A simple model kinetic equation for inelastic Maxwell particles

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Abstract. The model of inelastic Maxwell particles (IMP) allows one to derive some exact results which show the strong influence of inelasticity on the nonequilibrium properties of a granular gas. The aim of this work is to propose a simple model kinetic equation that preserves the most relevant properties of the Boltzmann equation (BE) for IMP and reduces to the BGK kinetic model in the elastic limit. In the proposed kinetic model the collision operator is replaced by a relaxation-time term toward a reference Maxwellian distribution plus a term representing the action of a friction force. It contains three parameters (the relaxation rate, the effective temperature of the reference Maxwellian, and the friction coefficient) which are determined by imposing consistency with basic exact properties of the BE for IMP. As a consequence, the kinetic model reproduces the true shear viscosity and predicts accurate expressions for the transport coefficients associated with the heat flux. The model can be exactly solved for the homogeneous cooling state, the solution exhibiting an algebraic high-energy tail with an exponent in fair agreement with the correct one.

Keywords: Granular gases; Boltzmann equation; inelastic Maxwell particles; kinetic models
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INTRODUCTION

The prototype model for the description of granular media in the regime of rapid flow consists of an assembly of (smooth) inelastic hard spheres (IHS) with a constant coefficient of normal restitution $\alpha < 1$. In the low density limit the velocity distribution function obeys the Boltzmann equation (BE), modified to account for the inelasticity of collisions [1]. Because of the mathematical intricacy of the BE for IHS, a simpler model of inelastic Maxwell particles (IMP) has been proposed [2–4], where the collision rate is assumed to be independent of the relative speed of the colliding pair. Apart from its interest as a model of granular gases, the IMP model is interesting by itself since it allows the derivation of some *exact* results for any dimensionality d, showing unambiguously the strong influence of inelasticity on the nonequilibrium properties of the gas.

While the BE for IMP is considerably simpler than for IHS, it is still a formidable task to solve it in a closed form, even in the case of the homogeneous cooling state (HCS). The aim of this work is to propose a simple model kinetic equation that preserves the most relevant properties of the BE for IMP and reduces to the celebrated Bhatnagar–Gross–Krook (BGK) model in the elastic limit $\alpha \to 1$. In the proposed kinetic model the collision operator is replaced by a relaxation-time term toward a reference Maxwellian distribution plus a term representing the action of a dissipative friction force. The kinetic model contains three parameters: a relaxation rate modified by a factor $\beta(\alpha)$ with respect to its elastic value, an effective reference temperature modified by a factor $\theta(\alpha)$ with respect to the actual granular temperature, and a friction coefficient $\gamma(\alpha)$. The model is a hybrid of two previous models [5, 6] originally proposed for IHS, reducing to them if either $\gamma(\alpha) = 0$ or $\theta(\alpha) = 1$, respectively. The three parameters $\beta(\alpha)$, $\theta(\alpha)$, and $\gamma(\alpha)$ are determined by imposing consistency with basic exact properties derived from the BE for IMP. As a consequence, the model reproduces the true shear viscosity and predicts accurate expressions for the transport coefficients associated with the heat flux. Moreover, it can be exactly solved for the HCS, the solution exhibiting an algebraic high-energy tail with an exponent in fair agreement with the correct one, especially for high inelasticity.

THE BOLTZMANN EQUATION FOR INELASTIC MAXWELL PARTICLES

The BE for IMP [2–4] can be obtained from the BE for IHS by replacing the term $|\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}}|$ in the collision rate (where $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity and $\widehat{\boldsymbol{\sigma}}$ is the unit vector directed along the centers of the two colliding spheres) by an *average* value proportional to the thermal speed $\sqrt{2T/m}$ (where T is the granular temperature and m is the mass of a particle). In the version of the IMP model first proposed by Bobylev *et al.* [2] the collision rate has the same dependence on the scalar product $\widehat{\mathbf{g}}\cdot\widehat{\boldsymbol{\sigma}}$ as in the case of hard spheres. In a simpler version [3, 4], the collision rate is assumed to be independent of $\widehat{\mathbf{g}}\cdot\widehat{\boldsymbol{\sigma}}$. In this latter case, the corresponding BE reads

