_j+\frac{\partial}{\partial\boldsymbol{r}}\cdot n_j\boldsymbol{v}_j=\nu_j n_j,\\
&\frac{\partial}{\partial t}\boldsymbol{v}_j+\boldsymbol{v}_j\cdot\frac{\partial}{\partial\boldsymbol{r}}\boldsymbol{v}_j=\boldsymbol{a}_j+\frac{\boldsymbol{F}}{m}.
\end{aligned}\;. \tag{47}$$

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