Generalized Boltzmann Physical Kinetics: Theory, Main Results and Problems

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Abstract. This paper addresses the fundamental principles of generalized Boltzmann physical kinetics, which introduces terms accounting for the variation of the distribution function over times of the order of the collision time into the Boltzmann equation. The paper is primarily aimed at clarifying the qualitative aspects of the theory whose mathematical formalism was developed in the author's earlier work. The application of the generalized Boltzmann equation to certain classical transport processes is discussed.

Introduction

In 1872 L Boltzmann published his famous kinetic equation for the one-particle distribution function (DF) $f(\mathbf{r},\mathbf{v},\mathbf{t})$ [1,2]. He expressed the equation in the form

$$Df/Dt = J^{st}(f), \tag{1}$$

where J^{st} is the collision integral, and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathrm{v}_\alpha \cdot \frac{\partial}{\partial \mathrm{r}} + \mathrm{F}_\alpha \cdot \frac{\partial}{\partial \mathrm{v}_\alpha}$ is the substantial (particle) derivative, $\mathbf{v}$ and $\mathbf{r}$ being the velocity and radius vector of the particle, respectively. Equation (1) governs the transport processes in a one-component gas, which is sufficiently rarefied that only binary collisions between particles are of importance and valid only for two character scales, connected with the hydrodynamic time-scale and the time-scale between particle collisions. While we are not concerned here with the explicit form of the collision integral, note that it should satisfy conservation laws of point-like particles in binary collisions. Integrals of the distribution function (i.e. its moments) determine the macroscopic hydrodynamic characteristics of the system, in particular the number density of particles n and the temperature T. The Boltzmann equation (BE) is not of course as simple as its symbolic form above might suggest, and it is in only a few special cases that it is amenable to a solution. One example is that of a Maxwellian distribution in a locally, thermodynamically equilibrium gas in the event when no external forces are present. In this case $J^{st} = 0$ and $f = f^{(0)}$ is met, giving the Maxwellian DF function $f^{(0)}$.

A weak point of the classical Boltzmann kinetic theory is the way it treats the dynamic properties of interacting particles. On the one hand, as the so-called “physical” derivation of the BE suggests Boltzmann particles are treated as material points; on the other hand, the collision integral in the BE brings into existence the cross sections for collisions between particles. A rigorous approach to the derivation of the kinetic equation for f (noted in following as KE_f) is based on the hierarchy of the Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) [3,4] equations. A KE_f obtained by the multi-scale method turns into the BE if one ignores the change of the distribution function (DF) over a time of the order of the collision time (or, equivalently, over a length of the order of the particle interaction radius). It is important to note [5-10] that accounting for the third of the scales mentioned above leads (*prior* to introducing any approximation destined to break the Bogolyubov chain) to additional terms, generally of the same order of magnitude, appear in the BE. If the correlation functions is used to derive KE_f from the BBGKY equations, then the passage to the BE means the neglect of non-local and time delay effects. Given the above difficulties of the Boltzmann kinetic theory (BKT), the following clearly inter related questions arise. First, what is a physically infinitesimal volume and how does its introduction (and, as the consequence, the unavoidable smoothing out of the DF) affect the kinetic equation? And second, how does a systematic account for the proper diameter of the particle in the derivation of the KE_f affect the Boltzmann equation? In the theory developed here I shall refer to the corresponding

KE_f as the GBE. The derivation of the GBE and the applications of BKT are presented, in particular, in [10]. Accordingly, our purpose is first to explain the essence of the physical generalization of the BE.

Main Ideas of the Generalized Boltzmann Physical Kinetics

About twenty years ago it was shown (see for example [10]), that taking into account the variation of the distribution function over times of the order of the collision time leads to additional terms in Boltzmann equation (BE), which are proportional to mean time τ *between* collisions of particles and therefore to Knudsen number and viscosity in hydrodynamic limit of the theory. Moreover it turns out that these terms cannot be omitted in the case of small Knudsen numbers because these terms contain small parameters in front of senior derivatives. Then these terms should be conserved in the theory in all diapason of evolution of Knudsen numbers. Consequently, the additional GBE terms (as compared to the BE) are significant for any Kn, and the order of magnitude of the difference between the BE and GBE solutions is impossible to tell beforehand. The corresponding kinetic equation is known as generalized Boltzmann equation (GBE) and from another point of view the additional terms can be considered as local approximation of collisional non-local terms with the time-delay. GBE leads to system of the generalized hydrodynamic equations (GHE), which are very effective in the theory of turbulent flows, physics of ionized gases, acoustics, physics of liquids and another applications of the theory of transport processes [10].

