On the existence of a shock wave solution for the Boltmann equation with a modified collision term

Akira Sakurai and Shuji Takahashi

Department of Mathematical Sciences, Tokyo Denki Univ., Hatoyama, Saitama,350-0394

Abstract. Although the existence problem of the shock wave solution for the Boltzmann equation has been much studied in many decades, this problem has been still open. Additionally even for any approximated equation, e.g., BGK model, no satisfactory answer has obtained (see Nikolaenko (1977), Caflisch and Nikolaenko (1982), Bose, Illner and Ukai (1998)). We consider the existence of shock wave solution for the Boltzmann equation with a modified collision term in which the distribution function is approximated by its 13 moments, originally proposed by Oguchi (1996). The proof is performed in two steps as follows. First we represent the solution in the form of series expansion of the Hermite polynomials, similarly to Grad (1949), postulating u and T appropriately. Utilizing the orthogonality of the Hermite polynomials, we can obtain the equation for the coefficients. We show that the square summation of the coefficients are convergent. Secondary we establish the solution of moment equations to determine the above u and T. We utilize the fact that the collision term here is expressed in an explicit function of the molecular velocity so that we can have a closed system of equations for five moments, that is, we can derive five equations for five unknown functions of space coordinate. As a result, these equations are reduced to a first-order differential equation that is solved to provide the well-known transition of density from its upstream supersonic value to subsonic downstream value.

Keywords: shock wave solution, Boltzmann equation, modefied collision term, Hermite polynomials
PACS: 51.10.+y, 52.35.Tc

THE SHOCK STRUCTURE PROBLEM OF THE BOLTMANN EQUATION WITH A MODIFIED COLLISION TERM

We consider the shock structure problem for a steady planar shock wave whose front is normal to the x-dimension. We will show the existence of the shock wave solution for the Boltmann equation with a modified collision term prposed by Oguchi (1996), which becomes to the present case of steady one-dimensional flow,

$$\begin{aligned} &v_x \frac{\partial F}{\partial x} = \Pi, \\ &\Pi = -\frac{n\nu}{(\pi T)^{3/2}} \exp\left(-\frac{V^2}{T}\right) \left[\frac{3p_{xx}}{4pT}\left(V_x^2 - \frac{V_y^2+V_z^2}{2}\right) + \frac{q_x V_x}{pT}\left(\frac{2V_x^2}{5T} - 1\right)\right] \end{aligned} \tag{1}$$

$$\begin{aligned} n(x) &= \int F d\mathbf{v}, \quad u(x) = \frac{1}{n}\int F v_x d\mathbf{v}, \\ T(x) &= \frac{2}{3n}\int F V^2 d\mathbf{v}, \quad p = nT, \\ p_{xx}(x) &= 2\int F V_x^2 d\mathbf{v} - p, \quad q_x(x) = \int F V^2 V_x d\mathbf{v}. \end{aligned} \tag{2}$$

The distribution function F becomes the Maxwell distribution function $F_\pm$ in the uniform subsonic and supersonic regions as $x \to \pm\infty$ or

$$F(\mathbf{v},x) \to F_\pm \equiv \frac{n_\pm}{(\pi T_\pm)^{3/2}} \exp\left(-\frac{V^2}{T_\pm}\right),$$

$$V_x = v_x - u, \quad V^2 = (v_x - u)^2 + v_y^2 + v_z^2 \tag{3}$$

as $x \to \pm\infty$ where $n_\pm$, $T_\pm$ and $u_\pm$ are the values for $x \to \pm\infty$. They are related with each other through the Rankine-Hugoniot condition defined as usual. We may use the quantities at $x = -\infty$ as reference values to produce $n_- = p_- = T_- = 1$, $M_- = \sqrt{6/5}u_-$.

CONSTRUCTION OF DISTRIBUTION FUNCTION F

We first look for a solution $F$of Eq. (1) subject to the boundary conditions of Eq. (3) provided that u and T are given appropriately as postulated below in Eq.(10). We represent the solution by the Hermite expansion and show that its coefficient is square summable.