$$(\partial_t+\mathbf{v}_1\cdot\nabla)f(\mathbf{r},\mathbf{v}_1;t)=J[\mathbf{r},\mathbf{v}_1;t|f]\equiv\frac{d+2}{2}\frac{\nu_0(\mathbf{r},t)}{n(\mathbf{r},t)\Omega_d}\int d\widehat{\boldsymbol{\sigma}}\int d\mathbf{v}_2\left(\alpha^{-1}\mathfrak{b}^{-1}-1\right)f(\mathbf{r},\mathbf{v}_1;t)f(\mathbf{r},\mathbf{v}_2;t), \tag{1}$$

where n is the number density, $\nu_0\propto nT^{1/2}$ is an effective collision frequency whose specific form will not be needed, $\Omega_d=2\pi^{d/2}/\Gamma(d/2)$ is the total solid angle in d dimensions, and $\mathfrak{b}$ is the operator transforming pre-collision velocities into post-collision ones, namely, $\mathfrak{b}\mathbf{v}_{1,2}=\mathbf{v}_{1,2}\mp(1+\alpha)(\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}/2$. The Boltzmann collision operator $J[f]$ conserves mass and momentum but not energy. The collisional moments of second and third degree are [7]

$$\frac{m}{d}\int d\mathbf{v}\,V^2J[f]=-\zeta p,\quad m\int d\mathbf{v}\left(V_iV_j-d^{-1}V^2\delta_{ij}\right)J[f]=-\nu_\eta\left(P_{ij}-p\delta_{ij}\right),\quad \frac{m}{2}\int d\mathbf{v}\,V^2\mathbf{V}J[f]=-\nu_\kappa\mathbf{q}. \tag{2}$$

Here, $\mathbf{V}\equiv\mathbf{v}-\mathbf{u}$ is the peculiar velocity, where $\mathbf{u}$ is the flow velocity, P is the pressure tensor, $p=nT=d^{-1}\mathrm{tr}\,\mathsf{P}$ is the hydrostatic pressure, and $\mathbf{q}$ is the heat flux. The exact expressions for the cooling rate ζ and the rates of change ν_η and ν_κ are [7]

$$\zeta^*\equiv\zeta/\nu_0=\frac{d+2}{4d}\left(1-\alpha^2\right),\quad \nu_\eta^*\equiv\nu_\eta/\nu_0=\frac{(1+\alpha)^2}{4}+\zeta^*,\quad \nu_\kappa^*\equiv\nu_\kappa/\nu_0=\frac{(1+\alpha)^2(d-1)}{4d}+\frac{3}{2}\zeta^*. \tag{3}$$

In isotropic states, the collisional moment of fourth degree is

$$n^{-1}\int d\mathbf{v}\,V^4J[f]=-\nu_2M_2+\lambda\nu_0(2T/m)^2,\quad M_\ell\equiv n^{-1}\int d\mathbf{v}\,V^{2\ell}f, \tag{4}$$

where the exact expressions for the rate of change ν_2 and the coefficient λ are [7]

$$\nu_2^*\equiv\nu_2/\nu_0=(1+\alpha)^2(4d-7+6\alpha-3\alpha^2)/16d+2\zeta^*,\quad \lambda=(1+\alpha)^2(d+2)(4d-1-6\alpha+3\alpha^2)/64. \tag{5}$$

In the case of the uniform, free cooling state, the BE (1) and the evolution equations for the second- and fourth-degree moments become

$$\partial_tf(\mathbf{v})=J[\mathbf{v}|f],\quad \partial_tT=-\zeta T,\quad \partial_tM_2=-\nu_2M_2+\lambda\nu_0(2T/m)^2. \tag{6}$$

