New Forms of the Boltzmann Collision Integral

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Abstract. Rotation matrix technique is described. New matrix elements of the collision operator are obtained. An expression of the Boltzmann collision integral through the hypergeometric function on Casimir operator is constructed. It is shown that the Boltzmann collision operator for gas mixtures near the equilibrium can be well approximated by the Landau-Fokker-Plank collision integral (for the simple gas the transport collision frequency should be extra corrected).

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Parameterization of Collisions in the Boltzmann Equation by a Rotation Matrix

In order to construct the collision integral in the Boltzmann kinetic equation, a collision of two particles is usually determined by setting a direction $\boldsymbol{n}, \left(n^2 = 1\right)$ of the post-collisional relative velocity [1]. When a particle of mass m_1 and velocity $\boldsymbol{v}$ collides with a particle of mass m_2 and velocity $\boldsymbol{u}$, the post-collisional velocities are:

$$\begin{aligned} \boldsymbol{v}' &= \frac{m\boldsymbol{v} + \boldsymbol{u} + |\boldsymbol{v} - \boldsymbol{u}|\boldsymbol{n}}{1+m}, \\ \boldsymbol{u}' &= \frac{m\boldsymbol{v} + \boldsymbol{u} - m|\boldsymbol{v} - \boldsymbol{u}|\boldsymbol{n}}{1+m}, \end{aligned} \tag{1}$$

where $m = \dfrac{m_1}{m_2}$. The collision integral reads:

$$I(f,\psi) = \int \mathrm{v}\,\sigma_\theta(\mathrm{v}^2, \frac{\mathbf{v}\cdot\boldsymbol{n}}{\mathrm{v}})\big[f(\boldsymbol{v}')\psi(\boldsymbol{u}') - f(\boldsymbol{v})\psi(\boldsymbol{u})\big]d\Omega_{\boldsymbol{n}} d\boldsymbol{u}, \tag{2}$$

where $\mathbf{v} = \boldsymbol{v} - \boldsymbol{u}$ and σ_θ is the differential collision cross section.

In our approach [2-3], we propose to parameterize a collision by a rotation matrix $\hat{R} \in O_3$. In this case, the transformation of the velocities due to the collision becomes a linear one, which is different from eq.(1):

$$\begin{aligned} \boldsymbol{v}' &= \frac{m\boldsymbol{v} + \boldsymbol{u} + \hat{R}(\boldsymbol{v} - \boldsymbol{u})}{1+m}, \\ \boldsymbol{u}' &= \frac{m\boldsymbol{v} + \boldsymbol{u} - m\hat{R}(\boldsymbol{v} - \boldsymbol{u})}{1+m}. \end{aligned} \tag{3}$$

The velocities $\boldsymbol{v}'$ and $\boldsymbol{u}'$ are determined by a partitioned (2×2 cells) scattering matrix $\hat{S}$. The size of each cell is obviously (3×3):

$$\xi' = \hat{S}\xi, \qquad \xi = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{pmatrix}, \qquad \hat{S} = \begin{pmatrix} \dfrac{m+\hat{R}}{1+m} & \dfrac{1-\hat{R}}{1+m} \\ \dfrac{m(1-\hat{R})}{1+m} & \dfrac{1+m\hat{R}}{1+m} \end{pmatrix}, \tag{4}$$

where, ξ is a 6-dimensional bivector consisting of the components $\boldsymbol{v}$ and $\boldsymbol{u}$. The scattering matrixes $\hat{S}(\hat{R})\ \left(\hat{R} \in O_3\right)$ constitute a group that is isomorphic to the group of orthogonal matrixes O_3 (the group of rotations including improper rotations):

$$\hat{S}(\hat{R}_1)\cdot\hat{S}(\hat{R}_2)=\hat{S}(\hat{R}_1\cdot\hat{R}_2),\qquad \hat{S}^{-1}(\hat{R})=\hat{S}(\hat{R}^{-1})\,. \tag{5}$$

$$\left|\det\hat{S}\right|=1,\qquad d\boldsymbol{v}d\boldsymbol{u}=d\boldsymbol{v}'d\boldsymbol{u}'\,. \tag{6}$$

Relation (6) is much simpler than $d\boldsymbol{v}d\boldsymbol{u}d\Omega'=d\boldsymbol{v}'d\boldsymbol{u}'d\Omega$ for the conventional [1] parameterization. Here, it should be noted that parameterization of a collision by the direction $\boldsymbol{n}$ normal to the plane of reflection of the relative velocity ($\mathbf{v}'=\mathbf{v}-2\boldsymbol{n}\,\boldsymbol{n}\cdot\mathbf{v}$) [4] also provides a linear transformation of the particles' velocities. In this case, however, scattering matrixes do not constitute a group, which is crucial for our further consideration.