The usual Hermite basis is defined by

$$H_i(x) = (-1)^i \frac{\omega^{(i)}}{\omega}, \quad \omega(x) = \frac{1}{\sqrt{2}\pi} e^{-\frac{x^2}{2}}.$$

We represent $F = F(x, v_x, v_y, v_z)$ by a Hermite series expansion in the form of

$$\begin{aligned} F &= \left(\frac{2}{T}\right)^{\frac{3}{2}} \Omega(W) \sum_{i,j,k=0}^{\infty} a_{ijk}(x) H_{ijk}(W), \\ (W_x, W_y, W_z) &= \sqrt{\frac{2}{T}}(V_x, V_y, V_z), \\ \Omega(W) &= \omega(W_x)\omega(W_y)\omega(W_z), \quad H_{ijk}(W) = H_i(W_x)H_j(W_y)H_k(W_z). \end{aligned} \tag{4}$$

We will show the square summability of a_{ijk}. Since moments n, u, p, p_{xx}, q_x are represented as

$$\begin{aligned} n(x) &= a_{000}, \\ nu(x) &= ua_{000} + \sqrt{\tfrac{T}{2}} a_{100}, \\ p + p_{xx} &= T(2a_{200} + a_{000}), \\ q_x &= \left(\tfrac{T}{2}\right)^{\frac{3}{2}} (6a_{300} + 2(a_{120} + a_{102}) + 5a_{100}), \end{aligned}$$

the collision term is represented as

$$\begin{aligned} \Pi &= -\frac{3}{8} \nu a_{200} \omega(2H_{200} - H_{020} - H_{002}) \\ &\quad - \frac{1}{20} \nu \{6a_{300} + 2(a_{120} + a_{102})\} \omega(H_{300} + H_{120} + H_{102}). \end{aligned} \tag{5}$$

By Eq. (1) we set

$$\int H_{ijk} \frac{\partial}{\partial x}(v_x F) dv = \int H_{ijk} \Pi dv. \tag{6}$$

Noting that $v = V + u(x)$, $dv = dV$, $\dfrac{dV}{dx} = -u'$, the left hand side of Eq. (6) becomes

$$i!j!k! \left\{ b'_{ijk} + u' b_{i-1,j,k} + \frac{T'}{4}(b_{i-2,j,k} + b_{i,j-2,k} + b_{i,j,k+2}) \right\} \tag{7}$$

for

$$b_{ijk} = \left(\sqrt{\frac{T}{2}}\right)^{i+j+k} \left\{ \sqrt{\frac{T}{2}}(i+1) a_{i+1,j,k} + u a_{ijk} + \sqrt{\frac{T}{2}} a_{i-1,j,k} \right\}.$$

Since the right hand side of Eq. (6) is zero otherwise $i + j + k$ is 2 or 3, it follows that

$$\begin{aligned} & b'_{000} = 0, \ ua_{000} = C_1, \\ & b'_{100} + u' b_{000} = 0, \ b_{100} + ub_{000} = C_2, \\ & b_{200} + b_{020} + b_{002} + ub_{100} + \tfrac{1}{2}(u^2 + \tfrac{3}{2}T) b_{000} = \tfrac{C_3}{2}. \end{aligned}$$

Since

$$q_x = 2(b_{200} + b_{020} + b_{002}), \quad p + p_{xx} = 2(C_2 - C_1 u),$$

the above coinsides with three relations obtained by integrating Eq. (1) over $\mathbf{v}$, using the weights 1, v_x and v^2 and utilizing the conservation laws of mass, momentum and energy as

$$\begin{cases} nu = C_1 \\ nu^2 + \dfrac{1}{2}(p + p_{xx}) = C_2 \\ nu^3 + \dfrac{5}{2}pu + up_{xx} + q_x = C_3 \end{cases} \tag{8}$$

where $C_1,\ C_2,\ C_3$ are given by the Rankine-Hugoniot condition.

