# Grad's Thirteen Moment Method for a Heterogeneous Dispersed Medium

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**Abstract.** From pollution problems to jet propulsion, two-phase dispersed media appear in a wide range of engineering systems and industrial processes. Consequently, the identification of equations allowing a good prediction of the behaviour of such media is very important. The aim of this paper is to provide an Eulerian description for a heterogeneous suspension constituted of different solid particle species. In this work, the carrier fluid is assumed to be a viscous gas and the collision between particles are supposed to be instantaneous, binary, weakly inelastic and non punctual. Kinetic equations for such media have been written in [1]. As in the usual kinetic theory, the method of Grad's thirteen moments [2] is used to obtain an approximation of the collision integral of the transport equation. Thus, a macroscopic mass conservation equation for each species and global balance equations for momentum and energy can be obtained.

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## INTRODUCTION

Polydispersed media appear in a large variety of environmental and industrial applications like the pneumatic conveying of coal powder for instance. Consequently, the identification of relevant equations for these media is a great subject of investigation for the scientific community. In the case of coal powder, the particles are quasi spherical and their diameters vary from $2\ 10^{-4}\ m$ to $2\ 10^{-3}\ m$. The volumetric masses of the carrying gas and of the material of the particles are very different ($\rho_{gas} = 1{,}205\ kg \cdot m^{-3}$ and $\rho_{coal} = 1035\ kg \cdot m^{-3}$) and the volume fraction of the particles is very small: It varies from $5\ 10^{-4}$ to $4\ 10^{-2}$ [3]. The aim of this paper is to introduce an Eulerian description for such suspensions of heterogeneous spheres by a method resulting from the kinetic theory.

In the next, section we present the main assumptions about the suspension and the collisional process and we recall, briefly, the main results on the kinetic description of a suspension of heterogeneous spheres having instantaneous, binary and inelastic collisions [1]. In the following section, as in the usual kinetic theory, the method of Grad's thirteen moments [4, 5, 2] is used to obtain an approximation of the collision integral of the transport equation. Thus, a macroscopic mass conservation equation for each species and global balance equations for momentum and energy are introduced. Besides the conservation laws, the existence of an H-theorem for this description, that is the compatibility of the kinetic equation with the thermodynamic second principle, is analysed. The last section is devoted to a short conclusion

## KINETIC DESCRIPTION OF THE SUSPENSION

### Main properties and assumptions

In this work, the carrier fluid is assumed to be a viscous gas and the collision between particles are supposed to be instantaneous, binary, weakly inelastic and non punctual. The suspension is assumed to be heterogenous and to be made up of $N$ particle species. A particle $\alpha$ ($\alpha = 1, ..., N$) is assumed to be a sphere of diameter $\sigma_\alpha$ and of mass $m_\alpha$. It is centered at position $\mathbf{x}_\alpha$ and has the velocity $\mathbf{v}_\alpha$. First, the collision of two spheres $P_\alpha$ and $P_\beta$ is considered. We put $\sigma_{\alpha\beta} = (\sigma_\alpha + \sigma_\beta)/2$ and we suppose that, before the collision the particles $P_\alpha$ and $P_\beta$ are centered at position $\mathbf{x}_\alpha$ and

$\mathbf{x}_\beta$ and have the velocities $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$. The relative velocity $\mathbf{g}_{\beta\alpha}$ and the impact vector $\mathbf{k}$ are defined by:

$$\mathbf{g}_{\beta\alpha} = \mathbf{v}_\beta - \mathbf{v}_\alpha \qquad \text{and} \qquad \mathbf{k} = \frac{\mathbf{x}_\alpha - \mathbf{x}_\beta}{|\, \mathbf{x}_\alpha - \mathbf{x}_\beta \,|} \tag{1}$$

