Dual Variational Principles Of the Linear Boltzmann Equation

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Abstract. The stationary problem of flow around a heat-conducting body of arbitrary shape is considered. For solving this problem two variational principles are formulated. First of them is the maximal variational principle. The second is the minimum variational principle. Efficiency of these dual variational principles is shown in example of the resistance coefficient calculation for a heat-conducting sphere on the basis of Krook equation.

Key words: The linear Boltzmann equation, the even or the odd part of distribution function, dual variation principles, the trial function.

INTRODUCTION

In papers [1, 2] for linear stationary Boltzmann equation two minimax variational principles are constructed. First of them [1] is based on description of a state of gas by means of the even part of the distribution function and components of the macroscopic velocity. The second variational principle [2] is based on the odd part of the distribution function, pressure and temperature of gas. For a problem of slow flow around a body the extreme values of functionals of these variational principles are equal the strength of body resistance force. Variational principles are an effective method for calculating extreme value of a functional as the error in a trial function ε causes a deviation of a functional from its extreme value by magnitude ε^2.

The greatest success in calculation of some magnitude by a variation method arises when it is possible to construct two variation principles such, that extreme values of their functionals are equal to the required magnitude, and one of them is a maximum principle, and another - a principle of a minimum. The evaluation of functionals on the coordinated trial functions gives an approximate value of the required magnitude with excess and with shortage. Their half-sum clarifies the obtained result as a deviation of functionals from exact value are distributed oppositly and in general cancel each other. In paper [3] the variational principles are applied to the calculation of a slip coefficient which is equal to the maximum value of a functional of a variational principle [1] and to the minimum value of a functional of a variational principle [2].

In the this work it is shown that in general case under some requirements to the sets of varied functions, the variation principle [1] is the maximum principle, and the variation principle [2] is the principle of the minimum for a strength of force of resistance. It was ascertained, that variational principles have dual character and supplement each other. As an example of use of dual principles there is calculation of resistance coefficient for a heat-conducting sphere at the arbitrary Knudsen numbers.

THE VARIATION MAXIMUM PRINCIPLE

The stationary problem of slow flow around a heat-conducting body by the uniform stream of a rarefied gas is considered on the basis of perturbation of a distribution function $\varphi(\vec{x},\vec{c})$ [1, 4]. Let's designate the restricted spatial domain in R^3, occupied by a body, as D_w, its boundary as Σ_w, region of gas movement as $D = R^3 / D_w$, 3-D space of velocities as Ω, the vector of the velocity $\vec{c} \in \Omega$.

The state of gas is described on the basis of Boltzmann equation with a linearized operator of collisions L. The model of solid spheres is used. We have

$$c_j \frac{\partial \varphi}{\partial x_j} = L\varphi \qquad (1)$$

On the surface of the body there are the boundary conditions of interaction of gas with the body and equality of normal streams of an energy from the body to gas. Inside the body the perturbation of temperature τ_w satisfies the heat conduction equation

$$\frac{\partial}{\partial x_j}(\lambda \frac{\partial \tau_w}{\partial x_j}) = 0, \ \ \vec{x} \in D_w$$

The thermal conductivity of the body λ is given.

Let's present $\varphi(\vec{x},\vec{c})$ in the form of the sum of even $g(\vec{x},\vec{c})$ and odd $h(\vec{x},\vec{c})$ parts with respect to the vector of the velocity $\vec{c}$ – $\varphi = g + h,\ g = 0.5[\varphi(\vec{c}) + \varphi(-\vec{c})],\ h = 0.5[\varphi(\vec{c}) - \varphi(-\vec{c})]$. Transferring in (1) from function φ to functions g and h, we obtain the set of equations

$$c_j \frac{\partial g}{\partial x_j} = L_-(h),\ c_j \frac{\partial h}{\partial x_j} = L_+(g) \qquad (2)$$

Here L_- is the operator L, acting on a set of odd functions h, and L_+ – the operator L, acting on a set of even functions g.

