Stable Equations to Second Order in the Knudsen number for the Fluid Dynamics Variables

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Abstract. Two methods to formulate well posed equations which are consistent approximations of the Burnett equations are given. In the first method, attention is drawn to the Chapman-Enskog replacement of time derivatives by the approximate conservation laws. It is shown that partly restoring the time derivatives in the Burnett equations results in well posed equations. In the second method the linearized Burnett equations are studied by a multiple time analysis. All resulting equations are well posed. It is also shown that the full set of perturbation equations can to within third order terms be summarized as well posed equations for transformed fields. .

Keywords: Burnett equations, Bobylev's instability, Stabilization, Multiple times
PACS: 51.10.+y, 47.50.+d

I. ON THE RANGE OF VALIDITY OF THE BURNETT EQUATIONS

Let us start by a general discussion of the Burnett equations. They are derived from the Boltzmann equation by the Chapman-Enskog procedure and are the results to second order in the Knudsen number. It is well known that the Burnett equations are plagued with an unphysical instability of the equilibrium state. Does this mean that the Burnett equations are simply useless? No, they are correct to second order in the Knudsen number. The unphysical instability occurs for wavelengths of the order of the mean free path and smaller. This means that the effective Knudsen number is of the order of 1 and larger. The instability is thus outside the range of validity of the equation. It is an instability of higher order terms in the Knudsen number. There are many possible equations that agree to second order in the Knudsen number but differ to higher order terms. Some of them are well posed and some of them are not. Regularizing the equations means adding higher order terms to convert the Burnett equations to a set of well posed equations. This is precisely the object of the two approaches in the present contribution.

II. PARTLY RESTORING TIME DERIVATIVES IN THE BURNETT EQUATIONS

Background and idea

It is well known that the Hilbert expansion is not valid for times characteristic of diffusion, see Chekmarev and Chekmareva, [3]. As a consequence, it is not possible to derive equations to the Navier-Stokes order for time dependent phenomena. The Chapman-Enskog method was a great step forward. For the first time the Navier-Stokes equations were derived from the Boltzmann equation. In this method, to each order a linear integral equation appears. The linearized collision operator acting on the unknown function equals a source which is given by the lower order contributions. In the Chapman-Enskog method, the time derivatives D/Dt in the source are (consistently) approximated from the corresponding conservation law at a lower level. D/Dt is thus replaced by D_0/Dt, see Chapman and Cowling [2]. As a consequence, the integrability conditions of the linearized collision operator are satisfied identically. In this way a series expansion of the fluid dynamics fields are avoided in the Chapman-Enskog method. This method works very well to first order in the Knudsen number, producing the Navier-Stokes equations. But the second, order, the Burnett equations are not well posed as was shown by Bobylev, [1].

The simple idea behind the present attack on the Burnett equation is it is consistent with the approximation to reinstall the real derivatives D/Dt instead of D_0/Dt in the Burnett terms. This did, however, turn out not to give well

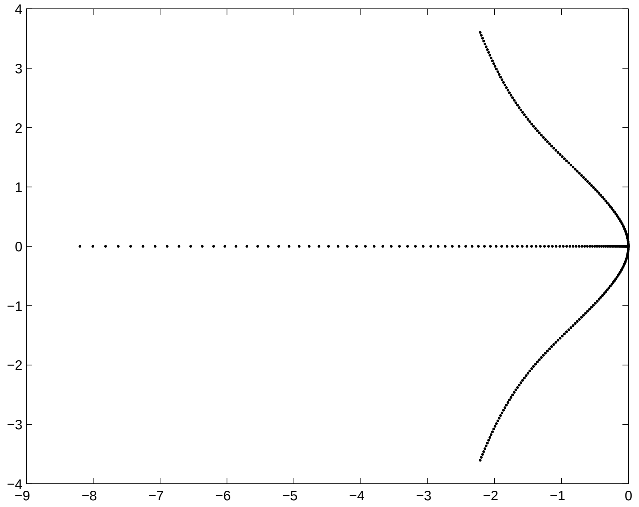

FIGURE 1. Growth factor Λ for $0 \leq k \leq 2$.