The solution to the cooling equation is $T(t)=T(0)[1+\zeta(0)t/2]^{-2}$ (Haff's law). If time is measured by the accumulated number of collisions per particle as $\tau(t)=\int_0^t dt'\,\nu_0(t')=(2/\zeta^*)\ln[1+\zeta(0)t/2]$, Haff's law becomes $T(\tau)=T(0)\exp(-\zeta^*\tau)$. It is convenient to introduce the *reduced* moments $M_\ell^*\equiv M_\ell/(2T/m)^\ell$, so that $M_1^*=d/2$ by definition, and the *reduced* distribution $f^*(\mathbf{c},\tau)$ defined by

$$f(\mathbf{v},t)=n[m/2T(t)]^{d/2}f^*(\mathbf{c},\tau),\quad \mathbf{c}=\mathbf{v}/\sqrt{2T(t)/m}. \tag{7}$$

Thus, Eq. (6) yields

$$\partial_\tau f^*(\mathbf{c})+(\zeta^*/2)\partial_{\mathbf{c}}\cdot\mathbf{c}f^*(\mathbf{c})=J^*[\mathbf{c}|f^*],\quad \partial_\tau M_2^*=-(\nu_2^*-2\zeta^*)M_2^*+\lambda, \tag{8}$$

where $J^*[\mathbf{c}|f^*]$ is the dimensionless version of the collision operator $J[\mathbf{v}|f]$. Except in the one-dimensional case, the reduced moment $M_2^*(\tau)$ converges in time to the well defined value $M_2^*(\infty)=\lambda/(\nu_2^*-2\zeta^*)$. In general, the distribution function reaches a *scaling form* [8], called homogeneous cooling state (HCS), which is the stationary solution of Eq. (8), i.e., $f^*(\mathbf{c},\tau)\to f^*(\mathbf{c},\infty)=f^*_{\text{hcs}}(\mathbf{c})$. Its exact expression is not known, except in the one-dimensional case, where $f^*_{\text{hcs}}(\mathbf{c})=(2^{3/2}/\pi)(1+2c^2)^{-2}$ [9]. For $d\geq 2$, the fourth cumulant (or kurtosis) $a_2\equiv 4M_2^*/d(d+2)-1$ of the distribution f^*_{hcs} is

$$a_2=\frac{4}{d(d+2)}\frac{\lambda}{\nu_2^*-2\zeta^*}-1=\frac{6(1-\alpha)^2}{4d-7+6\alpha-3\alpha^2}. \tag{9}$$

Therefore, $a_2\geq 0$ for IMP, in contrast to what happens in the case of IHS with $\alpha>\sqrt{2}/2$ [10]. It is also known that f^*_{hcs} exhibits an *algebraic* high-energy tail of the form [4]

$$f^*_{\text{hcs}}(\mathbf{c})\sim c^{-d-s(\alpha)}, \tag{10}$$

where the exponent $s(\alpha)$ is the solution of the transcendental equation

$$1-\frac{1-\alpha^2}{4d}s={}_2F_1[-s/2,1/2;d/2;1-(1-\alpha)^2/4]+\left(\frac{1+\alpha}{2}\right)^s\frac{\Gamma(s/2+1/2)\Gamma(d/2)}{\Gamma(s/2+d/2)\Gamma(1/2)}, \tag{11}$$

${}_2F_1[a,b;c;z]$ being a hypergeometric function. Equation (10) implies that those moments M_ℓ^* with $\ell\geq s(\alpha)/2$ are divergent.