The GBE for η-component gas mixture of neutral gases has the form

$$\frac{Df_\alpha}{Dt} - \frac{D}{Dt}\left(\tau_\alpha \frac{Df_\alpha}{Dt}\right) = J_\alpha \, , \qquad (2)$$

written with the help of the substantial (particle) derivative containing the external force $\mathbf{F}_\alpha$, J_α is the classical (Boltzmann) collision integral, $f_\alpha(\mathrm{r}, \mathrm{v}_\alpha, t)$ $(\alpha = 1,\ldots,\eta)$ is the one-particle distribution function (DF) . There is the through approximation for τ_α in gas, plasma and liquid. For gas τ_α is the mean time between the particle collisions for α-component, for plasma τ_α corresponds to the mean time between close collisions and for liquids τ_α is the mean time of particle residence in the Frenkel cell. In the hydrodynamic regime τ_α can be expressed in terms of viscosity μ_α, static pressure p_α, and the coefficient Π dependent on the particle interaction model. GBE (1) can be written formally in the Boltzmann form

$$\frac{Df_\alpha^a}{Dt} = J_\alpha(f) \qquad (3)$$

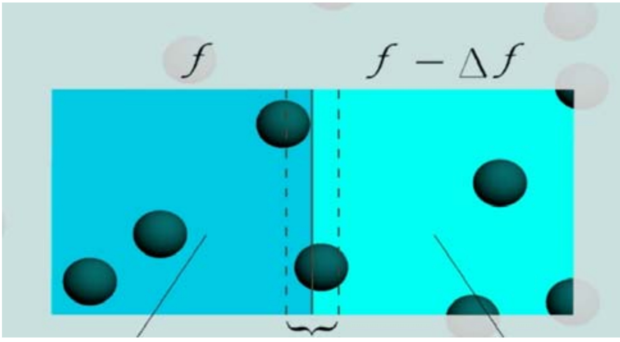
$$f_\alpha^a = f_\alpha - \tau_\alpha \frac{Df_\alpha}{Dt} \qquad (4)$$

It means that DF contains the fluctuation terms and in the Boltzmann theory we use for mixture of particles of finite sizes the simplest approximation

$$f_\alpha^a = f_\alpha \, , \qquad (5)$$

Let a particle of finite radius be characterized, as before, by the position vector $\mathbf{r}$ and velocity $\mathbf{v}$ of its center of mass at some instant of time t. Let us introduce physically small volume (PhSV) as element of measurement of macroscopic characteristics of physical system for a point containing in this PhSV. We should hope that PhSV contains sufficient particles N_{ph} for statistical description of the system. Every PhSV contains entire quantity of *point-like* Boltzmann particles and the same DF f is prescribed for whole PhSV. Therefore Boltzmann particles are fully "packed" in the reference volume. For the particles of finite size (PFS) we have on principle another situation. The fact that center of mass of a PFS is in $\mathbf{PhSV_1}$ (see Figure 1) does not mean that all of the particle is there. Symbolic picture imagines also the particles, which are partly can be found in adjoining $\mathbf{PhSV}_2$. In other words, at any given point in time there are always particles, which are partly inside and partly outside of the reference volume. This fact unavoidably leads to fluctuations in mass and hence in other hydrodynamic quantities. Suppose that DF f corresponds to $\mathbf{PhSV_1}$ and DF $f - \Delta f$ is connected with $\mathbf{PhSV}_2$ for Boltzmann particles. In the boundary area (the width of which is of order of the particle diameter for the model of hard spheres) a particle of finite size with the position of its center of mass in $\mathbf{PhSV}_2$ will be placed partly in $\mathbf{PhSV_1}$ and leads to changing of f in $\mathbf{PhSV_1}$. Correspondingly a particle with the center of mass in $\mathbf{PhSV_1}$

(and then belonging to $\mathbf{PhSV_1}$) in this area will be placed partly in $\mathbf{PhSV}_2$ and will influence on DF in $\mathbf{PhSV}_2$. It is the demonstration of the time-delay, time-ahead, non-locality effects.