Let's introduce a scalar product of functions φ_1 and φ_2

$$(\varphi_1, \varphi_2) = \int \varphi_1 \varphi_2 d\vec{\omega},\ d\vec{\omega} = \pi^{-3/2} \exp(-\vec{c}^{\,2})$$

For the first equation be solvable relative to h it is rather necessary, that the element $c_j \frac{\partial g}{\partial x_j}$ be orthogonal to the functions $c_i,\ i = 1,2,3$, forming the kernel of the operator L, i. e. for the components of the stress tensor $P_{ij} = \int 2 c_i c_j g d\vec{\omega}$ the equalities should be fulfilled

$$\frac{\partial P_{ij}}{\partial x_j} = 0,\ \ i = 1,2,3 \qquad (3)$$

Let function g be satisfied by (3). Then

$$h = 2c_i V_i + L_-^{-1}(c_j \frac{\partial g}{\partial x_j}) \qquad (4)$$

where $V_i(\vec{x})$ – components of the vector of the velocity, L_-^{-1} – the operator, inverse to L_-. Substituting expression (4) for h in the second equation (2), we obtain

$$c_j \frac{\partial}{\partial x_j} L_-^{-1}(c_i \frac{\partial g}{\partial x_i}) - L_+(g) + 2c_i c_j \frac{\partial V_i}{\partial x_j} = 0 \tag{5}$$

Let's multiply the equation (5) scalarly by 1, we obtain, that the equation (5) is solvable, if

$$\frac{\partial V_i}{\partial x_i} = 0 \tag{6}$$

So, in the regin D, occupied by gas, the desired functions g and V_i are defined by the equations (3), (5), and (6). On the body surface at $\vec{x} \in \Sigma_w$ boundary conditions [1] take place

$$\frac{c_n}{|c_n|} L_-^{-1}(c_j \frac{\partial g}{\partial x_j}) + 2\frac{c_n}{|c_n|} c_j V_j + B\chi = 0, V_n = 0, \lambda \frac{\partial \tau_w}{\partial n} = (B\chi, (\vec{c}^2 - 5/2)_w$$

Here $\chi = g - \tau_w(\vec{c}^2 - 5/2)$, the operator $B = (E+M)^{-1}(E-M)$; $Mg = \int H(\vec{w},\vec{c})g(\vec{w})d\vec{w}$, $H(\vec{w},\vec{c}) = 1/2|w_n|e^{-\vec{w}^2}(|c_n|e^{-\vec{c}^2})^{-1}R(\vec{z},\vec{u})$, $\vec{z} = -\vec{w}sign(w_n)$, $\vec{u} = \vec{c}sign(c_n)$, E – the unit operator. A scalar product $(g_1, g_2)_w = \int |c_n| g_1 g_2 d\vec{\omega}$. Here $R(\vec{z},\vec{u})$ is a kernel of gas scattering by the body surface [4]. It is obvious, that $M(1) = 1$, therefore $B(1) = 0$. At $r = |\vec{x}| \to \infty$ there are the following asymptotic requirements $V_j = V_{0j} + U_j$, $U_j = O(r^{-2})$, $V_{0j} = S_j - \frac{k_i}{8\pi\mu r}(\delta_{ij} - \frac{x_i x_j}{r^2})$, $j = 1,2,3,$ $g = p + \tau(\vec{c}^2 - 5/2) + \frac{B(\vec{c}^2)}{2\mu} c_i c_j \overline{p}_{ij} + O(r^{-3})$, $p = -\frac{k_i x_i}{4\pi r^2} + O(r^{-2})$, $\tau = -d_i x_i r^{-3} + O(r^{-3})$, $\overline{p}_{ij} = -p(\delta_{ij} + 3\frac{x_i x_j}{r^2}) + O(r^{-3})$, $k_i = const$, $d_i = const$.

Here p and τ – perturbations of pressure and temperature; S_j – components of the velocity of the incident flow; $B(\vec{c}^2)c_i c_j \overline{p}_{ij}/(2\mu)$ – the even part of the Navier-Stokes distribution function, μ – coefficient of viscosity of gas, $\overline{p}_{ij} = P_{ij} - p\delta_{ij}$.

The problem of calculation of components of body resistance force F_i is stated in the form of a variational maximum principle

$$F_i S_i = \max_{u \in N \times T} K_1(u), u = \{g, \tau_w\} \tag{7}$$

$K_1(u) = 2\Phi_1(u) - \dot{S}(u)$, $\dot{S}(u) = -\int_D [(g, L_+ g) + (c_i \frac{\partial g}{\partial x_i}, L_-^{-1}(c_i \frac{\partial g}{\partial x_i}))]d\vec{x} + \int_{\Sigma_w} (\chi, B\chi)d\sigma + \int_{D_w} \lambda (\frac{\partial \tau_w}{\partial x_j})^2 d\vec{x} \geq 0,$