Finally, the exact expressions for the transport coefficients are [7]

$$\eta=\frac{p}{\nu_0}\frac{1}{\nu_\eta^*-\zeta^*/2},\quad \kappa=\frac{d+2}{2}\frac{p}{m\nu_0}\frac{1+2a_2}{\nu_\kappa^*-2\zeta^*},\quad \mu=\frac{2T}{n}(\kappa-\kappa'),\quad \kappa'=\frac{d+2}{2}\frac{p}{m\nu_0}\frac{1+3a_2/2}{\nu_\kappa^*-3\zeta^*/2}. \tag{12}$$

Here, η is the shear viscosity, κ is the thermal conductivity, κ' is a modified thermal conductivity, and μ is a coefficient relating $\mathbf{q}$ and ∇n. As can be seen from (3), $\nu_\eta^*>\zeta^*/2$ and $\nu_\kappa^*>3\zeta^*/2$, so that η and κ' are well defined for all α. On the other hand, κ and μ are not positive definite if $\nu_\kappa^*\leq 2\zeta^*$, i.e., if $\alpha\leq\alpha_h\equiv(4-d)/3d$. Therefore, if $d<4$ and $\alpha\leq\alpha_h$, there is no hydrodynamic behavior since the heat flux does not relax to a Fourier-law form.

MODEL KINETIC EQUATION

Although the BE for IMP is more manageable than for IHS and some important properties are accessible in an exact way, its explicit solution is not known, even for the HCS. It is then natural to wonder whether a simple generalization of the well-known BGK model kinetic equation to the case of IMP can be proposed. The model considered in this paper is constructed by the replacement

$$J[\mathbf{r},\mathbf{v};t|f]\to\widetilde{J}[\mathbf{r},\mathbf{v};t|f]\equiv-\beta(\alpha)\nu_0(\mathbf{r},t)\left[f(\mathbf{r},\mathbf{v};t)-f_0(\mathbf{r},\mathbf{v};t)\right]+\gamma(\alpha)\nu_0(\mathbf{r},t)\partial_{\mathbf{v}}\cdot\mathbf{V}f(\mathbf{r},\mathbf{v};t), \tag{13}$$

where

$$f_0(\mathbf{r},\mathbf{v};t)=n(\mathbf{r},t)\left[m/2\pi\theta(\alpha)T(\mathbf{r},t)\right]^{d/2}\exp\left[-mV^2/2\theta(\alpha)T(\mathbf{r},t)\right] \tag{14}$$

is a local equilibrium distribution parameterized by the temperature $\theta(\alpha)T$. The effect of the inelastic collisions in the original BE is played in the model (13) by a relaxation term toward the distribution f_0 at an effective temperature θT, plus an external friction term. The model contains three free parameters: the factor $\beta(\alpha)>0$ modifying the relaxation rate with respect to its elastic value, the factor $\theta(\alpha)<1$ modifying the granular temperature in the reference state f_0, and the friction coefficient $\gamma(\alpha)>0$. These three parameters will be chosen in the next section to optimize the agreement with the most important properties of the BE for IMP. If one particularizes to $\gamma(\alpha)=0$, the model (13) reduces to the one proposed by Brey, Moreno, and Dufty (BMD) [5], while the choice $\theta(\alpha)=1$ yields a simplified version of the model proposed by Brey, Dufty, and Santos (BDS) [6]. From that point of view, the model (13) can be seen as a hybrid of the BMD and BDS models. Although the two latter models were originally proposed for IHS, Eq. (13) is proposed here as a model for IMP, not IHS. In fact, it has been shown [11, 12] that the BMD model shares more features with the BE for IMP than for IHS.

Let us now obtain the basic physical properties of the model operator $\widetilde{J}$. First, it is straightforward to check that it conserves mass and momentum. However, since $\theta\neq 1$ and $\gamma\neq 0$, energy is not conserved by collisions. More specifically, the cooling rate and the rates of change defined by Eq. (2) are given (in reduced units) by