PhSV$_1$ **d** **PhSV$_2$**

Figure 1. To the qualitative derivation of the generalized Boltzmann equation (GBE).

The mathematical expectation of the number of particles moving through the boundary reference surface strictly perpendicular to it is zero. Therefore, in the first approximation, fluctuations will be proportional to the mean free path (or, equivalently, to the mean time between the collisions). In the boundary layer of the λ -width in the open $\mathbf{PhSV_1}$ physical system there particles originated because of the existence of $\mathbf{PhSV}_2$. In accordance with the ergodic Boltzmann hypothesis, the averaged statistical values of physical parameters are equal to its mean values in time and perturbation of DF in $\mathbf{PhSV_1}$ (which is equal to $[(f - \Delta f) - f]/\Delta t$) will be proportional to mean time *between* collisions τ . Then for the open PhSV the correction for DF should be introduced which leads to Eqs (3), (4).

Important to notice that it is only *qualitative* explanation of GBE derivation obtained earlier (see for example [10]) by different strict methods from the BBGKY – chain of kinetic equations. The structure of the KE_f is generally as follows

$$Df/Dt = J^B + J^{td}, \tag{6}$$

where J^{td} is the non-local integral term incorporating the time delay effect. The generalized Boltzmann physical kinetics, in essence, involves a local approximation

$$J^{td} = \frac{D}{Dt}\left(\tau \frac{Df}{Dt}\right) \tag{7}$$

for the second collision integral, here τ being the mean time *between* the particle collisions. We can draw here an analogy with the Bhatnager - Gross - Krook (BGK) approximation for J^B,

$$J^B = \left(f^{(0)} - f\right)/\tau, \tag{8}$$

which popularity as a means to represent the Boltzmann collision integral is due to the huge simplifications it offers. The ratio of the second to the first term on the right-hand side of Eqn (6) is given to an order of magnitude as $J^{td}/J^B \approx O(Kn^2)$ and at large Knudsen numbers (defining as ratio of mean free path of particles to the character hydrodynamic length) these terms become of the same order of magnitude. It would seem that at small Knudsen numbers answering to hydrodynamic description the contribution from the second term on the right-hand side of Eqn (6) is negligible. This is not the case, however. When one goes over to the hydrodynamic approximation (by multiplying the kinetic equation by collision invariants and then integrating over velocities), the Boltzmann integral part vanishes, and the second term on the right-hand side of Eqn (6) gives a single-order contribution in the generalized Navier - Stokes description. Mathematically, we cannot neglect a term with a small parameter in front of the higher derivative. Physically, the appearing additional terms are due to viscosity and they correspond to the small-scale Kolmogorov turbulence [10]. The integral term J^{td}, thus, turns out to be important both at small and large Knudsen numbers in the theory of transport processes. Thus, $\tau Df/Dt$ is the distribution function fluctuation and writing Eqn (3) without taking into account Eqn (4) makes the BE non-closed. From the viewpoint of the fluctuation theory, Boltzmann employed the simplest possible closure procedure $f^a = f$. For GBE the generalized H-theorem is proven [7].

Some Applications of the Generalized Boltzmann Physical Kinetics

Let us consider now some aspects of GBE application beginning with hydrodynamic aspects of the theory. Returning to Figure 1 we can state that the number of particles in reference volume is proportional to cube of the character length L of volume, the number of particles in the surface layer is proportional to λL^2, and as result all effect of fluctuation can be estimated as ratio of two mentioned values or as $\lambda / L = Kn$.

Obviously the hydrodynamic equations will explicitly involve fluctuations proportional to τ. For example, the continuity equation changes its form and will contain terms proportional to viscosity. However, we will here attempt to “guess” the structure of the generalized continuity equation using the arguments outlined above. Neglecting fluctuations, the continuity equation should have the classical form with

$$\rho^a = \rho - \tau A, \; (\rho \mathbf{v}_0)^a = \rho \mathbf{v}_0 - \tau \mathbf{B},$$

where ρ is density and $\mathbf{v}_0$ is hydrodynamic velocity. Strictly speaking, the factors A and $\mathbf{B}$ can be obtained from the generalized kinetic equation, in our case, from the GBE. Still, we can guess their form without appeal to the KE_f. Indeed, let us write the generalized continuity equation

$$\frac{\partial}{\partial t}(\rho - \tau A) + \frac{\partial}{\partial \mathbf{r}} \cdot (\rho \mathbf{v}_0 - \tau \mathbf{B}) = 0 \tag{9}$$