The summability of

$$\sum_{n=0}^{\infty} |a_n|^2, \quad a_n = \frac{1}{n!} \sum_{i+j+k=n} a_{ijk} \tag{9}$$

is shown as follows. Eq. (6) yields for $n \geq 4$ that

$$|b_n|_{L^\infty} \leq |u'|_{L^1} |b_{n-1}|_{L^\infty} + \left|\frac{T'}{4}\right|_{L^1} |b_{n-2}|_{L^\infty},$$

which yields

$$|b_n|_{L^\infty}^2 \leq |u'|_{L^1} \left(|u'|_{L^1} + \left|\frac{T'}{4}\right|_{L^1} \right) |b_{n-1}|_{L^\infty}^2 + \left|\frac{T'}{4}\right|_{L^1} \left(\left|\frac{T'}{4}\right|_{L^1} + |u'|_{L^1} \right) |b_{n-2}|_{L^\infty}^2.$$

Postulating here that

$$|u'|_{L^1} + \left|\frac{T'}{4}\right|_{L^1} < 1, \tag{10}$$

we have

$$\sum_{n \geq 4} |b_n|_{L^\infty}^2 \leq C(|b_3|_{L^\infty}^2 + |b_2|_{L^\infty}^2).$$

Since we may assume $T \geq T_- = 1$,

$$\sum \left| \frac{i+1}{\sqrt{2}^{i+j+k+1}} a_{i+1,j,k} \right|_{L^\infty}^2 < \infty,$$

which yields the summability of Eq. (9).

CONSTRUCTION OF VELOCITY U AND P

Although Eq. (1) is of integro-differential type as the original Boltzmann kinetic equation, its collision term is expressed in an explicit function of $\mathbf{V}$ given by five moments, n, u, p (or T), p_{xx} and q_x in Eq. (1) as functions of x only. Accordingly, we may derive relations between these moments by integrating Eq. (1) with certain weights, from which we may determine these variables. In addition to Eq. (8), we consider two more moment integrals given as

$$\int \frac{\partial F}{\partial x} d\mathbf{v} = (P.V.) \int \frac{\Pi}{v_x} d\mathbf{v}, \tag{11}$$

$$\int V^2 \frac{\partial F}{\partial x} d\mathbf{v} = (P.V.) \int \frac{V^2}{v_x} \Pi d\mathbf{v}, \tag{12}$$

where (P.V.) in right hand sides indicates the Cauchy principal values. These are simplified to

$$\begin{aligned} \frac{dn}{dx} &= -\frac{n\nu}{pT} \left[\frac{3}{4} u p_{xx} \left\{ -1 + \left(\xi^2 - \frac{1}{2} \right) \frac{I}{\xi} \right\} \right. \\ &\qquad \left. + \frac{2}{5} q_x \left\{ \xi^2 - 1 - \xi^2 \left(\xi^2 - \frac{3}{2} \right) \frac{I}{\xi} \right\} \right] \\ &\equiv A p_{xx} + B q_x, \end{aligned} \tag{13}$$

$$
\begin{aligned}
\frac{dp}{dx} &= \frac{2}{3}\Bigg[\left(u^2+T\right)\frac{dn}{dx} \\
&\quad + \frac{n\nu}{p}\left\{\frac{3}{8}up_{xx}\frac{I}{\xi}+\frac{2}{5}q_x\left(\xi I-1\right)\right\}\Bigg] \\
&\equiv Cp_{xx}+Dq_x,
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
I &= \sqrt{\pi}^{-1}(P.V.)\int_{-\infty}^{\infty}\exp\left(-\frac{V_x^2}{T}\right)\frac{dv_x}{V_x+u} \\
&= 2\sqrt{\pi}^{-1}\mathrm{e}^{-\xi^2}\int_0^{\infty}\exp\left(-\eta^2\right)\frac{\sinh(2\xi\eta)}{\eta}d\eta,
\end{aligned}
$$

with $\xi=\dfrac{u}{\sqrt{T}}$, $\eta=\dfrac{v_x}{\sqrt{T}}$, from which we obtain