$$\widetilde{\zeta}^*(\alpha)=\beta(\alpha)[1-\theta(\alpha)]+2\gamma(\alpha),\quad \widetilde{\nu}_\eta^*(\alpha)=\beta(\alpha)\theta(\alpha)+\widetilde{\zeta}^*(\alpha),\quad \widetilde{\nu}_\kappa^*(\alpha)=\frac{\beta(\alpha)}{2}[3\theta(\alpha)-1]+\frac{3}{2}\widetilde{\zeta}^*(\alpha), \tag{15}$$

where henceforth a tilde means that the corresponding quantity has been evaluated with the model operator $\widetilde{J}$. Moreover,

$$n^{-1}\nu_0^{-1}\int d\mathbf{v}V^{2\ell}\widetilde{J}[f]=-(\beta+2\ell\gamma)M_\ell+\beta\frac{\Gamma(\ell+d/2)}{\Gamma(d/2)}\left(\frac{2\theta T}{m}\right)^\ell. \tag{16}$$

In particular, setting $\ell=2$ we reobtain Eq. (4) with

$$\widetilde{\nu}_2^*(\alpha)=\beta(\alpha)[2\theta(\alpha)-1]+2\widetilde{\zeta}^*(\alpha),\quad \widetilde{\lambda}(\alpha)=\beta(\alpha)\theta^2(\alpha)d(d+2)/4. \tag{17}$$

Let us now consider the free cooling problem. As shown by Eq. (8), the necessary and sufficient condition to reach a scaling solution (HCS) with a finite fourth-degree velocity moment is $\widetilde{\nu}_2^* > 2\widetilde{\zeta}^*$, i.e., $\theta > 1/2$. In that case, the first equality of Eq. (9) implies that the fourth cumulant of the scaling solution is

$$\widetilde{a}_2(\alpha) = [1-\theta(\alpha)]^2/[2\theta(\alpha)-1]. \tag{18}$$

Therefore, we have $\widetilde{a}_2 > 0$, regardless of the precise values of the parameters β, θ, and γ of the model. As a consequence, the model (13) can never capture the negative values exhibited by a_2 in the case of IHS for $\alpha > \sqrt{2}/2$. From Eq. (16) it is easy to obtain the time-dependence of the reduced velocity moments in the free cooling problem:

$$M_\ell^*(\tau) = \mathrm{e}^{-\beta[1-\ell(1-\theta)]\tau} M_\ell^*(0) + \left\{1-\mathrm{e}^{-\beta[1-\ell(1-\theta)]\tau}\right\} \frac{\theta^\ell}{1-\ell(1-\theta)} \frac{\Gamma(\ell+d/2)}{\Gamma(d/2)}. \tag{19}$$

Since, on physical grounds, $\theta < 1$, it turns out that the reduced moments $M_\ell^*(\tau)$ *diverge* in time if $\ell \geq (1-\theta)^{-1}$. This implies that the scaling solution presents a high-energy tail of the form (10) with the exponent

$$\widetilde{s}(\alpha) = 2[1-\theta(\alpha)]^{-1}. \tag{20}$$

Finally, the transport coefficients of the model are given by (12), except for the obvious replacements $\zeta^* \to \widetilde{\zeta}^*$, $\nu_\eta^* \to \widetilde{\nu}_\eta^*$, $\nu_\kappa^* \to \widetilde{\nu}_\kappa^*$, and $a_2 \to \widetilde{a}_2$.

The main advantage of a kinetic model lies in the possibility of obtaining the explicit form of the velocity distribution function. Let us illustrate this in the free cooling case. According to the model (13), the first equation of (8) becomes

$$\left\{\beta^{-1}\partial_\tau + 1 + [(1-\theta)/2]\,\partial_{\mathbf{c}}\cdot\mathbf{c}\right\} f^*(\mathbf{c}) = (\pi\theta)^{-d/2}\exp(-c^2/\theta). \tag{21}$$