Boltzmann Collision Integral

To rewrite the collision integral (2) for the case when a collision is parameterized by a rotation matrix, integration over directions of the vector $\boldsymbol{n}$ should be replaced by integration on the invariant measure [5] over the group O_3^+ ($d\Omega_n/4\pi\to d\hat{R}/8\pi^2$):

$$d\hat{R}=d\hat{R}_0\hat{R}=d\hat{R}\hat{R}_0=d\hat{R}^{-1},\qquad d\hat{R}=2(1-\cos\phi)d\phi d\Omega_{\boldsymbol{n}},\quad \int d\hat{R}=8\pi^2\,. \tag{7}$$

In accordance with the usual rules [5] of the Lie groups theory, we can also easily construct a representation of the scattering group in the Hilbert space of functions on bivector $(\boldsymbol{v},\boldsymbol{u})$:

$$e^{-\phi\hat{\sigma}}f(\boldsymbol{v})\psi(\boldsymbol{u})=f(\boldsymbol{v}')\psi(\boldsymbol{u}')\,.$$

The collision integral now takes the following form:

$$I(f,\psi)=\int d\boldsymbol{u}\,\hat{\chi}f(\boldsymbol{v})\psi(\boldsymbol{u})\,,\qquad \hat{\chi}=\int\frac{d\hat{R}}{2\pi}\left[e^{-\phi\hat{\sigma}}-1\right]b(\mathrm{v},\mu)\,, \tag{8}$$

where $b(\mathrm{v},\mu)=\mathrm{v}\,\sigma_\theta(\mathrm{v},\mu)$ is a scattering indicatrix, $\mu\left(\hat{R},\mathbf{v}\right)=\dfrac{\mathbf{v}\cdot\hat{R}\mathbf{v}}{\mathrm{v}^2}$ is the cosine of the scattering angle. The rotation matrixes are parameterized by the angle $(0\le\phi\le\pi)$ of rotation and by the direction $\boldsymbol{n}$ of the rotation axis ($\boldsymbol{n}^2=1$). It is seen from eq. (8), that the rather simple generator $\hat{\sigma}$ determines all general properties of the Boltzmann collision operator:

$$\hat{\sigma}=\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}},\qquad \hat{\boldsymbol{\sigma}}=\frac{\partial}{\partial\mathbf{v}}\times\mathbf{v}=-\mathbf{v}\times\frac{\partial}{\partial\mathbf{v}}=-\frac{1}{1+m}\mathbf{v}\times\left(\frac{\partial}{\partial\boldsymbol{v}}-m\frac{\partial}{\partial\boldsymbol{u}}\right), \tag{9}$$

$$\hat{\boldsymbol{\sigma}}^2=\frac{\partial}{\partial\mathbf{v}}\left(\mathrm{v}^2-\cdot\mathbf{v}\mathbf{v}\cdot\right)\frac{\partial}{\partial\mathbf{v}}=\mathrm{v}^2\left(\frac{\partial}{\partial\mathbf{v}}\right)^2-\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}}\left(1+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}}\right)=\left(\frac{\partial}{\partial\mathbf{v}}\right)^2\mathrm{v}^2+\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{v}\left(1-\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{v}\right). \tag{10}$$

The product of two Maxwellian distributions is an eigenfunction of the operator $\hat{\sigma}$ corresponding to the zero eigenvalue:

$$\hat{\sigma}\,e^{-\frac{m_1v^2}{2kT}}e^{-\frac{m_2u^2}{2kT}}=0\,. \tag{11}$$

The generator $\hat{\sigma}$ commutates with the three invariants and obviously commutates with the Casimir operator $\hat{\boldsymbol{\sigma}}^2$:

$$\left[\hat{\sigma},\mathrm{v}^2\right]=0,\qquad [\hat{\sigma},\mu]=0,\qquad [\hat{\sigma},(m\boldsymbol{v}+\boldsymbol{u})]=0,\qquad \left[\hat{\sigma},\hat{\boldsymbol{\sigma}}^2\right]=0\,. \tag{12}$$