$\Phi_1(u) = \int_\Sigma S_i P_{ij} n_j d\sigma$; $N: g \in H^1_{2\nu}(D \times \Omega), \frac{\partial P_{ij}}{\partial x_j} = 0, i = 1,2,3; T: \tau_w \in W_2^1, \|\tau_w\|^2 = \int_{D_w} [\tau_w^2 + (\frac{\partial \tau_w}{\partial x_j})^2]d\vec{x}$. Σ – an arbitrary surface enveloping the body. $H^1_{2\nu}(D \times \Omega)$ is the complete space of the functions $g(\vec{x},\vec{c})$ possessing the generalized first derivative $\frac{c_i}{\nu}\frac{\partial g}{\partial x_i}$ and having quadrate of the norm $\|g\|^2 = \int_{D\times\Omega} [g_0^2 + (\frac{c_i}{\nu}\frac{\partial g}{\partial x_i})^2]\nu d\vec{\omega} d\vec{x}$ (ν is the collision frequency of molecules). Here $g_0 = g - p - \tau(\vec{c}^2 - 5/2)$. The squared of the norm of an element $u \in N \times T$ is determined by the relation $\|u\|^2 = \|g\|^2 + \|\tau_w\|^2$. It is possible to show [5], that the functional $\dot{S}(u)$ is limited from above and from below in the norm of the basic space $N \times T$, i. d. $C_1\|u\|^2 \le \dot{S}(u) \le C_2\|u\|^2$, $C_1 > 0$.

Let's introduce the square of «energy» norm, having assumed $[u,u] = \dot{S}(u)$, [6]. For a linear functional $\Phi_1(u)$ it is proved that $|\Phi_1(u)| \le C_3 \lfloor u,u \rfloor^{1/2}$. Therefore under the theorem of Riesz [6] there is a unique element $u_0 \in N \times T$ such, that $\Phi_1(u) = [u, u_0]$. Then $K_1(u) = [u_0, u_0] - [u - u_o, u - u_0]$. It follows that the maximum of the functional $K_1(u)$ is reached on the element u_0. It must be pointed out that the functional (7) does not contain the components of the vector of the velocity V_i.

THE VARIATIONAL PRINCIPLE OF A MINIMUM

The problem of the stationary flow of the heat-conducting body can be formulated on the basis of desired function h and τ_w, having excluded function g from the set of equations (2) and boundary conditions. In this case searching of the solution of the problem is reduced to the determination of element $u = \{h, \tau_w\}$ ensuring the minimum to the functional

$$F_i S_i = \min_{u \in G \times T} \dot{S}(u) \tag{8}$$

$\dot{S}(u) = -\int_D [(h, L\_h) + (c_i \frac{\partial h}{\partial x_i}, L_+^{-1} c_i \frac{\partial h}{\partial x_i})] d\vec{x} + \int_{\Sigma_w} (\frac{c_n}{|c_n|} h, A \frac{c_n}{|c_n|} h)_w d\sigma + \int_{D_w} \lambda ((\frac{\partial \tau_w}{\partial x_j})^2 d\vec{x} \ge 0,\ A = (E - M)^{-1}(E + M)$. Here by G the set of functions h is designed which obey requirements

$$\frac{\partial V_i}{\partial x_i} = 0,\ \frac{\partial q_i}{\partial x_i} = 0,\ \vec{x} \in D; V_n = 0,\ q_n = -\lambda \frac{\partial \tau_w}{\partial n},\ \vec{x} \in \Sigma_w \tag{9}$$

and h can be presented as $h = 2c_i V_i + h_0$, where components of the velocity V_i possess the asymptotics of the flow problem $V_i = S_i - \frac{k_i}{8\pi\mu r}(\delta_{ij} - \frac{x_i x_j}{r^2}) + O(r^{-2})$, and function h_0 is such, that $(h_0, c_i) = 0$. Function h belongs to the space $H^1_{2\nu}(D \times \Omega)$. The space of functions $H^1_{2\nu}(D \times \Omega)$ consists of the odd functions h possessing the generalized first derivative $\frac{c_i}{\nu}\frac{\partial h}{\partial x_i}$. The norm square is defined an integral $\|h\|^2 = \int_{D \times \Omega} [h_0^2 + (\frac{c_i}{\nu}\frac{\partial h}{\partial x_i})^2] \nu d\vec{\omega} d\vec{x}$. Components of the heat flow q_i are defined by the scalar product $q_i = (c_i(\vec{c}^2 - 5/2), h)$. The set of functions T consists of $\tau_w \in W_2^1(D_w)$ such, that $\frac{\partial}{\partial x_j}(\lambda \frac{\partial \tau_w}{\partial x_j}) = 0$ at $\vec{x} \in D_w$, and at $\vec{x} \in \Sigma_w$ satisfy the second requirement (9). For the variational principle of minimum (8) it is possible also to prove existence of an element u_0 on which the minimum of the functional is reached.