$$
\frac{dp}{dn}=\frac{Cp_{xx}+Dq_x}{Ap_{xx}+Bq_x}. \tag{15}
$$

Eq. (15) becomes singular as 0/0 at two ends of $n=n_{\pm}$, since both p_{xx} and q_x should vanish there, which correspond to $x=\pm\infty$. We examine the nature of the solution there. We will find that the nature of the solution near $u=u_+$ is of node type while it is of saddle point type near $u=u_-$. Accordingly (i) we have two regular solution curves starting from $(p_-,\, u_-)$ of which one stretches towards to an area near $(p_+,\, u_+)$ while (ii) any solution curve near $u=u_+$ approaches to the point $(p_+,\, u_+)$. (iii) Further, solution passing a point $(\tilde{p},\tilde{u})$ of $u_+<\tilde{u}<u_-$, $p<\tilde{p}<p_+$ is regular and unique in $u_+<u<u_-$ as far as the magnitude of the parameter M_- is close to 1 or M_--1 is small enough, from which we have $(a,\, b,\, c,\, d)$ above are almost constant to the present case. Combine all three facts of (i), (ii), (iii) above, we can see that there exist a unique smooth solution of Eq. (15) starting from $(p_-,\, n_-)$ and ending at $(p_+,\, n_+)$. These procedures above are illustrated in the simplified model below.

We consider an equation in interval (x_1,x_2) which has singularities at the boundaries x_1 and x_2 and for which $(x_1,y(x_1))$ is a saddle point and $(x_2,y(x_2))$ is a nod, that is, we consider of the form

$$
\begin{aligned}
&\frac{dy}{dx}=\frac{c(x-x_1)(x-x_2)+d(x+y-\lambda)}{a(x-x_1)(x-x_2)+b(x+y-\lambda)}, \\
&y(x_1)=y_1=\lambda-kx_1\ y(x_2)=y_2=\lambda-kx_2.
\end{aligned}
\tag{16}
$$

We here call (x_0,y_0) saddle point if $\Delta(x_0)>0$ and $\mathbf{D}(x_0)<0$ and nod if $\Delta(x_0)>0$ and $\mathbf{D}(x_0)>0$, where $\Delta(x)=(a-d)^2+4bc$ and $\mathbf{D}(x)=ad-bc$. Since

$$
\mathbf{D}(x_1)=(ad-bc)(x_1-x_2),
$$

if a,b,c,d are constant, it follows that

$$
\mathbf{D}(x_1)=-\mathbf{D}(x_2).
$$

Since (a,b,c,d) is almost constant in our problem, we show the local existence for the equation of the form

$$
\frac{dy}{dx}=\frac{cx+dy+g(x,y)}{ax+by+f(x,y)}\equiv\frac{Q(x,y)}{P(x,y)}, \tag{17}
$$

with constant values a,b,c,d provided that $x=0$ is a saddle point or a nod and f/x^2 and g/x^2 are bounded near $x=0$.

Function $v=y/x$ satisfies

$$
v'=\frac{1}{x}\left(\frac{c+dv}{a+bv}-v\right)+\frac{\zeta}{x}, \quad \zeta=\frac{Q}{P}-\frac{c+dv}{a+bv}.
$$

For case $\zeta=0$, if $\Delta>0$, for

$$
p,q=\frac{d-a\pm\sqrt{\Delta}}{2b}, \ \ \alpha+\beta=1, \ \ -\alpha q-\beta p=\frac{a}{b},
$$

the solution is written in the form of

$$(v-p)^{\alpha}(v-q)^{\beta}x = V \tag{18}$$

for integral constant V and $\alpha\beta > 0$ for saddle point type and $\alpha\beta < 0$ for nod type. In case $\zeta \neq 0$, for $V = (x,v)$ defined by (18),

$$\begin{aligned}
\frac{dV}{dx} &= \frac{\partial V}{\partial x} + \frac{\partial V}{\partial v}\frac{dv}{dx} \\
&= v\frac{v+\frac{a}{b}}{(v-p)(v-q)}\frac{\zeta}{x} \\
&= V^{1-\beta}x^{\frac{1}{\beta}}(v+\frac{a}{b})(v-p)^{\frac{\alpha}{\beta}-1}\frac{\zeta}{x}. \\
\beta(V^{\frac{1}{\beta}})' &= x^{\frac{1}{\beta}}(v+\frac{a}{b})(v-p)^{\frac{\alpha}{\beta}-1}\frac{\zeta}{x}.
\end{aligned} \tag{19}$$