Matrix Elements

Taking into consideration that

$$\delta(\boldsymbol{v}'-\boldsymbol{v}_0)\delta(\boldsymbol{u}'-\boldsymbol{u}_0)=\delta(\widehat{S}\xi-\xi_0)=\delta\left(\xi-\widehat{S}(\widehat{R}^{-1})\xi_0\right) \tag{13}$$

we have decomposition formula for matrix elements:

$$I(\delta(\boldsymbol{v}-\boldsymbol{v}_0),\delta(\boldsymbol{u}-\boldsymbol{u}_0))=I(\delta(\xi-\xi_0))=\int\frac{d\widehat{R}}{4\pi}b\left(\mathbf{v}_0,\widehat{R}\right)\left[\delta(\boldsymbol{v}-\boldsymbol{v}_0')-\delta(\boldsymbol{v}-\boldsymbol{v}_0)\right]. \tag{14}$$

With the help of the Green's function for the Helmholtz equation

$$\left(\nabla_{\boldsymbol{v}}^2-\kappa^2\right)\frac{-e^{-\kappa|\boldsymbol{v}-\boldsymbol{v}_0|}}{4\pi|\boldsymbol{v}-\boldsymbol{v}_0|}=\delta(\boldsymbol{v}-\boldsymbol{v}_0)\,, \tag{15}$$

we obtain from (14) the decomposition of matrix elements on the basis of Yukawa potentials in the velocity space:

$$I(\delta(\boldsymbol{v}-\boldsymbol{v}_0),\delta(\boldsymbol{u}-\boldsymbol{u}_0)) = -\left(\nabla_{\boldsymbol{v}}^2-\kappa^2\right)\int\frac{d\hat{R}}{2\pi}b(\mathrm{v}_0,\mu_0)\left(\frac{e^{-\kappa|\boldsymbol{v}-\boldsymbol{v}_0'|}}{4\pi|\boldsymbol{v}-\boldsymbol{v}_0'|}-\frac{e^{-\kappa|\boldsymbol{v}-\boldsymbol{v}_0|}}{4\pi|\boldsymbol{v}-\boldsymbol{v}_0|}\right). \tag{16}$$

First few moments of the Yukawa potential are:

$$\int\frac{e^{-\kappa|\boldsymbol{v}-\boldsymbol{v}_0|}}{4\pi|\boldsymbol{v}-\boldsymbol{v}_0|}d\boldsymbol{v}=\frac{1}{\kappa^2};\quad \int\frac{e^{-\kappa|\boldsymbol{v}-\boldsymbol{v}_0|}}{4\pi|\boldsymbol{v}-\boldsymbol{v}_0|}\boldsymbol{v}d\boldsymbol{v}=\frac{\boldsymbol{v}_0}{\kappa^2};\quad \int\frac{e^{-\kappa|\boldsymbol{v}-\boldsymbol{v}_*|}}{4\pi|\boldsymbol{v}-\boldsymbol{v}_*|}v^2d\boldsymbol{v}=\frac{v_0^2}{\kappa^2}+\frac{6}{\kappa^4}; \tag{17}$$

When $\kappa = 0$ eq.(16) coincides with the result of C. Villani [6].

Scattering Operator $\hat{\chi}$

The collision integral has the simple representation through the scattering operator $\hat{\chi}$:

$$I(f,\psi)=\int d\boldsymbol{u}\,\hat{\chi}f(\boldsymbol{v})\psi(\boldsymbol{u}),\qquad \hat{\chi}=\int\frac{d\hat{R}}{2\pi}\left[e^{-\phi\hat{\sigma}}-1\right]b(\mathrm{v},\mu). \tag{18}$$