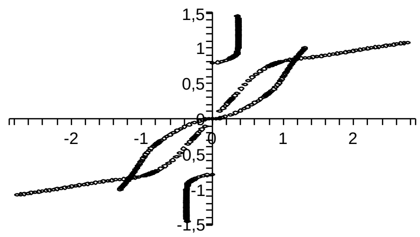

FIGURE 2. Complex wave number k for $0 \leq \omega \leq 6$.

posed equations. But the more general replacement

$$D_0/Dt \rightarrow (1-\alpha)D/Dt + \alpha D_0/Dt \tag{1}$$

is also consistent with the approximation For suitably chosen α well posed equations result.$\alpha = 1$ gives the Burnett equations. For more details, see [4]

Linear stability analysis

It turns out that it is sufficient to make the above mentioned replacement in the term $D_0\mathbf{S}/Dt$ contributing to the viscous pressure. In the linearized momentum equation for a longitudinal wave we write $T = T_0[1+\tilde{T}],\ \rho = \rho_0[1+\tilde{\rho}]$

$$\frac{\partial v_\parallel}{\partial t} = -\frac{1+\alpha\omega_2\frac{\mu^2}{\rho p}\frac{2}{3}k^2}{1+\omega_2\frac{\mu^2}{p\rho}(\alpha-1)\frac{2}{3}k^2}\frac{k_BT}{m\rho}\nabla_\parallel\rho - \frac{1+\frac{\mu^2}{\rho p}(\alpha\omega_2-\omega_3)\frac{2}{3}k^2}{1+\omega_2\frac{\mu^2}{p\rho}(\alpha-1)\frac{2}{3}k^2}\frac{k_B}{m}\nabla_\parallel T + \frac{1}{1+\omega_2\frac{\mu^2}{p\rho}(\alpha-1)\frac{2}{3}k^2}\frac{4\mu}{3\rho}\triangle v_\parallel \tag{2}$$

We see that it can be interpreted as the momentum equation of the linearized Navier-Stokes equations for different fluids for different wave vectors k. The same applies to the full system of equations. This is correct as long as the coefficients are positive and nonsingular. This means that the equations are linearly stable as long as $\alpha \geq \omega_3/\omega_2$. As $0 < \omega_2 < \omega_3$ for Maxwell molecules as well as hard spheres this rules out the Burnett equations, which correspond to $\alpha = 1$. We then choose $\alpha = \omega_3/\omega_2$. The resulting equations we name hybrid Burnett equations.

We now look for solutions

$$exp(ikx + \Lambda t).$$

In Fig. 1 we see that for real k the growth factor has a non-negative real part, which means stability. We also consider spatial stability, plotting for $\Lambda = -i\omega$, where ω is real the complex wave numer k. See Fig. 2. We see that the waves are damped in the direction of propagation, which means spatial stability. In the figures we use dimensionless variables, where the unit of length is of the order of the mean free path

$$x = x^* \frac{\mu_0}{\rho_0}\sqrt{\frac{m}{k_B T_0}},\ t = t^* \frac{\mu_0}{\rho_0}\frac{m}{k_B T_0}.$$

Let us now write down the resulting hybrid Burnett expression for the viscous pressure tensor, which we denote by $\mathbf{P}_h$.

$$\begin{aligned}\mathbf{P}_h = p\mathbf{1} - 2\mu\mathbf{S} + \varpi_1 \frac{\mu^2}{p}(\nabla\cdot\mathbf{v})\mathbf{S} + (\varpi_2 - \varpi_3)\frac{\mu^2}{p}\frac{D\mathbf{S}}{Dt} \\ -2\varpi_2\frac{\mu^2}{p}\langle \mathbf{S}\cdot(\nabla\mathbf{v})\rangle - \varpi_3\frac{\mu^2}{\rho^2}\langle\nabla\nabla\rho\rangle + \varpi_3\frac{\mu^2}{\rho T}\langle -\frac{1}{\rho}\nabla T\nabla\rho + \frac{T}{\rho^2}\nabla\rho\nabla\rho - \frac{p}{\rho T}(\nabla\mathbf{v})^2\rangle \\ +\varpi_4\frac{\mu^2}{\rho p T}\langle\nabla p\nabla T\rangle + \varpi_5\frac{\mu^2}{\rho T^2}\langle\nabla T\nabla T\rangle + \varpi_6\frac{\mu^2}{p}\langle\mathbf{S}\cdot\mathbf{S}\rangle.\end{aligned}$$

The heat current then is the conventional one, $\mathbf{q}_c$.

Now we have our hybrid Burnett equations.

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0, \tag{3}$$

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla\cdot\mathbf{P}_h, \tag{4}$$

$$\rho\frac{3k_B}{2m}\frac{DT}{Dt} = -\mathbf{P}_h : \nabla\mathbf{v} - \nabla\cdot\mathbf{q}_c. \tag{5}$$

To more easily see the structure of the hybrid Burnett momentum equation we denote the nonlinear Burnett terms by dots.

$$\rho\{\mathbf{1} - (\omega_3 - \omega_2)\frac{\mu^2}{p\rho}[\frac{1}{2}(\triangle\mathbf{1} - \nabla\nabla) + \frac{2}{3}\nabla\nabla\}\cdot\frac{D\mathbf{v}}{Dt} = -\nabla p + 2\nabla\cdot(\mu\mathbf{S}) + \omega_3\frac{\mu^2}{\rho^2}\frac{2}{3}\triangle\nabla\rho + \ldots \tag{6}$$

The equation of energy only differs from the energy equation of the Burnett equations by nonlinear Burnett terms.