CALCULATION OF RESISTANCE COEFFICIENT OF A SPHERE

The formulated variational principles are applied to calculation of the resistance coefficient of a heat-conducting sphere on the basis of linearized Krook model.

The success in calculation of resistance force of the sphere by means of variational principles, as a rule, is based on a successful choice of trial functions. For the variational maximum principle the desired functions g and τ_w were taken in the form $g = p + \tau(\vec{c}^2 - 5/2) + \overline{p}_{ij} c_c c_j + (a c_3 sign(\vec{x}, \vec{c}) H(f) + b c_3 / |\vec{c}| sign(\vec{x}, \vec{c}) \sqrt{f}) e^{-\alpha r / Kn}$:

$$f = (\vec{x}, \vec{c}) - \vec{c}^2 (r^2 - 1);\ H(f) = \begin{cases} 1, f \geq 0 \\ 0, f < 0 \end{cases};\ \tau_w = d x_3$$

In this expression $p(\vec{x}),\ \tau(\vec{x}),\ \overline{p}_{ij}(\vec{x})$ are the varied functions, $a,\ b,\ d$ – varied constants. The desired functions $p(\vec{x}),\ \tau(\vec{x})$ and $\overline{p}_{ij}(\vec{x})$ were formed as solution of the Euler equations. At that these solutions strictly satisfied conditions $\dfrac{\partial P_{ij}}{\partial x_j} = 0, i = 1, 2, 3$.

At the calculation of resistance force on the basis of the minimum variational principle the desired function h was assumed as $h = 2 c_i V_i + (a H(f) + b\sqrt{f}) c_3 e^{-\alpha r / Kn} + e e^{-\alpha r / Kn} H(f) sign(\vec{x}, \vec{c})(\vec{c}^2 - 2)(x_3 - c_3 / \vec{c}^2 (\vec{x}, \vec{c})) - $ $- \dfrac{2}{5\sqrt{\pi}} e c_i (\vec{c}^2 - 5/2) \dfrac{Y_i^2}{r^3} (1 - e^{-\alpha r / Kn})$. The desired functions $V_i(\vec{x})$ and constants $a,\ b,\ d$ and e were formed as a solution of the Euler equations at the satisfaction of conditions $\dfrac{\partial V_i}{\partial x_i} = 0,\ \dfrac{\partial q_i}{\partial x_i} = 0$ at $\vec{x} \in D$ and $V_n = 0,\ q_n = -\lambda \dfrac{\partial \tau_w}{\partial n}$ at $\vec{x} \in \Sigma_w$.

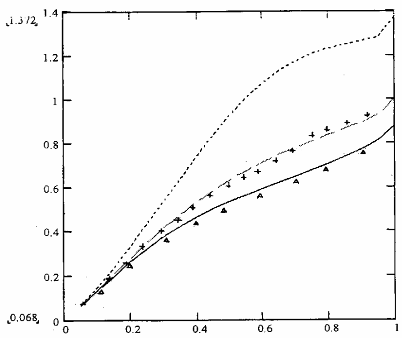

FIGURE 1

In Figure 1 the calculated resistance coefficient as function of magnitude $z = Kn/(1+Kn)$ is presented. Magnitude F can be presented in the form of $F(z) = \beta F_{\max}(z) + (1-\beta) F_{\min}(z)$ where $F_{\max}(z)$ and $F_{\min}(z)$ are dependences of resistance coefficient constructed for the maximum principles and the minimum principles respectively. The constant β was defined by the requirement, that at the Knudsen number equal to infinity ($z = 1$), $F(1)$ be strictly equal to the known free-molecular value of resistance coefficient. The upper curve deals with the dependence $F_{\min}(z)$, and the lower curve deals with the dependence $F_{\max}(z)$. Magnitude $F(z)$ was compared to the results of experiment of R. Millikena [4].

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