In respect to the scalar product:

$$(\Phi,\mathrm{F})=\int d\boldsymbol{v}d\boldsymbol{u}\,\Phi(\boldsymbol{v},\boldsymbol{u})\,\mathrm{F}(\boldsymbol{v},\boldsymbol{u}), \tag{19}$$

the operator $\hat{\chi}$ is symmetric and the generator $\hat{\sigma}$ is antisymmetric:

$$\tilde{\hat{\chi}}=\hat{\chi},\quad \tilde{\hat{\sigma}}=-\hat{\sigma}. \tag{20}$$

Moments of the Boltzmann collision integral are expressed with the help of the operator $\hat{\chi}$ as follows:

$$\int d\boldsymbol{v}M(\boldsymbol{v})I(f,\psi)=(\hat{\chi}M,f\psi). \tag{21}$$

The operator $\hat{\chi}$ commutates with the product of Maxwellians $\hat{\chi}\,e^{-\frac{m_1v^2}{2kT}}e^{-\frac{m_2u^2}{2kT}}=e^{-\frac{m_1v^2}{2kT}}e^{-\frac{m_2u^2}{2kT}}\hat{\chi}$ that provides:

$$\int d\boldsymbol{u}\,\hat{\chi}\,e^{-\frac{m_1v^2}{2kT}}\overline{f}(\boldsymbol{v})e^{-\frac{m_2u^2}{2kT}}\overline{\psi}(\boldsymbol{u})=e^{-\frac{m_1v^2}{2kT}}\int d\boldsymbol{u}e^{-\frac{m_2u^2}{2kT}}\hat{\chi}\overline{f}(\boldsymbol{v})\overline{\psi}(\boldsymbol{u}). \tag{22}$$

Renormalization of the Boltzmann Collision Integral

There are two independent approaches [6,2]. To renormalize the collision integral, we expand the rotation operator $e^{\phi\hat{\sigma}}$ in the Taylor series with a residual term:

$$e^{\phi\hat{\sigma}}=1+\phi\hat{\sigma}...+\frac{1}{(n-1)!}(\phi\hat{\sigma})^{n-1}+\frac{1}{n!}(\phi\hat{\sigma})^n\int_0^1 d\alpha\,q_n(\alpha)e^{\alpha\phi\hat{\sigma}}, \tag{23}$$

where $q_n(\alpha)=n(1-\alpha)^{n-1}$, $\int_0^1 d\alpha\,q_n(\alpha)=1$ and can be considered as a distribution function of the parameter α. The measure $b(\mathrm{v},\mu)d\hat{R}$ is invariant under the replacement $\hat{R}\to\hat{R}^{-1}$. Hence, the operator $\hat{\chi}$ for any given $n>0$ has the following exact representation:

$$\hat{\chi}=\sum_{k=1}^{2k<n}\frac{1}{(2k)!}\left\langle\phi^{2k}n_{i_1}...n_{i_{2k}}\right\rangle\hat{\sigma}_{i_1}...\hat{\sigma}_{i_{2k}}+\frac{1}{n!}\left\langle\frac{\phi^n\hat{\sigma}^n}{2}\left[e^{\alpha\phi\hat{\sigma}}+(-1)^n e^{-\alpha\phi\hat{\sigma}}\right]\right\rangle, \tag{24}$$

$$\langle[...]\rangle=\int d\alpha q_n(\alpha)d\Omega_{\boldsymbol{n}}\frac{d\phi}{\pi}(1-\cos\phi)b(\mathrm{v},\mu)[...] \tag{25}$$

and the collision operator can be presented in a divergence form $I(f,\psi)=-\frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{J}(f,\psi)$ [2]. As a result, the Boltzmann equation can be rewritten in the form of the Liouville Equation:

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f}{\partial\boldsymbol{r}}+\frac{\partial}{\partial\boldsymbol{v}}\cdot\left(\frac{e_1}{m_1}\boldsymbol{E}+\frac{e_1}{m_1c}\boldsymbol{v}\times\boldsymbol{B}+\boldsymbol{g}+\frac{1}{m_1}\boldsymbol{F_{coll}}\right)f=0, \tag{26}$$

where, in addition to the usual electromagnetic and gravity forces, we have nonlocal kinetic force $\boldsymbol{F}_{coll}=m_1\boldsymbol{J}/f(\boldsymbol{v})$, which depends on the distribution functions $f(\boldsymbol{v})$ and $\psi(\boldsymbol{v})$. Equation (26) allows us to consider the distribution function $f(\boldsymbol{v},\boldsymbol{r},t)$ as a density of points in the phase space, which are moving along continues paths under the influence of the nonlocal force. The points do not jump any more, as it was in the case of the classical Boltzmann equation. This equation provides new opportunities for numerical simulation of gas flows. We also arrive to the ***important physical conclusion:*** It is possible considerably simplify a microdynamics of the gas dynamic system while the evolution of a distribution function remains unchanged.