We construct a solution with

$$\lim_{x\to 0} v(x) = q.$$

For

$$\varepsilon(x) = v(x) - q,$$

it follows that

$$\begin{aligned}
|\varepsilon(x)|_{\infty} &\leq \frac{x}{\beta+1}|(v+\frac{a}{b})(v-p)^{-1}\frac{\zeta}{x}|_{\infty} \\
&\leq \frac{x}{b(\beta+1)}\frac{N_1|\varepsilon|_{\infty}+N_2}{|(\varepsilon+q-p)(b\varepsilon+a+bq+\frac{f}{x})|_{\infty}},
\end{aligned} \tag{20}$$

with

$$N_1 = |b||\frac{g}{x^2}|_{\infty} + |d||\frac{f}{x^2}|_{\infty}, \quad N_2 = |a+bq||\frac{g}{x^2}|_{\infty} + |c+dq||\frac{f}{x^2}|_{\infty}.$$

Thus $|\varepsilon(x)|_{L^{\infty}(0,x_0)}$ is bounded for sufficiently small x_0, although N_1 and N_2 include ε. Since $v = y/x$ and $\varepsilon(x) = v(x) - q$,

$$|y(x) - qx| < Lx$$

for $x \in (0,x_0)$ with some L. Thus we can construct a local solution if $f(x,y)/x^2$ and $g(x,y)/x^2$ are bounded for (x,y) with $|y(x) - qx| < Lx$ and $x \in [0,x_0]$.

In our problem, it follows that

$$\begin{aligned}
\mathbf{D}_- &= (ad-bc)|_{u=u_-}\cdot(x_2-x_1) \\
\mathbf{D}_+ &= -(ad-bc)|_{u=u_+}\cdot(x_2-x_1) \\
ad-bc &= -4\frac{C_1^2}{u^2}(AD-BC).
\end{aligned} \tag{21}$$

Since

$$AD-BC = -\frac{1}{20Tu}(\frac{n\nu}{p})^2\frac{1}{\xi}\{\xi(6\xi^2-5)I^2+2(-5\xi^2+2)I+4\xi\},$$

we have

$$\mathbf{D}_+ > 0 > \mathbf{D}_-$$

for $M_- = \sqrt{6/5}$. Boundedness of f/x^2 and g/x^2 provided that $|y(x) - q| < Lx$ is shown by

$$\begin{aligned}
&\xi^2 - \xi_-^2 = C_1(\frac{u}{p} - \frac{u_-}{p_-}) = C_1\frac{p_-x - u_-y}{(y+p_-)p_-} = O(x), \\
&\xi^n - \xi_-^n = O(x), \quad n = 1,2,3,\cdots, \\
&I(\xi) - I(\xi_-) = I'(\xi_0)(\xi - \xi_-), \quad \xi_0 \in (\xi_-,\xi).
\end{aligned} \tag{22}$$

Since there are no singularities between saddle and nod points, we can construct a solution connecting these points (see Nagai(2003)). Thus we can construct p, so that Eq.(14) yields n.

Notice that postulation (10) is satisfied if $M_- - 1$ is sufficiently small.

REFERENCES

1. B. Nikolaenko, *Applications of Bifurcation Theory* , Academic Press, 333–357, (1977).
2. R.E. Caflisch and B. Nikolaenko *Commun. Math. Phys.* **86**, 161–194, (1982)
3. C. Bose, R. Illner and S. Ukai *Transport Theory and Statistical Physics* **27**, 35–66, (1998) .
4. H. Oguchi and T. Soga *Rarefied Gas Dynamics* in Shen C, et al. (eds) , Peking University Press, Beijing, 1997, 285–297.
5. H. Grad *Comm. Pure Appl. Math.* **2**, 331-407, (1949) .
6. R. Nagai, H. Homma, K. Maeno and A. Sakurai *Shock Waves* (DOI)10.1007/s00193-003-0208-z, (2003)