Consider a reference contour drawn at a distance of the order of a particle mean free path λ from the cavity wall. On the other hand if the reference volume extends over the whole of the cavity, then the classical conservation laws should be obeyed and there are no fluctuations on the walls. In other words, the classical equations of continuity and motion must be satisfied at the wall. Using hydrodynamic terminology, we note that the conditions $A = 0$, $\mathbf{B} = 0$ correspond to a laminar sub-layer in a turbulent flow. Now if a local Maxwellian distribution is assumed, then the generalized equation of continuity in the Euler approximation is written as

$$\frac{\partial}{\partial t}\left\{\rho - \tau\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\rho \mathbf{v}_0)\right]\right\} + \frac{\partial}{\partial \mathbf{r}} \cdot \left\{\rho \mathbf{v}_0 - \tau\left[\frac{\partial}{\partial t}(\rho \mathbf{v}_0) + \frac{\partial}{\partial \mathbf{r}} \cdot \rho \mathbf{v}_0 \mathbf{v}_0 + \overleftrightarrow{I} \cdot \frac{\partial p}{\partial \mathbf{r}} - \rho \mathbf{a}\right]\right\} = 0, \tag{10}$$

where $\overleftrightarrow{I}$ is the unit tensor. In the hydrodynamic approximation, the mean time τ between the collisions is related to the dynamic viscosity μ by $\tau p = \Pi \mu$, where the factor Π depends on the choice of a collision model and is $\Pi = 0.8$ for the particular case of neutral gas comprising hard spheres [3]. The generalized hydrodynamic equations (GHE) of energy and motion are much more difficult to guess in this way, making the GBE indispensable.

Now several remarks of principal significance:

1. All fluctuations are found from the strict kinetic considerations and tabulated [10]. The appearing additional terms in GHE are due to viscosity and they correspond to the small-scale Kolmogorov turbulence. The neglect of formally small terms is equivalent, in particular, to dropping the (small-scale) Kolmogorov turbulence from consideration and is the origin of all principal difficulties in usual turbulent theory.

Fluctuations on the wall are equal to zero, from the physical point of view this fact corresponds to laminar sub-layer. Mathematically it leads to additional boundary conditions for GHE. Major difficulties arose when the question of existence and uniqueness of solutions of the Navier - Stokes equations was addressed. O A Ladyzhenskaya has shown for three-dimensional flows that under smooth initial conditions a unique solution is only possible over a finite time interval. Ladyzhenskaya even introduced a “correction” into the Navier - Stokes equations in order that its unique solvability could be proved (see discussion in [11]). GHE do not lead to these difficulties.

2. It would appear that in continuum mechanics the idea of discreteness can be abandoned altogether and the medium under study be considered as a continuum in the literal sense of the word. Such an approach is of course possible and indeed leads to Euler equations in hydrodynamics. But when the viscosity and thermal conductivity effects are to be included, a totally different situation arises. As is well known, the dynamical viscosity is proportional to the mean time τ between the particle collisions, and a continuum

medium in the Euler model with $\tau = 0$ implies that neither viscosity nor thermal conductivity are possible.
3. Many GHE applications were realized for calculation of turbulent flows with the good coincidence with the bench-mark experiments [see for example, 10]. GHE are working with good accuracy even in the theory of sound propagation in the rarefied gases where all moment equations based on the classical BE lead to unsatisfactory results.
4. New effects can be found by investigation of atmosphere perturbations as result of cyclone and anti-cyclone evolution. It is known from acoustics that classical Euler equations can lead to wave sound solutions even the initial perturbation has not the wave character. For this situation the Navier Stokes description, involving viscosity, leads to relaxation with damping of perturbations. The corresponding GHE description for evolution of the one-dimensional perturbation leads to following equation for velocity v ($\tau p^{(0)} = \Pi\mu$)

$$\tau \frac{\partial^3 v}{\partial t^3} - \frac{\partial^2 v}{\partial t^2} - \tau c^2 \frac{\partial^3 v}{\partial x^2 \partial t} + c^2 \frac{\partial^2 v}{\partial x^2} = 0 \quad , \tag{11}$$