Expression of the Boltzmann Collision Integral through the Hypergeometric Function.

The scattering operator $\hat{\chi}$ is invariant under the group of rotations and its eigenfunctions are the Legendre polynomials formed with the help of an arbitrary vector $\mathbf{v}_0$:

$$\hat{\chi}\, P_l\left(\frac{\mathbf{v}\cdot\mathbf{v}_0}{\mathrm{v}\,\mathrm{v}_0}\right) = \lambda_l P_l\left(\frac{\mathbf{v}\cdot\mathbf{v}_0}{\mathrm{v}\,\mathrm{v}_0}\right). \tag{27}$$

Eigenvalues of the operator $\hat{\chi}$ are given by the expression:

$$\lambda_l = \int \frac{d\hat{R}}{2\pi} b(\mathrm{v},\mu)[P_l(\mu)-1] = 2\pi\int_{-1}^{1} d\mu\, b(\mathrm{v},\mu)[P_l(\mu)-1] = \langle P_l(\mu)-1\rangle . \tag{28}$$

The differential operator $\hat{\boldsymbol{\sigma}}^2$ that is the Casimir operator for the group of rotations commutates with generators of this group and due to this has the same eigenfunctions:

$$\hat{\boldsymbol{\sigma}}^2 P_l\left(\frac{\mathbf{v}\cdot\mathbf{v}_0}{\mathrm{v}\,\mathrm{v}_0}\right) = \sigma^2(l)\, P_l\left(\frac{\mathbf{v}\cdot\mathbf{v}_0}{\mathrm{v}\,\mathrm{v}_0}\right). \tag{29}$$

Its eigenvalues $\sigma^2(l)$ can be expressed through the index l and vice versa:

$$\sigma^2 = -l(l+1), \qquad l = \frac{-1\pm\sqrt{1-4\sigma^2}}{2}. \tag{30}$$

So that, with the help of eq.(30), we get the way to express the eigenvalues λ_l of the scattering operator $\hat{\chi}$ through the eigenvalues σ^2 of the operator $\hat{\boldsymbol{\sigma}}^2$:

$$\lambda_l = 2\pi\int_{-1}^{1} d\mu\, b(\mathrm{v},\mu)[P_l(\mu)-1] = 2\pi\int_{-1}^{1} d\mu\, b(\mathrm{v},\mu)\left[P_{\frac{-1\pm\sqrt{1-4\sigma^2}}{2}}(\mu)-1\right]. \tag{31}$$

It is well known, that the Legendre polynomials have simple expression through the hypergeometric function:

$$P_l(\mu) = {}_2F_1(a,\, b;\, c;\, x), \qquad a=-l=\frac{1-\sqrt{1-4\sigma^2}}{2},\ b=l+1=\frac{1+\sqrt{1-4\sigma^2}}{2},\ c=1,\ x=\frac{1-\mu}{2}. \tag{32}$$

Substituting the eigenvalues σ^2 by the operator $\hat{\boldsymbol{\sigma}}^2$ itself, we obtain the analytical expression of the scattering operator $\hat{\chi}$ through the operator $\hat{\boldsymbol{\sigma}}^2$:

$$\begin{aligned}\hat{\chi} &= 2\pi\int_{-1}^{1} d\mu\, b(\mathrm{v},\mu)\left[P_{\frac{-1+\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2}}(\mu)-1\right] \\ &= 2\pi\int_{-1}^{1} d\mu\, b(\mathrm{v},\mu)\left[{}_2F_1\left(\frac{1-\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2},\frac{1+\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2};1;\frac{1-\mu}{2}\right)-1\right].\end{aligned} \tag{33}$$