It is interesting to note that the parameter γ does not intervene explicitly in Eq. (21), so that it is formally equivalent to the equation obtained from the BMD model [5]. Given an arbitrary initial condition $f^*(\mathbf{c},0)$, the exact solution to Eq. (21) is [12, 13]

$$f^*(\mathbf{c},\tau) = \mathrm{e}^{-\beta[1+d(1-\theta)/2]\tau} f^*\left(\mathrm{e}^{-\beta(1-\theta)\tau/2}\mathbf{c},0\right) + (\pi\theta)^{-d/2}\int_0^{\beta\tau} \mathrm{d}y\, \exp\left\{-[1+d(1-\theta)/2]y - c^2 e^{-(1-\theta)y}/\theta\right\}. \tag{22}$$

The HCS is obtained taking the limit $\tau \to \infty$ with the result [5, 12, 13]

$$f_{\mathrm{hcs}}^*(\mathbf{c}) = (\pi\theta)^{-d/2}(1-\theta)^{-1}(\theta/c^2)^{d/2+1/(1-\theta)}\left[\Gamma(d/2+1/(1-\theta)) - \Gamma\left(d/2+1/(1-\theta), c^2/\theta\right)\right], \tag{23}$$

where the change of variable $x = c^2\exp[-(1-\theta)y]/\theta$ has been made and $\Gamma(z,a) = \int_a^\infty \mathrm{d}x\, x^{z-1}\mathrm{e}^{-x}$ is the incomplete gamma function. Note that the whole dependence of f_{hcs}^* on inelasticity appears through the parameter θ only. In the high-energy limit (actually, if $c^2 \gg \theta$), $\Gamma\left(d/2+1/(1-\theta), c^2/\theta\right) \to 0$, so that we recover the tail (10) with the exponent (20). The transient from the initial distribution $f^*(\mathbf{c},0)$ to the asymptotic distribution $f_{\mathrm{hcs}}^*(\mathbf{c})$ is described by Eq. (22), which can be rewritten in a simpler form by introducing the deviation $\delta f^*(\mathbf{c},\tau) = f^*(\mathbf{c},\tau) - f_{\mathrm{hcs}}^*(\mathbf{c})$. The results for δf^* and its moments δM_ℓ^* with $\ell < (1-\theta)^{-1}$ are

$$\delta f^*(\mathbf{c},\tau) = \mathrm{e}^{-\beta[1+d(1-\theta)/2]\tau}\delta f^*\left(\mathrm{e}^{-\beta(1-\theta)\tau/2}\mathbf{c},0\right), \quad \delta M_\ell^*(\tau) = \mathrm{e}^{-\beta[1-\ell(1-\theta)]\tau}\delta M_\ell^*(0). \tag{24}$$

While $f^*(\mathbf{c},\tau)$ relaxes uniformly to $f_{\mathrm{hcs}}^*(\mathbf{c})$ at fixed $\mathbf{c}$ with a relaxation rate $\beta[1+d(1-\theta)/2]$, the moment $M_\ell^*(\tau)$ relaxes to its HCS value with a shorter relaxation rate $\beta\,[1-\ell(1-\theta)]$ which goes to zero as ℓ approaches the threshold value $(1-\theta)^{-1}$ from below. This paradoxical property [13] is a consequence of the non-commutability of the limits $c \to \infty$ and $\tau \to \infty$ in Eq. (24). To analyze this with more detail, let us suppose that the initial distribution $f^*(\mathbf{c},0)$ has a high-energy tail much weaker than that of $f_{\mathrm{hcs}}^*(\mathbf{c})$, so that $\delta f^*(\mathbf{c},0) \approx -f_{\mathrm{hcs}}^*(\mathbf{c}) \approx -Ac^{-d-2/(1-\theta)}$ in the region $c^2 \gg \theta$, where the amplitude A is known from Eq. (23). In that case, at fixed τ, one has

$$\delta f^*(\mathbf{c},\tau) \approx -Ac^{-d-2/(1-\theta)} \approx -f_{\mathrm{hcs}}^*(\mathbf{c}), \quad c^2 \gg \theta e^{\beta(1-\theta)\tau}. \tag{25}$$

This means that, at any fixed time τ, there always exists an infinite range of large speeds, $c^2 \gg \theta e^{\beta(1-\theta)\tau}$, where the deviation of the velocity distribution function from its asymptotic HCS value is of 100%. This region is eventually responsible for the divergence of moments with $\ell \geq (1-\theta)^{-1}$.