which has obvious wave solutions

$$v = F_1(x - ct) + F_2(x + ct), \tag{12}$$

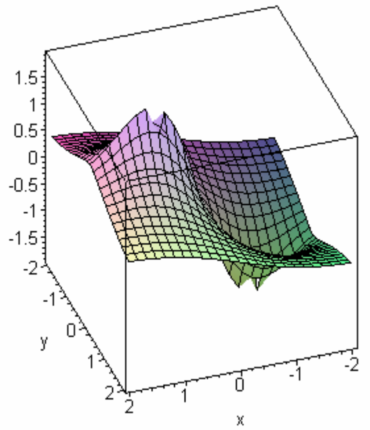
where c is isothermal sound velocity. Non-linear numerical investigation confirms the formation of sound wave fronts of infra-sound diapason. This fact can explain the origin of the meteorological dependence of human beings.

The kinetic effects listed above will always be relevant to a kinetic theory using one particle description – including, in particular, applications to liquids or plasmas, where self-consistent forces with appropriately cut-off radius of their action are introduced to expand the capability of GBE. The application of the above principles also leads to the modification of the system of the Maxwell electro-dynamic equations (ME). While the traditional formulation of this system does not involve the continuity equation (like (9) but for the charge density ρ^a and the current density $\mathbf{j}^a$), nevertheless the ME derivation employs continuity equation and leads to appearance of fluctuations (proportional to τ) of charge density and the current density. In rarefied media both effects lead to Johnson's flicker noise observed in 1925 for the first time by J.B. Johnson by the measurement of current fluctuations of thermo-electron emission. For plasma τ is the mean time between "close" collisions of charged particles [9,10].

As example the classical problem in plasma physics connected with Landau damping can be considered from the positions of GBE as the corresponding asymptotic solution of GBE. Let us remind the classical formulation of the problem. In doing so, we will make the same assumptions that were used in the Landau BE-based derivation [12] namely: (a) the integral collision term is neglected; (b) the evolution of electrons in a self-consistent electric field corresponds to a non-stationary one-dimensional model; (c) ions are in rest, the distribution functions for electrons f_e deviates small from the maxwellian function; (d) a wave number k and complex frequency ω are appropriate to the wave mode considered; (e) the quadratic terms determining the deviation from the equilibrium DF are neglected. In this case effect of Landau collisionless damping should be considered as asymptotic of the corresponding GBE solution and corresponding dispersion equation is the same as in Landau theory. In the usual nomenclature (k is the wave number and r_D is Debye radius)

$$\int_{-\infty}^{+\infty} \frac{e^{-t^2}}{z_0 - t} dt - \frac{\sqrt{\pi}}{z_0} = r_D^2 k^2 \sqrt{\pi} \, . \tag{13}$$

Eq. (13) contains dimensionless values – particle velocity $t = u\sqrt{m_e}\big/\sqrt{2k_B T}$ and the complex frequency $z_0 = x + iy = \hat{\omega} = \hat{\omega}' + i\hat{\omega}''$. $\hat{\omega}' = \omega'\sqrt{m_e}\big/\left(k\sqrt{2k_B T}\right)$, $\hat{\omega}'' = \omega''\sqrt{m_e}\big/\left(k\sqrt{2k_B T}\right)$. The physical origin of the collisionless Landau wave damping is simple. Really, if individual electron of mass m_e moves in the periodic electric field, this electron can diminish its energy (electron velocity larger than phase velocity of wave) or receive additional energy from the wave (electron velocity less than phase velocity of wave). Then the total energy balance for a swarm of electrons depends on quantity of "cold" and "hot" electrons. For the Maxwellian function, the quantity of "cold" electrons is more than quantity of "hot" electrons. This fact leads to, so-called, the collisionless Landau damping of the electric field perturbation. It needs to obtain the solution of dispersion equation (13) for finding of decrement $\gamma = -\omega''$ of wave attenuation. From the first glance it is the simple problem connected only with the estimation of the Landau integral $L(z_0) = U + iV = \int\limits_{-\infty}^{+\infty} \frac{e^{-t^2}}{z_0 - t} dt$. Integral $L(z_0)$ can be evaluated by numerical way, the results of corresponding calculations for real part $Re\, L(z_0)$ and $Im\, L(z_0)$ are shown on Figures 2 and 3.