From(33), we easy obtain the expansion with residual term of the scattering operator on polynomials on $\hat{\boldsymbol{\sigma}}^2$:

$$\begin{aligned}\hat{\chi} &= \left\langle\frac{1-\mu}{2}\right\rangle\hat{\boldsymbol{\sigma}}^2 + \left\langle\left(\frac{1-\mu}{2}\right)^2\right\rangle\frac{\hat{\boldsymbol{\sigma}}^2\left(\hat{\boldsymbol{\sigma}}^2+2\right)}{2^2} + \ldots + \left\langle\left(\frac{1-\mu}{2}\right)^{n-1}\right\rangle\frac{\hat{\boldsymbol{\sigma}}^2\ldots\left(\hat{\boldsymbol{\sigma}}^2+(n-2)(n-1)\right)}{1^2 2^2\ldots(n-1)^2} \\ &+ \frac{\hat{\boldsymbol{\sigma}}^2\ldots\left[\hat{\boldsymbol{\sigma}}^2+(n-1)n\right]}{1^2 2^2\ldots n^2}\int_{-1}^{1} 2\pi d\mu\, b_n(\mu)\, {}_2F_1\left(\frac{2n+1-\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2},\frac{2n+1+\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2};1+n;\frac{1-\mu}{2}\right),\end{aligned} \tag{34}$$

$$b_n(\mu) = n\int_{-1}^{\mu}\left(\frac{\mu-\mu_1}{2}\right)^{n-1} b(\mu_1)\frac{d\mu_1}{2}, \qquad \left\langle\frac{1-\mu}{2}\right\rangle^n = \int_{-1}^{1}\left(\frac{1-\mu}{2}\right)^n b(\mu)\,d\mu\,. \tag{35}$$

The general term of series on $\langle(1-\mu)^n\rangle$ of the collision integral has been earlier obtained by A.V. Bobylev in different form [9,10].

$$I(f,\psi)=\int d\boldsymbol{u}\,\hat{\chi}f(\boldsymbol{v})\psi(\boldsymbol{u})=-\frac{\partial}{\partial\boldsymbol{v}}\cdot\hat{\boldsymbol{J}}\,f(\boldsymbol{v})\psi(\boldsymbol{u})\,. \tag{36}$$

Case $n=1$.

$$\hat{\chi}=\hat{\boldsymbol{\sigma}}^2\int_{-1}^{1}d\mu\,b_1(\mu)\,{}_2F_1\left(\frac{3-\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2},\frac{3+\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2};2;\frac{1-\mu}{2}\right),\qquad b_1(\mu)=\int_{-1}^{\mu}b(\mu_1)\frac{d\mu_1}{2}. \tag{37}$$

$$\hat{\boldsymbol{J}}=-\frac{1}{1+m}\int d\boldsymbol{u}\left(\mathrm{v}^2-\mathbf{vv}\,\cdot\right)\frac{\partial}{\partial\mathbf{v}}\int_{-1}^{1}d\mu\,b_1(\mu)\,{}_2F_1\left(\frac{3-\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2},\frac{3+\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2};2;\frac{1-\mu}{2}\right), \tag{38}$$

$$\begin{aligned}\hat{\boldsymbol{\sigma}}^2&=\frac{\partial}{\partial\mathbf{v}}\left(\mathrm{v}^2-\cdot\mathbf{vv}\,\cdot\right)\frac{\partial}{\partial\mathbf{v}}=\mathrm{v}^2\left(\frac{\partial}{\partial\mathbf{v}}\right)^2-\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}}\left(1+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\\&=\left(\frac{\partial}{\partial\mathbf{v}}\right)^2\mathrm{v}^2+\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{v}\left(1-\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{v}\right)=\frac{\partial}{\partial\,\mathrm{v}_i}\frac{\partial}{\partial\,\mathrm{v}_k}\left(\delta_{ik}\,\mathrm{v}^2-\mathrm{v}_i\,\mathrm{v}_k\right)+2\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{v}.\end{aligned} \tag{39}$$

Case $n=2$.