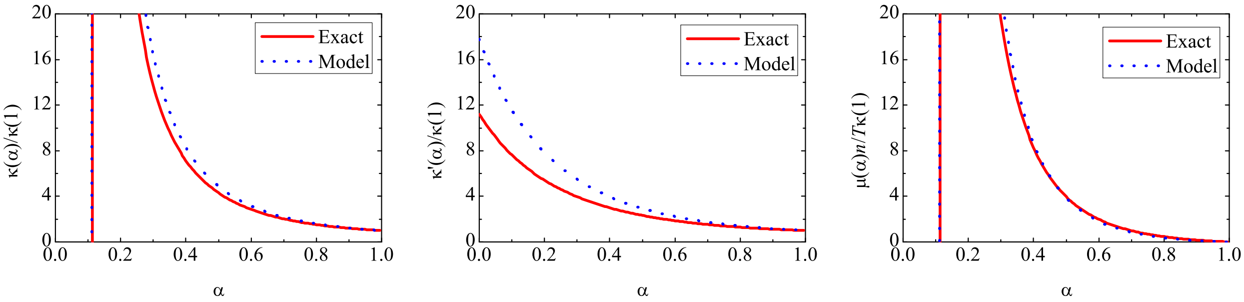

FIGURE 1. Transport coefficients $\kappa(\alpha)$, $\kappa'(\alpha)$, and $\mu(\alpha)$ for $d=3$.

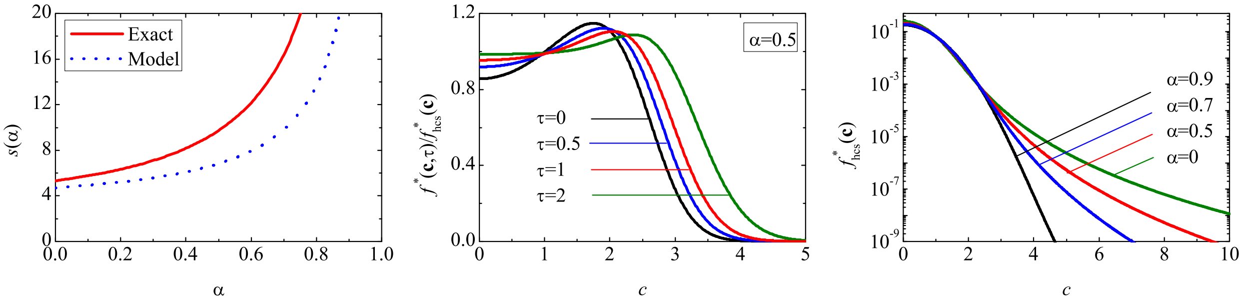

FIGURE 2. Left panel: High-energy exponent $s(\alpha)$ for $d=3$. Middle panel: Time evolution of the ratio $f^*(\mathbf{c},\tau)/f^*_{\text{hcs}}(\mathbf{c})$, starting from an initial equilibrium distribution, for $\alpha=0.5$ and $d=3$. Right panel: Plot of $f^*_{\text{hcs}}(\mathbf{c})$ for $d=3$ and $\alpha=0$, 0.5, 0.7, and 0.9.