III. A MULTIPLE TIME APPROACH TO THE BURNETT EQUATIONS

We write down the linearized Burnett equations. It is now convenient to use dimensionless variables based on the macrosopic length scale, L, given by the problem and correspondingly for time. We have

$$\begin{aligned}\rho_t + v_x &= 0, \\ v_t &= -(\rho + T)_x + \varepsilon\frac{4}{3}v_{xx} + \varepsilon^2\frac{2}{3}\omega_2\rho_{xxx} - \varepsilon^2\frac{2}{3}(\omega_3 - \omega_2)T_{xxx}, \\ \frac{3}{2}T_t &= -v_x + \varepsilon\frac{3}{2}fT_{xx} - \varepsilon^2\frac{2}{3}(\theta_4 - \theta_2)v_{xxx}.\end{aligned}$$

$$\varepsilon = \frac{\mu_0}{\rho_0 L}\sqrt{\frac{m}{k_B T_0}}$$

is essentially the Knudsen number.

Three time scales

It is convenient to use the entropy s and pressure p together with v as field. They are expanded in series in ε. It is well known that a direct series expansion breaks down for times of the order of the diffusion time. What happens is that a first order term could appear which is proportional to εt. For $t \sim \varepsilon^{-1}$ it is just as large as the zero order term and the expansion breaks down. Such a term is called secular. See Nayfeh [5]. A well known method to solve this difficulty is to introduce multiples times which are formally considered to be independent variables, $t_0 = t, t_1 = \varepsilon t, t_2 = \varepsilon^2 t, \ldots$ The point is that time t_n is of importance when it is of ~ 1. In other words for $t \sim \varepsilon^{-n}$. In particular, t_0 is a fast time scale for sound propagatio. Diffusion is governed by the slow time t_1. Finally, sound dispersion is charactized by the very slow time t_2. So

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \ldots \tag{7}$$

When this and the series expansion of the fields is introduced into the linearized Burnett equations, equations to each order in ε appear.

Zero order equations. The zero order solution is the same as in an ordinary perturbation expansion, it is a function of t_0. A wave equation and conservation of entropy to this order (a subscript 0 after a comma means $\partial/\partial t_0$ and similarly for the other times)

$$\begin{aligned} v_{0,0} + p_{0,x} &= 0, \\ p_{0,0} + \frac{5}{3} v_{0,x} &= 0, \\ s_{0,0} &= 0 \end{aligned} \tag{8}$$

First order equations. But there is a difference now. The constants in the zero order solution are arbitrary functions of the slower times $t_1, \ldots$ But it turns out that the dependence on t_1 is completely determined by the condition that no secular terms should appear. So we obtain the slow evolution of the zero order fields,

$$\begin{aligned} -p_{0,1} + (\frac{1}{5} f + \frac{2}{3}) p_{0,xx} = 0 \\ -v_{0,1} + (\frac{1}{5} f + \frac{2}{3}) v_{0,xx} = 0 \\ -s_{0,1} + \frac{3}{5} f s_{0,xx} = 0. \end{aligned} \tag{9}$$

We have obtained diffusion equations.

The fast evolution of the first order fields is most easily expressed by introducing

$$\begin{aligned} V_1 &= v_1 - \frac{6}{25} f s_{0,x}, \\ P_1 &= p_1 + (\frac{1}{5} f - \frac{2}{3}) v_{0,x}, \\ S_1 &= s_1 + \frac{3}{5} f v_{0,x} \end{aligned} \tag{10}$$

The equations are then

$$\begin{aligned} V_{1,0} + P_{1,x} &= 0, \\ P_{1,0} + \frac{5}{3} V_{1,x} &= 0. \\ S_{1,0} = 0 & \end{aligned} \tag{11}$$

For the new combinations of the fields, the equations are the same as the lowest order equations.