$$\hat{\chi}=\left\langle\frac{1-\mu}{2}\right\rangle\hat{\boldsymbol{\sigma}}^2+\frac{\hat{\boldsymbol{\sigma}}^2\left(\hat{\boldsymbol{\sigma}}^2+2\right)}{4}\int_{-1}^{1}d\mu\,b_2(\mu)\,{}_2F_1\left(\frac{5-\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2},\frac{5+\sqrt{1-4\hat{\boldsymbol{\sigma}}^2}}{2};3;\frac{1-\mu}{2}\right), \tag{40}$$

$$b_2(\mu)=\int_{-1}^{\mu}\left(\frac{\mu-\mu_1}{2}\right)b(\mu_1)\,d\mu_1\,. \tag{41}$$

For the Coulomb collisions the collision indicatrix reads as:

$$b(\mathrm{v},\mu)=(1+m)^2\left(\frac{e_1e_2}{m_1}\right)^2\frac{1}{\mathrm{v}^3(1-\mu)^2} \tag{42}$$

and has nonintegrable singularity at $\mu=1$. Renormalization procedure (38), (40) reduces this singularity:

$$b_1(\mu)=\int_{-1}^{\mu}b(\mu_1)\frac{d\mu_1}{2}=\frac{1}{2}\left[\frac{1}{1-\mu}-\frac{1}{2}\right], \tag{43}$$

$$b_2(\mu)=\int_{-1}^{\mu}\left(\frac{\mu-\mu_1}{2}\right)b(\mu_1)\,d\mu_1=\frac{(1+m)^2}{\mathrm{v}^3}\left(\frac{e_1e_2}{m_1}\right)^2\frac{1}{2}\left[\ln\frac{2}{(1-\mu)}-\frac{1+\mu}{2}\right], \tag{44}$$

$$\left\langle\left(\frac{1-\mu}{2}\right)^n\right\rangle_2=2\pi\int_{-1}^{1}\left(\frac{1-\mu}{2}\right)^n b_1(\mu)\,d\mu=\frac{2\pi}{(n+1)^2(n+2)}\frac{(1+m)^2}{\mathrm{v}^3}\left(\frac{e_1e_2}{m_1}\right)^2. \tag{45}$$

As it is seen from eq.(45), the renormalized indicatrix $b_2(\mu)$ has finite moments:

Boltzmann Collision Operator near Equilibrium. Landau Integral.

The equilibrium two-particle distribution function $F_M=f_M(\boldsymbol{v})\psi_M(\boldsymbol{u})$ is a product of two Maxwellians. In the variables: a relative velocity $\mathbf{v}$ and a center of mass velocity $\boldsymbol{w}$, the function F_M depends on $\mathbf{v}$ through the vector squared only: $F_M=F_M(\mathbf{v}^2)$. Near the equilibrium a small anisotropy of the two-particles distribution function can be approximated by the finite sum of spherical harmonics[8]:

$$\tilde{F}_L(\mathbf{v})=F_0(\mathbf{v}^2)+\boldsymbol{Y}_1(\mathbf{v})\cdot\boldsymbol{F}_1(\mathbf{v}^2)+...+\boldsymbol{Y}_L(\mathbf{v})\cdot\boldsymbol{F}_L(\mathbf{v}^2)\,. \tag{46}$$

It is remarkable, that the action of the scattering operator $\hat{\chi}$ on the functions $\tilde{F}_L(\mathbf{v})$ is equivalent to the action of the L first terms in Taylor series of hypergeometric function (34):

$$\hat{\chi}\tilde{F}_L=\left[\left\langle\frac{1-\mu}{2}\right\rangle\hat{\boldsymbol{\sigma}}^2+\left\langle\left(\frac{1-\mu}{2}\right)^2\right\rangle\frac{\hat{\boldsymbol{\sigma}}^2\left(\hat{\boldsymbol{\sigma}}^2+2\right)}{1^22^2}+...+\left\langle\left(\frac{1-\mu}{2}\right)^L\right\rangle\frac{\hat{\boldsymbol{\sigma}}^2...\left(\hat{\boldsymbol{\sigma}}^2+(L-1)L\right)}{1^22^2...L^2}\right]\tilde{F}_L\,. \tag{47}$$