After the collision, the particles are still centered in $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ and they have the velocities $\mathbf{v}'_\alpha$ and $\mathbf{v}'_\beta$. The relative velocity after collision is $\mathbf{g}'_{\beta\alpha} = \mathbf{v}'_\beta - \mathbf{v}'_\alpha$. Furthermore, because of the inelasticity of the collisions, we introduce the coefficient of restitution $e_{\alpha\beta}$:

$$\mathbf{g'}_{\beta\alpha} \cdot \mathbf{k} = -e_{\alpha\beta} \left( \mathbf{g}_{\beta\alpha} \cdot \mathbf{k} \right) \tag{2}$$

with $0 < e_{\alpha\beta} = e_{\beta\alpha} \leq 1$. From experimental works [3], in the case of coal powder, the collisions are weakly inelastic and the restitution coefficient is $e_{\alpha\beta} \simeq 0,95$. If $e_{\alpha\beta} = 1$, the collisions are elastic and energy is conserved during the collisions. By considering the momentum balance equation and the previous assumptions, it's easy to express the velocities after collision in term of those before:

$$\mathbf{v}'_\alpha = \mathbf{v}_\alpha + \frac{(1+e_{\alpha\beta})\, m_\beta}{m_\alpha + m_\beta} \left( \mathbf{g}_{\beta\alpha} \cdot \mathbf{k} \right) \mathbf{k} \qquad \text{and} \qquad \mathbf{v}'_\beta = \mathbf{v}_\beta - \frac{(1+e_{\alpha\beta})\, m_\alpha}{m_\alpha + m_\beta} \left( \mathbf{g}_{\beta\alpha} \cdot \mathbf{k} \right) \mathbf{k} \tag{3}$$

and vice-versa:

$$\mathbf{v}_\alpha = \mathbf{v}'_\alpha + \frac{(1+e_{\alpha\beta})\, m_\beta}{(m_\alpha + m_\beta) e_{\alpha\beta}} \left( \mathbf{g}'_{\beta\alpha} \cdot \mathbf{k} \right) \mathbf{k} \qquad \text{and} \qquad \mathbf{v}_\beta = \mathbf{v}'_\beta - \frac{(1+e_{\alpha\beta})\, m_\alpha}{(m_\alpha + m_\beta) e_{\alpha\beta}} \left( \mathbf{g}'_{\beta\alpha} \cdot \mathbf{k} \right) \mathbf{k} \tag{4}$$

## Boltzman equation for the suspension

In order to provide a kinetic description for a heterogeneous suspension constituted of $N$ different solid particle species, it is convenient to introduce a distribution function $f_\alpha(\mathbf{v}_\alpha, \mathbf{x}_\alpha, t)$ for each species and to write a Boltzmann equation for each of these $N$ distribution functions. We recall the Boltzmann equation introduced in a previous work [1] for the species $\alpha$:

$$\frac{\partial f_\alpha}{\partial t}(\mathbf{v}_\alpha, \mathbf{x}_\alpha, t) + \mathbf{v}_\alpha \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}_\alpha}(\mathbf{v}_\alpha, \mathbf{x}_\alpha, t) + \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot [\mathbf{F}_\alpha\, f_\alpha(\mathbf{v}_\alpha, \mathbf{x}_\alpha, t)] = \sum_{\beta=1}^{N} J_{\alpha\beta} \left[ f_\alpha(\mathbf{v}_\alpha, \mathbf{x}_\alpha, t) \,,\, f_\beta \left( \mathbf{v}_\beta, \mathbf{x}_\beta, t \right) \right] \tag{5}$$

where $J_{\alpha\beta}(f_\alpha\,,\, f_\beta)$ is the part of the collision operator associated with the collisions between the particles $\alpha$ and $\beta$ and $\mathbf{F}_\alpha$ is the external force per unit of mass acting on a particle $\alpha$. This force depends on the velocities and on the nature of the fluid and of the particle. It can be taken as [3]:

$$\mathbf{F}_\alpha = \frac{18 \rho_f\, \nu_f}{\hat{\rho}_\alpha\, \sigma_\alpha^2} \left[ 1 + 0.15 \left( \frac{\sigma_\alpha}{\nu_f} \,|\, \mathbf{v}_\alpha - \mathbf{u}_f \,| \right)^{0.687} \right] (\mathbf{v}_\alpha - \mathbf{u}_f) \tag{6}$$

where $\rho_f$ and $\hat{\rho}_p$ are the volumetric masses of the fluid and of the particle material, $\nu_f$ the kinematic viscosity of the fluid and $\mathbf{u}_f$ the fluid velocity.