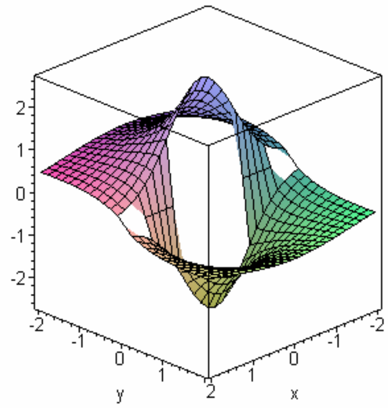


Figure 2. $Re\, L(z_0) = \int\limits_{-\infty}^{+\infty} \frac{(x-t)e^{-t^2}}{(x-t)^2 + y^2} dt$ in domain ($x = -2,...,2;\, y = -2,...,2$), (on the left).

Figure 3. $Im\, L(z_0) = -y \int\limits_{-\infty}^{+\infty} \frac{(x-t)e^{-t^2}}{(x-t)^2 + y^2} dt$ in domain ($x = -2,...,2;\, y = -2,...,2$), (on the right).

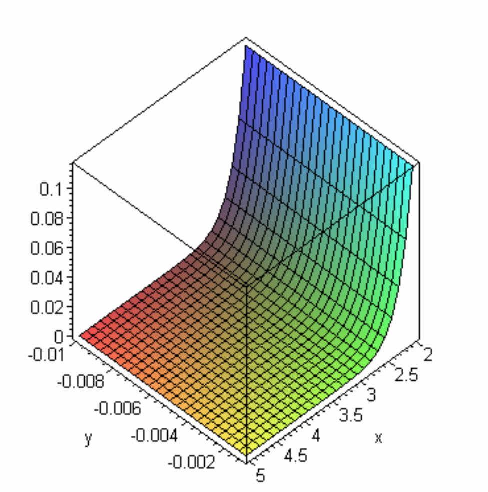
As we see the integral surfaces have very complicated character. But the problem is not only in finding of numerical solution of (13). The ideology of Landau damping penetrates in all physics (not only in plasma physics) and hundreds thousands of references connected with the consideration of this problem. This fact defines also the importance of analytical solution of the Landau problem. In his original paper Landau practically delivers $L(z_0)$ for $\omega'' < 0$ as the sum of semi-residual and first three terms of the $L(z_0)$ series in the approximation of "large z_0"

$$L_L(z_0) = -i\pi\, exp\left(-z_0^2\right) + \frac{\sqrt{\pi}}{z_0}\left(1 + 0.5\frac{1}{z_0^2} + 0.75\frac{1}{z_0^4} + ...\right). \tag{14}$$

Obviously Landau approximation $L_L(z_0)$ does not work by the small z_0; for small $|y|$ and rather large x the results of the difference $Im(L(z_0) - L_L(z_0))$ calculation are presented on Figure 4. From the first glance it is not bad approximation for large z_0, where hemi-residual $-\pi i\, exp\left(-z_0^2\right)$ gives very small correction to the approximation of "large z_0". But the function V has discontinuity by the pass through axis x in the lower half plane and the right approximation should be written in another form $L_{ap}(z_0)$, namely

$$L_{ap}(z_0) = i\pi\, exp\left(-z_0^2\right) + \frac{\sqrt{\pi}}{z_0}\left(1 + 0.5\frac{1}{z_0^2} + 0.75\frac{1}{z_0^4} + ...\right). \tag{15}$$

The right approximation is working much better and this fact is reflected in Figure 5 for the difference $Im(L(z_0) - L_{ap}(z_0))$ for the same area of x, y as in Figure 4.



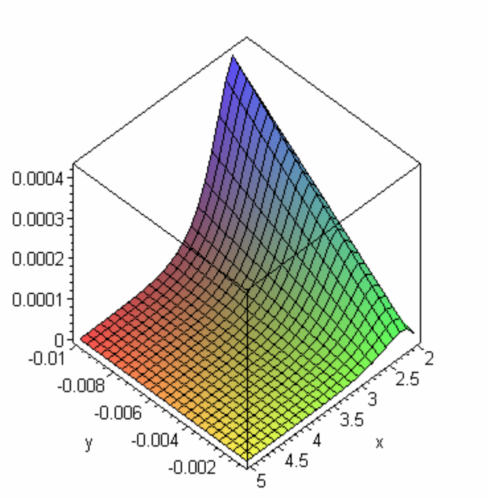


Figure 4. Integral surface for $Im\left[L(z_0)+\pi i e^{-z_0^2}-\frac{\sqrt{\pi}}{z_0}\left(1+\frac{0.5}{z_0^2}+\frac{0.75}{z_0^2}\right)\right]$ in domain $(x=2,...,+5;\ y=-0.01,...,-0.001)$. The difference for the Landau approximation, (on the left).