The Boltzmann collision integral reduces to the following:

$$I(\tilde{F}_L) = \int d\boldsymbol{u}\left[\left\langle\frac{1-\mu}{2}\right\rangle\hat{\boldsymbol{\sigma}}^2 + \left\langle\left(\frac{1-\mu}{2}\right)^2\right\rangle\frac{\hat{\boldsymbol{\sigma}}^2\left(\hat{\boldsymbol{\sigma}}^2+2\right)}{1^2 2^2} + \ldots + \left\langle\left(\frac{1-\mu}{2}\right)^L\right\rangle\frac{\hat{\boldsymbol{\sigma}}^2\ldots\left(\hat{\boldsymbol{\sigma}}^2+(L-1)L\right)}{1^2 2^2 \ldots L^2}\right]\tilde{F}_L \,. \qquad (48)$$

The first term in eq.(48)

$$\frac{1}{2}\int d\boldsymbol{u}\langle 1-\mu\rangle\boldsymbol{\sigma}^2\tilde{F} = \frac{1}{2(1+m)^2}\frac{\partial}{\partial v_i}\int du\langle 1-\mu\rangle\left(\mathrm{v}^2\,\delta_{ik} - \mathrm{v}_i\,\mathrm{v}_k\right)\left(\frac{\partial}{\partial v_k} - m\frac{\partial}{\partial u_k}\right)\tilde{f}(\boldsymbol{v})\tilde{\psi}(\boldsymbol{u}) \qquad (49)$$

is exactly (for Rutherford cross section with cut-off) the Landau-Fokker-Plank collision integral [7] . Note, that this integral has appeared here without any assumption of scattering on the small angles. Eq.(48) manifests the universality of the dynamics near the equilibrium. The only one moment of collision indicatrix, the transport collision frequency $\langle 1-\mu\rangle$, plays a major role near the equilibrium for gas mixtures.

For a simple gas the situation is slightly different. We have $f = \psi$ and this leads to $F(-\mathbf{v}) = F(\mathbf{v})$. Hence, the odd terms in eq.(46) are equal to zero identically, $\boldsymbol{F}_{2k+1}(\mathbf{v}^2) = 0,\ k = 0,1,\ldots$. Due to this, only the even eigenvalues λ_{2k} of the scattering operator are important for simple gas. Near the equilibrium the two-particles distribution function can be approximated by the sum of two spherical harmonics with $l = 0$ and $l = 2$:

$$\tilde{F}_2(\mathbf{v}) = F_0(\mathbf{v}^2) + \left(\mathrm{v}_i\mathrm{v}_k - \frac{\mathrm{v}^2}{3}\delta_{ik}\right)F_{ik}(\mathbf{v}^2) \qquad (50)$$

and the only two eigenvalues of the scattering operator

$$\lambda_0 = 0 \quad \text{and} \quad \lambda_2 = \langle P_2(\mu) - 1\rangle = \frac{3}{2}\left\langle\mu^2 - 1\right\rangle \qquad (51)$$

will play a major role. It is easy to see, that the first two even eigenvalues of the operator $\frac{1}{4}\left\langle 1-\mu^2\right\rangle\hat{\boldsymbol{\sigma}}^2$ coincides with eigenvalues (51):

$$(\hat{\chi}-1)\tilde{F}_2 = \frac{1}{4}\left\langle 1-\mu^2\right\rangle\hat{\boldsymbol{\sigma}}^2\tilde{F}_2\,. \qquad (52)$$

It is also very important that other all his eigenvalues are negative and provide a stability. Hence, near the equilibrium the Boltzmann collision integral for a simple gas can be substituted by the Landau-Fokker-Plank collision integral with the corrected collision frequency $\frac{1}{2}\left\langle 1-\mu^2\right\rangle$:

$$I(\tilde{F}_2) = \int d\boldsymbol{u}\left\langle\frac{1-\mu^2}{4}\right\rangle\hat{\boldsymbol{\sigma}}^2\tilde{F}_2 = \frac{1}{16}\frac{\partial}{\partial v_i}\int du\left\langle 1-\mu^2\right\rangle\left(\mathrm{v}^2\,\delta_{ik} - \mathrm{v}_i\,\mathrm{v}_k\right)\left(\frac{\partial}{\partial v_k} - \frac{\partial}{\partial u_k}\right)\tilde{f}(\boldsymbol{v})\tilde{f}(\boldsymbol{u}) \qquad (53)$$

This equation has appeared (with different motivation) and been discussed in the interesting preprint [10].

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