In order to express the collision integral, we introduce the distribution function $f^{(2)}_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{x}_\alpha, \mathbf{v}_\beta, \mathbf{x}_\beta, t)$ which characterizes the statistic of binary collisions and depends on the velocities and positions of two particles $\alpha$ and $\beta$ and time. It is defined so that $f^{(2)}_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{x}_\alpha, \mathbf{v}_\beta, \mathbf{x}_\beta, t)\, d\mathbf{v}_\alpha\, d\mathbf{v}_\beta\, d\mathbf{x}_\alpha\, d\mathbf{x}_\beta$ is the probable number of pairs of particles which are, at time $t$, in $(\mathbf{x}_\alpha, d\mathbf{x}_\alpha)$ and $(\mathbf{x}_\beta, d\mathbf{x}_\beta)$ with velocities respectively in $(\mathbf{v}_\alpha, d\mathbf{v}_\alpha)$ and $(\mathbf{v}_\beta, d\mathbf{v}_\beta)$.We introduce also the pseudo-inverse collision [2, 6, 7], that is, the collision during which the two particles $P_\alpha$ and $P_\beta$ collide with the velocities $\mathbf{v}''_\alpha$ and $\mathbf{v}''_\beta$ before collision and $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$ after collision. Under these assumptions, we have:

$$\begin{aligned} J_{\alpha\beta}(\mathbf{x}_\alpha, t) \quad &= \quad \frac{\sigma^2_{\alpha\beta}}{e^2_{\alpha\beta}} \int_{(\mathbf{g}_{\beta\alpha} \cdot \mathbf{k}) > 0} \left[ f^{(2)}_{\alpha\beta}(\mathbf{v}''_\alpha, \mathbf{x}_\alpha, \mathbf{v}''_\beta, \mathbf{x}_\alpha + \sigma_{\alpha\beta} \mathbf{k}, t) \right. \\ &- \quad \left. f^{(2)}_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{x}_\alpha, \mathbf{v}_\beta, \mathbf{x}_\alpha - \sigma_{\alpha\beta} \mathbf{k}, t) \right] (\mathbf{g}_{\beta\alpha} \cdot \mathbf{k}) d\mathbf{k} d\mathbf{v}_\beta \end{aligned} \tag{7}$$

In order to obtain a full modelling of the Boltzmann collision operator, the distribution function $f^{(2)}_{\alpha\beta}(\mathbf{v}_\alpha,\mathbf{x}_\alpha,\mathbf{v}_\beta,\mathbf{x}_\beta,t)$ has to be modelled. We put [8]:

$$f^{(2)}_{\alpha\beta}(\mathbf{v}_\alpha,\mathbf{x}_\alpha,\mathbf{v}_\beta,\mathbf{x}_\beta,t)=\chi_{\alpha\beta}(\mathbf{x}_\alpha,\mathbf{x}_\beta)\,f_\alpha(\mathbf{v}_\alpha,\mathbf{x}_\alpha,t)\,f_\beta(\mathbf{v}_\beta,\mathbf{x}_\beta,t) \tag{8}$$

where $\chi_{\alpha\beta}$ is the radial distribution function that takes into account that the hard particles cannot penetrate each other. The expression of this distribution function has been discussed in several papers [8, 9, 10, 11, 12]. In the standard Enskog theory, $\chi_{\alpha\beta}$ is the local equilibrium value of the radial distribution function evaluated as a function of the local density $n$ at some particular point $\mathbf{X}$. Different choices are possible for this point: The point of contact of the two colliding particles, the midpoint of the line connecting the centres of the two colliding spheres or the center of mass of the two colliding spheres for instance. Obviously, in the case of a monodispersed suspension, these three choices coincide but, when the particles $P_\alpha$ and $P_\beta$ are different, each choice leads to a different solution of the kinetic equation and none is to be preferred because all of them lead to a conflict with irreversible thermodynamics [9]. Van Beijeren and Ernst [10] propose a revision of the Enskog theory in order to avoid this matter: $\chi_{\alpha\beta}$ has to be taken as the following non local functional of the density fields of the components of the heterogeneous medium:

$$\chi_{\alpha\beta}(\mathbf{x}_\alpha,\mathbf{x}_\beta)=1+\sum_{\delta=1}^{N}\int n_\delta(\mathbf{x}_\delta,t)\,V(\mathbf{x}_\alpha,\mathbf{x}_\beta\mid\mathbf{x}_\delta)\,d\mathbf{x}_\delta+\frac{1}{2}\sum_{\delta,\gamma=1}^{N}\int n_\delta(\mathbf{x}_\delta,t)\,n_\gamma(\mathbf{x}_\gamma,t)\,V(\mathbf{x}_\alpha,\mathbf{x}_\beta\mid\mathbf{x}_\delta,\mathbf{x}_\gamma)\,d\mathbf{x}_\delta\,d\mathbf{x}_\gamma+\ldots \tag{9}$$

where $n_\delta(\mathbf{x}_\delta,t)$ is the numerical density of the species $\delta$:

$$n_\delta(\mathbf{x}_\delta,t)=\int f_\delta(\mathbf{v}_\delta,\mathbf{x}_\delta,t)\,d\mathbf{v}_\delta \tag{10}$$

and where the Husimi functions $V(\mathbf{x}_\alpha,\mathbf{x}_\beta\mid\mathbf{x}_1,\ldots,\mathbf{x}_p)$ depend only on the relative distances between particles and represent the sum of all Mayer graphs of $p$ labelled points, which become bi-connected, when the $(\alpha,\beta)$ bond, that is the Mayer function $h_{\alpha\beta}$, is added. In the case of hard spheres:

$$h_{\alpha\beta}(x)=\begin{cases}-1 & \text{si} \quad x<\sigma_{\alpha\beta}\\ 0 & \text{si} \quad x\geq\sigma_{\alpha\beta}\end{cases}$$

These Husimi functions and more details on the theory of graphs are given in [13, 14].

## BALANCE EQUATIONS AND H-THEOREM

### Transport equation

First, we define in terms of the distribution function $f_\alpha$, the mean value $<\gamma>_\alpha$ of a given particle property $\gamma$ by the following expression:

$$<\gamma>_\alpha=\frac{1}{n_\alpha}\int\gamma\,f_\alpha(\mathbf{v}_\alpha,\mathbf{x},t)\,d\mathbf{v}_\alpha \tag{11}$$

The velocity fluctuation $\mathbf{c}_\alpha$ is defined so that $\mathbf{c}_\alpha=\mathbf{v}_\alpha-\mathbf{u}_\alpha$ where $\mathbf{u}_\alpha=<\mathbf{v}_\alpha>_\alpha$ is the mean velocity of the species $\alpha$. As in the usual kinetic theory [8, 15], the Boltzmann equation (5) is multiplied by a function $\psi_\alpha(\mathbf{c}_\alpha)$ and integrated on the $\mathbf{v}_\alpha$ velocity space in order to obtain a transport equation. Concerning the first member of this equation, we have obviously:

$$\begin{aligned}&\int\left\{\frac{\partial f_\alpha}{\partial t}(\mathbf{v}_\alpha,\mathbf{x},t)\,+\,\mathbf{v}_\alpha\cdot\frac{\partial f_\alpha}{\partial\mathbf{x}}(\mathbf{v}_\alpha,\mathbf{x},t)\,+\,\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot[\mathbf{F}_\alpha\,f_\alpha(\mathbf{v}_\alpha,\mathbf{x},t)]\right\}\psi_\alpha\,d\mathbf{v}_\alpha=\frac{\partial}{\partial t}[n_\alpha<\psi_\alpha>_\alpha]\\ +&\quad n_\alpha\frac{\partial u_{\alpha p}}{\partial t}\left\langle\frac{\partial\psi_\alpha}{\partial c_{\alpha p}}\right\rangle_\alpha+\frac{\partial}{\partial x_p}[n_\alpha<\psi_\alpha c_{\alpha p}>_\alpha]+\frac{\partial}{\partial x_p}[n_\alpha<\psi_\alpha>_\alpha u_{\alpha p}]+n_\alpha\frac{\partial u_{\alpha p}}{\partial x_q}\left\langle c_{\alpha q}\frac{\partial\psi_\alpha}{\partial c_{\alpha p}}\right\rangle_\alpha\\ +&\quad n_\alpha\frac{\partial u_{\alpha p}}{\partial x_q}u_{\alpha q}\left\langle\frac{\partial\psi_\alpha}{\partial c_{\alpha p}}\right\rangle_\alpha-n_\alpha\left\langle F_{\alpha p}\frac{\partial\psi_\alpha}{\partial c_{\alpha p}}\right\rangle_\alpha\end{aligned} \tag{12}$$