Second order equations. Now we come to the second order equations. To start with we have the very slow evolution of the zero order fields. They are obtained from the fact that no secular terms should appear in the third order fields.

$$
\begin{aligned}
v_{0,2} &= \alpha p_{0,xxx}, \\
p_{0,2} &= \frac{5}{3}\alpha v_{0,xxx}, \\
\alpha = [(\frac{1}{3}\omega_2 - \frac{2}{15}(\omega_3 + (\theta_4 - \theta_2)) - \frac{3}{10}[(\frac{1}{5}f - \frac{2}{3})^2 + f^2\frac{6}{25}] & \\
s_{0,2} &= 0
\end{aligned} \tag{12}
$$

The first pair of equations can be transformed as two linearized KdV equations. We then have the slow development of the first order terms. We can once more use the combinations above and write. .

$$
\begin{aligned}
V_{1,1} - (\frac{2}{3} + f\frac{1}{5})V_{1,xx} &= 0 \\
P_{1,1} - (\frac{2}{3} + f\frac{1}{5})P_{1,xx} &= 0, \\
S_{1,1} - \frac{3}{5}fS_{1,xx} &= 0.
\end{aligned} \tag{13}
$$

Hence these combinations satisfy the same equations in the slow time as the zero order terms

Finally, we have the fast evolution for the second order fields. We introduce the combinations

$$
\begin{aligned}
V_2 &= v_2 - f\frac{6}{25}(s_1 + \frac{3}{5}fv_{0,x})_{,x}, \\
P_2 = p_2 - (\frac{2}{3} - f\frac{1}{5})v_{1,x} - \{\frac{3}{10}[(\frac{1}{5}f - \frac{2}{3})^2 + f^2\frac{6}{25}] &+ \frac{1}{3}\omega_2 - \frac{2}{15}(\omega_3 - (\theta_4 - \theta_2))\}p_{0,xx} \\
&+[(\frac{2}{3} - f\frac{1}{5})\frac{6}{25}f + \frac{4}{15}\omega_3 + \frac{6}{25}f^2\frac{3}{5} - \frac{4}{3}\frac{6}{25}f]s_{0,xx}, \\
S_2 &= s_2 + \frac{3}{5}f(v_1 - \frac{6}{25}fs_{0,x})_{,x} - \frac{2}{5}(\theta_4 - \theta_2)p_{0,xx}
\end{aligned} \tag{14}
$$

For those combinations, the equations are simply the same as the zero order equations

$$
\begin{aligned}
S_{2,0} = 0 & \\
V_{2,0} + P_{2,x} &= 0 \\
P_{2,0} + \frac{5}{3}V_{2,x} &= 0,
\end{aligned} \tag{15}
$$

Well posedness

Let us write $S_0 = s_0$ and similarly for the other fields. $S_0, P_0, V_0, S_1, P_1, V_1, S_2, P_2, V_2$ satisfy the equations (9), (10), (11), (12), (14) and (15). All of these equations are well posed. Solving them, the fields s_i, p_i, v_i are then given by (11), (15). This then gives the physical field s, p, v up to second order in ε as sums. They will not have any instability of the kind of the Burnett equations.

In [3] Chekmarev and Chekmareva have combined a Hilbert expansion with multiple time scales. They limit themselves to sound waves. To find the second order equations they make an approximation corresponding to Maxwell molecules. When specialized to this case, our equations agree with those in, except that they did not derive the set of equations (14) and (15) giving the fast evolution of the second order fields. This suggests that the full set of equations we have derived from the Burnett equations should be possible to derive directly from the Boltzmann equation by an Hilbert expansion combined with multiple time scales. Hence, it should be possible to use these methods for time dependent fluid dynamics phenomena up to the Burnett level.

Summed equations for transformed fields

Let us now show that the resulting set of equations up to terms of order ε^3 kan be summed in a set of well posed equations in the ordinary physical time variable t. We write

$$S = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \ldots \tag{16}$$

and similarly for the other fields. We easily see that with an error ε^3 they satisfy the equations

$$\begin{aligned} S_{,t} &= \varepsilon \frac{3}{5} f S_{,xx}, \\ V_{,t} &= -P_{,x} + \varepsilon(\frac{2}{3} + f\frac{1}{5})V_{,xx} + \varepsilon^2 \alpha P_{,xxx} \\ P_{,t} &= -\frac{5}{3}V_{,x} + \varepsilon(\frac{2}{3} + f\frac{1}{5})P_{,xx} + \varepsilon^2 \frac{5}{3}\alpha V_{,xxx} \end{aligned} \tag{17}$$

The second pair is most clearly written for

$$V_{\pm,t} \pm \sqrt{\frac{5}{3}} V_{\pm,x} = \varepsilon(\frac{2}{3} + f\frac{1}{5})V_{\pm,xx} \pm \varepsilon^2 \sqrt{\frac{5}{3}} V_{\pm,xxx},$$

$$V_{\pm} = V \pm \sqrt{\frac{3}{5}} P$$

What we have found are the equations for the entropy mode and the two sound modes. Each of the equations has a quadratic Liapunov function.

Let us conclude by saying that a direct derivation from the Boltzmann equation of the equations presented is in progress.

ACKNOWLEDGMENTS

The author wishes to thank A.V. Bobylev and L. Arkeryd for hospitality and discussions.

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