RESULTS AND COMPARISON WITH THE BOLTZMANN EQUATION

So far, all the properties of the kinetic model (13) described in the preceding section are valid with independence of the specific expressions for the parameters $\beta(\alpha)$, $\theta(\alpha)$, and $\gamma(\alpha)$. Now we fix them by requiring the model to reproduce the basic properties of the original BE. The most characteristic consequence of inelasticity is the cooling rate, so that an obvious requirement is $\zeta^*=\widetilde{\zeta}^*$. The next requirement could be the agreement with either the relaxation rate ν_η^* or ν_κ^*. We cannot enforce both since that would imply $\theta=d/(d+2)$, which yields an unphysical model in the elastic limit. This impossibility of satisfying the shear viscosity and the thermal conductivity simultaneously, which also happens with the conventional BGK model, is remedied by a more sophisticated model [12]. In the case of the kinetic model (13), let us take $\nu_\eta^*=\widetilde{\nu}_\eta^*$ as a second condition. Finally, given the important role played by the kurtosis a_2 of the HCS, the third condition adopted here is $a_2=\widetilde{a}_2$. These three requirements yield

$$\beta(\alpha)=\frac{(1+\alpha)^2}{4\theta(\alpha)},\quad \gamma(\alpha)=\frac{d+2}{8d}(1-\alpha^2)-\frac{(1+\alpha)^2}{8}\left[\theta^{-1}(\alpha)-1\right],\quad \theta(\alpha)=1+a_2(\alpha)-\sqrt{a_2(\alpha)\left[1+a_2(\alpha)\right]}, \tag{26}$$

where $a_2(\alpha)$ is given by Eq. (9).

By construction, the model reproduces the exact shear viscosity η but not the transport coefficients associated with the heat flux. However, as Fig. 1 shows, the model captures reasonably well the rapid increase of $\kappa(\alpha)$, $\kappa'(\alpha)$, and $\mu(\alpha)$ with the inelasticity, the agreement being especially remarkable in the case of $\mu(\alpha)$. The model also predicts the existence of a threshold value α_h below which hydrodynamics no longer holds. The value of α_h is the solution of a quartic equation which for $d=3$ yields $\alpha_h\simeq 0.114$, in excellent agreement with the exact result $\alpha_h=1/9\simeq 0.111$.

As Fig. 2 shows, the model underestimates the exact exponent $s(\alpha)$ but the agreement improves as the inelasticity increases. The time evolution of the distribution function in the free cooling state, as well as the asymptotic scaling solution are also shown in Fig. 2 for a few representative cases.

CONCLUDING REMARKS

The model of IMP shares with the more realistic model of IHS the description of the important influence of inelasticity on the dynamical properties of a granular gas. However, that influence is magnified by the IMP model, giving rise to stronger departures from the Maxwellian distribution and even to the absence of hydrodynamics for sufficiently inelastic systems. Nonetheless, the fact that non-trivial exact results can be derived from the BE for IMP justifies its study in order to gain a broader perspective on the peculiar properties of dissipative gases. From that point of view, the model (13) proposed here can be useful to have access, at least at a semi-quantitative way, to relevant information (such as the velocity distribution function itself) not directly available from the BE for IMP.

The key ingredient of the kinetic model (13), also present in the BMD model [5], is the effective temperature $\theta(\alpha)T < T$ in the reference distribution f_0. Its existence leads to a non-trivial HCS solution with a positive definite kurtosis and an algebraic high-energy tail. The additional presence of the friction term relieves the effective temperature of fully accounting for the cooling rate, so that a closer contact with the BE for IMP is possible.

The analysis presented in this paper can be extended along a number of routes. The explicit solution of the kinetic model to other problems, such as the simple shear flow, is straightforward. Moreover, the flexibility of the model allows one to choose its parameters by a fit to other quantities, such as the effective collision frequency $\nu_\kappa(\alpha)$ and/or the exponent $s(\alpha)$, different from the ones considered in this paper. The Gaussian kinetic model proposed in Ref. [12], which is able to reproduce $\nu_\eta(\alpha)$ and $\nu_\kappa(\alpha)$ simultaneously, can be extended by the inclusion of a friction term, thus providing an extra parameter $\gamma(\alpha)$. It s also possible to generalize the model (13) to mixtures of IMP [14].

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