where the Einstein convention is used for the latin indices only. Concerning the right hand side member of the transport equation, we recall briefly the results presented in a previous paper [1]. After a few technical steps (change of notations, use of the pseudo-inverse collision, exchange between particles $P_\alpha$ and $P_\beta$), the collision integral of the transport equation is written in a form that can be easily used to prove exactly the conservation of mass for each species:

$$\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial x_p}\left(\rho_\alpha u_{\alpha p}\right) = 0 \tag{13}$$

An additionnal step of symmetrization of the collision operator is required to analyse the global balance of momentum and energy. The following property on the distribution function $\chi_{\alpha\beta}$ is used:

$$\chi_{\alpha\beta}(\mathbf{x}_\alpha, \mathbf{x}_\beta) = \chi_{\beta\alpha}(\mathbf{x}_\beta, \mathbf{x}_\alpha) \tag{14}$$

and all the particle diameters are assumed to be small compared to the characteritic length of the flow. Consequently, Taylor expansions allow to separate the contribution of the collisions in the balance equations in a flux term and in a source term. This process leads to the following expression of the right hand side of the transport equation [1]:

$$\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\int J_{\alpha\beta}(\mathbf{x},t)\,\psi_\alpha(\mathbf{c}_\alpha)\,d\mathbf{v}_\alpha = \lambda(\psi) + \frac{\partial}{\partial x_i}\theta_i(\psi) - \theta_i(\frac{\partial \psi}{\partial x_i}) \tag{15}$$

with:

$$\begin{aligned}\lambda(\psi) \quad = \quad & \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\frac{\sigma_{\alpha\beta}^2}{2}\int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0}\Big(\psi_\alpha(\mathbf{c}_\alpha^{'}) - \psi_\alpha(\mathbf{c}_\alpha) + \psi_\beta(\mathbf{c}_\beta^{'}) - \psi_\beta(\mathbf{c}_\beta)\Big) \\ & f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x}, \mathbf{v}_\beta, \mathbf{x} - \sigma_{\alpha\beta}\mathbf{k}, t)(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})\,d\mathbf{v}_\alpha\,d\mathbf{v}_\beta\,d\mathbf{k}\end{aligned} \tag{16}$$

and:

$$\begin{aligned}\theta_i(\psi) \quad = \quad & \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\frac{\sigma_{\alpha\beta}^3}{2}\int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0}\Big(\psi_\beta(\mathbf{c}_\beta^{'}) - \psi_\beta(\mathbf{c}_\beta)\Big)\,k_i\left\{1 - \frac{\sigma_{\alpha\beta}k_j}{2}\frac{\partial}{\partial x_j} + \frac{\sigma_{\alpha\beta}^2 k_j k_m}{6}\frac{\partial^2}{\partial x_j \partial x_m}\right\} \\ & f_{\beta\alpha}^{(2)}(\mathbf{v}_\beta, \mathbf{x}, \mathbf{v}_\alpha, \mathbf{x} + \sigma_{\alpha\beta}\mathbf{k}, t)(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})\,d\mathbf{v}_\alpha\,d\mathbf{v}_\beta\,d\mathbf{k}\ + O(\sigma^6)\end{aligned} \tag{17}$$

where $\sigma$ is the order of magnitude of the diameters of the particles. According to this expression, it's easy to observe that, because of the inelasticity of the collisions, energy is not conserved. Moreover, the source term of the momentum balance equation, $\lambda(m\mathbf{c})$, is exactly equal to zero. Obviously, $\theta_i(\frac{\partial m\mathbf{c}}{\partial x_i})$ is also exactly equal to zero and, consequently, the balance equation for the global momentum is reduced to:

$$\sum_{\alpha=1}^{N}\left[\rho_\alpha\left(\frac{\partial u_{\alpha i}}{\partial t} + u_{\alpha p}\frac{\partial u_{\alpha i}}{\partial x_p}\right) + \frac{\partial}{\partial x_p}\left(\rho_\alpha\left\langle c_{\alpha i}c_{\alpha p}\right\rangle_\alpha\right) - \rho_\alpha < F_{\alpha i} >_\alpha\right] = +\frac{\partial}{\partial x_j}\theta_j(m c_i) \tag{18}$$

## H-Theorem

In this section, we pay attention to the H-theorem. As usually, the transport equation is written with $\psi_\alpha = \ln f_\alpha$. By using (12) and (15) and by remarking that the flux term $\theta_i$ is of order $\sigma^3$, we can write:

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot\mathscr{H} = \lambda(\ln f)\ + O(\sigma^3) \tag{19}$$

with $H = \sum_{\alpha=1}^{N}\int f_\alpha \ln f_\alpha\, d\mathbf{v}_\alpha$ and $\mathscr{H} = \sum_{\alpha=1}^{N}\int \mathbf{v}_\alpha\ f_\alpha \ln f_\alpha\, d\mathbf{v}_\alpha$. If, in addition, we suppose $1 - e_{\alpha\beta} = O(1-e) << 1$, we have:

$$\begin{aligned}\frac{\partial H}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot\mathscr{H} \quad = \quad & \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\frac{\sigma_{\alpha\beta}^2}{4}\int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0}(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})\,\chi_{\alpha\beta}(\mathbf{x},\mathbf{x}_\beta)\,\ln\frac{f_\alpha^{'} f_\beta^{'}}{f_\alpha f_\beta}\Big(f_\alpha f_\beta - f_\alpha^{'} f_\beta^{'}\Big)\,d\mathbf{k}\,d\mathbf{v}_\beta d\mathbf{v}_\alpha \\ + \quad & O(\sigma^3) + O(1-e) \leq 0\end{aligned} \tag{20}$$

where $f'_\alpha = f_\alpha(\mathbf{v}'_\alpha, \mathbf{x}, t)$. The H-theorem is proved in the limit of punctual particles and of a very small inelasticity. Now let us find the distribution function such that the second member of (20) is zero. By proceeding like [8], we obtain the following Maxwellian solution:

$$\bar{f}_{\alpha o}(\mathbf{v}_\alpha, \mathbf{x}, t) = n_\alpha \left(\frac{1}{2\pi T}\right)^{\frac{3}{2}} exp\left[\frac{-(\mathbf{v}_\alpha - \mathbf{u})^2}{2T}\right] \quad \text{with} \quad \mathbf{u} = \sum_{\alpha=1}^{N} \frac{n_\alpha}{n} \mathbf{u}_\alpha \quad \text{and} \quad T = \sum_{\alpha=1}^{N} \frac{n_\alpha}{n} T_\alpha \tag{21}$$

with: $T_\alpha = < c_\alpha^2 >_\alpha /3$ and $n = \sum_{\alpha=1}^{N} n_\alpha$. It can be pointed out that $\mathbf{u}$ is the barycentric velocity of the suspension and T the global kinetic temperature. In other words, in the thermodynamical equilibrium state, there are a unique velocity $\mathbf{u}$ and a unique temperature $T$ for all the species : $\mathbf{u}_\alpha = \mathbf{u}$ and $T_\alpha = T$ for $\alpha = 1, ...N$. Nevertheless, in the following section, a Maxwellian distribution function will be associated with each species.

## Grad's thirteen moment method

Following Grad [4], we express the single particle distribution function of each species as the following expansion:

$$f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t) = f_\alpha = \left[1 + \frac{a_{\alpha ij}}{2T_\alpha^2} c_{\alpha i} c_{\alpha j} - \frac{a_{\alpha imm} c_{\alpha i}}{10 T_\alpha^2}\left(5 - \frac{c_\alpha^2}{T_\alpha}\right)\right] f_{\alpha o}(\mathbf{v}_\alpha, \mathbf{x}, t) \tag{22}$$

where $c_\alpha = \mid \mathbf{c}_\alpha \mid$ and where $f_{\alpha o}(\mathbf{v}_\alpha, \mathbf{x}, t)$ is the Maxwellian distribution function given in terms of the particle number density $n_\alpha$, the velocity fluctuation $\mathbf{c}_\alpha$ and the temperature $T_\alpha$ by:

$$f_{\alpha o}(\mathbf{v}_\alpha, \mathbf{x}, t) = \frac{n_\alpha(\mathbf{x}, t)}{(2\pi T_\alpha(\mathbf{x}, t))^{\frac{3}{2}}} exp\left[\frac{-(\mathbf{c}_\alpha(\mathbf{x}, t))^2}{2T_\alpha(\mathbf{x}, t)}\right] \tag{23}$$

The thirteen moments $a_{\alpha i}$, $a_{\alpha ij}$, $a_{\alpha imm}$ ... depend only on $\mathbf{x}$ and on time $t$. It is easy to see that $a_{\alpha i} = 0$ and $a_{\alpha ij} = a_{\alpha ji}$. In order to express the flux term in (18), we expand the two particle distribution function $f^{(2)}$ in (17) around $\mathbf{x}$. The first step consists in the expansion of the radial distribution function (9). By setting $\mathbf{x}_\beta = \mathbf{x} + \sigma_{\alpha\beta}\, \mathbf{k}$ we obtain:

$$\chi_{\alpha\beta}(\mathbf{x}, \mathbf{x}_\beta) = \chi_{\alpha\beta o}(\mathbf{x}, \mathbf{x}_\beta) \quad + \quad \sum_{\delta=1}^{N} \frac{\partial n_\delta}{\partial \mathbf{x}} \cdot \int (\mathbf{x}_\delta - \mathbf{x})\, H(\mathbf{x}, \mathbf{x}_\beta, \mathbf{x}_\delta)\, d\mathbf{x}_\delta \tag{24}$$

with

$$H(\mathbf{x}, \mathbf{x}_\beta, \mathbf{x}_\delta) = V(\mathbf{x}, \mathbf{x}_\beta \mid \mathbf{x}_\delta) + \sum_{\gamma=1}^{N} n_\gamma(\mathbf{x}) \int V(\mathbf{x}, \mathbf{x}_\beta \mid \mathbf{x}_\delta, \mathbf{x}_\gamma)\, d\mathbf{x}_\gamma + ... \tag{25}$$

and where $\chi_{\alpha\beta o}$ is the equilibrium radial distribution function for spheres of species $\alpha$ and $\beta$ in contact. It is evaluated as a function of the local density fields at the point $\mathbf{x}$:

$$\chi_{\alpha\beta o}(\mathbf{x}, \mathbf{x}_\beta) = 1 + \sum_{\delta=1}^{N} n_\delta(\mathbf{x}, t) \int V(\mathbf{x}, \mathbf{x}_\beta \mid \mathbf{x}_\delta)\, d\mathbf{x}_\delta + \sum_{\delta,\gamma=1}^{N} n_\delta(\mathbf{x}, t)\, n_\gamma(\mathbf{x}, t) \int V(\mathbf{x}_\alpha, \mathbf{x}_\beta \mid \mathbf{x}_\delta, \mathbf{x}_\gamma)\, d\mathbf{x}_\delta\, d\mathbf{x}_\gamma + ... \tag{26}$$

By ending the expansion of $f^{(2)}$ [1] and by using the collision relations (3), the flux term of the momentum balance equation can be written as:

$$\theta_i(m c_j) = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \left[\frac{(1 + e_{\alpha\beta})\, m_\alpha\, m_\beta}{2(m_\alpha + m_\beta)} \sum_{p=1}^{5} J_{\alpha\beta\, ij}^{(p)}\right] + O(\sigma^5) \tag{27}$$

with:

$$J_{\alpha\beta\, ij}^{(1)} = -\sigma_{\alpha\beta}^3\, \chi_{\alpha\beta o} \int_{(\mathbf{g}_{\beta\alpha} \cdot \mathbf{k}) > 0} (\mathbf{g}_{\beta\alpha} \cdot \mathbf{k})^2\, k_i k_j\, f_\alpha\, f_\beta \left[1 - \frac{\sigma_{\alpha\beta}}{2} k_p \frac{\partial}{\partial x_p}\left\{ln \frac{f_\alpha}{f_\beta}\right\}\right] d\mathbf{k}\, d\mathbf{v}_\beta d\mathbf{v}_\alpha \tag{28}$$

$$J^{(2)}_{\alpha\beta\, ij} = \frac{\sigma^4_{\alpha\beta}}{2}\chi_{\alpha\beta\, o}\int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0}(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})^2 k_i k_j k_p f_\alpha f_\beta \frac{\partial \chi_{\alpha\beta\, o}}{\partial x_p} d\mathbf{k}\, d\mathbf{v}_\beta d\mathbf{v}_\alpha \tag{29}$$

$$J^{(3)}_{\alpha\beta\, ij} = -\sigma^3_{\alpha\beta}\int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0}(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})^2 k_i k_j f_\alpha f_\beta \left[\sum_{\delta=1}^{N}\frac{\partial n_\delta}{\partial x_l}(\mathbf{x})\int (x_{\delta\, l}-x_l)\, H(\mathbf{x},\mathbf{x}_\beta,\mathbf{x}_\delta) d\mathbf{x}_\delta\right] d\mathbf{k}\, d\mathbf{v}_\beta d\mathbf{v}_\alpha \tag{30}$$

$$J^{(4)}_{\alpha\beta\, ij} = -\sigma^4_{\alpha\beta}\int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0}(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})^2 k_i k_j k_p f_\beta \frac{\partial f_\alpha}{\partial x_p}\left[\sum_{\delta=1}^{N}\frac{\partial n_\delta}{\partial x_l}(\mathbf{x})\int (x_{\delta\, l}-x_l)\, H(\mathbf{x},\mathbf{x}_\beta,\mathbf{x}_\delta) d\mathbf{x}_\delta\right] d\mathbf{k}\, d\mathbf{v}_\beta d\mathbf{v}_\alpha \tag{31}$$

$$J^{(5)}_{\alpha\beta\, ij} = \frac{\sigma^4_{\alpha\beta}}{2}\int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0}(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})^2 k_i k_j k_p \frac{\partial}{\partial x_p}\left\{ f_\alpha f_\beta \left[\sum_{\delta=1}^{N}\frac{\partial n_\delta}{\partial x_l}(\mathbf{x})\int (x_{\delta\, l}-x_l)\, H(\mathbf{x},\mathbf{x}_\beta,\mathbf{x}_\delta) d\mathbf{x}_\delta\right]\right\} d\mathbf{k}\, d\mathbf{v}_\beta d\mathbf{v}_\alpha \tag{32}$$

where $f_\alpha$ and $f_\beta$ are defined by (22). It seems that these integrals could be explicitely evaluated. For the calculation of $J^{(1)}_{\alpha\beta\, ij}$ and $J^{(2)}_{\alpha\beta\, ij}$, which is actually in progress, the results given in [8, 2] are used. For the other integrals, the results given in [10] will be usefull. The main difficulty of this evaluation is due to the amount of the calculations. In order to symplify the calculation, approximated formulations for the radial distribution function $\chi_{\alpha\beta}$ can be used [12]. The evaluation of the collision integral of the energy balance equation leads to similar expressions.

## CONCLUSION

We have used the method of Grad's thirteen moments in order to introduce an Eulerian description for a heterogeneous suspension constituted of $N$ different solid particle species. Thus, a macroscopic mass conservation equation for each species and global balance equations for momentum and energy have been introduced in the frame of the Revised Enskog Theory proposed by Van Beijeren and Ernst [10]. Because of the large amount of the calculations, an explicit expression of the collision integral of the transport equation is not easy to obtain.

Now, our main purpose is to finish the calculations which are still in progress. In the future, different asymptotic cases for a suspension with two types of particles will be analysed: the case of a great difference in the particle sizes and/or masses and the case of a great difference in the number densities of each species.

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