Figure 5. Integral surface for $Im\left[L(z_0)-\pi i e^{-z_0^2}-\frac{\sqrt{\pi}}{z_0}\left(1+\frac{0.5}{z_0^2}+\frac{0.75}{z_0^2}\right)\right]$ in domain $(x=2,...,+5;\ y=-0.01,...,-0.001)$. The difference for $L_{ap}(z_0)$ approximation, (on the right).

But changing the sign in the front of the first term in the right-hand side of equation (14) leads to the disappearance of effect of damping at all in the classical Landau formulae!

What can be done in this physical situation for searching of right analytical solutions? For the lower half plane the following formulae can be written

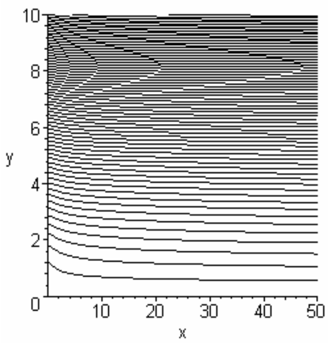
$$\int_{-R}^{R}\frac{f(x)}{z_0-x}dx=\int_{C_R}\frac{f(z)}{z_0-z}dz+2\pi i f(z_0), \tag{16}$$

where R and $-R$ are points on the real axis and C_R is a contour of integration for which z_0 is an interior point. If function $f(z)$ satisfies the special Cauchy's conditions and integral $\int_{C_R}\frac{f(z)}{z_0-z}dz$ over C_R approaches zero as $R\to\infty$, then

$$\int_{-\infty}^{+\infty}\frac{f(x)}{z_0-x}dx=2\pi i f(z_0) \tag{17}$$

for all z_0. Obviously the function $exp(-z^2)$ in general case does not satisfy the mentioned condition for arbitrary contour C_R. But from physical point of view the Landau problem consists in definition of an oscillation defining only by unique complex frequency $z_0=\hat{\omega}'+\hat{\omega}''=\hat{\omega}'-\hat{\gamma}$. The found solution should not depend on the other possible oscillations in the potential electrical field and therefore on integral $\int_{C_R}\frac{f(z)}{z_0-z}dz$. Then the relation (17) should be considered, as additional condition in the theory; then the question arises – is it possible to find the solution of Eq. (13) by the additional condition (17) generalizing the resonance Landau condition $u=\omega'/k$ for real axes? It turns out that that there is the discrete set of complex frequencies, which can be found analytically with the help of Lambert function [10]. Figures 6 and 7 reflect the result of calculations. For high levels this spectrum contains many very close practically straight lines, which human eyes can perceive as background. Moreover plotter from the technical point of view has no possibility to reflect the small curvature of lines approximating this curvature as a long step. My suggestion is to turn this shortcoming into merit in explication of topology of high quantum levels in quantum systems.

Really, extremely interesting that this (from the first glance) grave shortcoming of plotters lead to the automatic construction of approximation for derivatives $d(r_D k)/d\hat{\omega}'$ and $d(r_D k)/d\hat{\omega}''$. This effect has no attitude to the mathematical programming. You can see this very complicated topology of curves including the spectrum of the bell-like dispersion curves in Fig. 6 and Fig. 7, which also form the discrete spectrum. The results of calculations are in good agreement with results of direct mathematical experiment [13] even in Landau's linear formulation, when the Landau formulae for decrement calculation leads to strong disagreement with results of direct mathematical experiments [13].



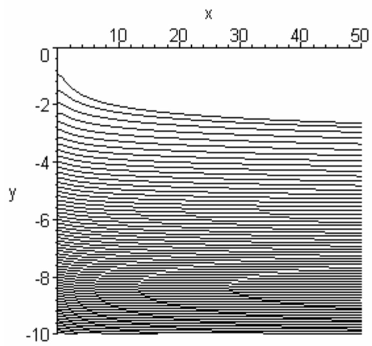


Figure 6. The dimensionless frequency $\hat{\omega}'$ (y axes) versus parameter $r_D k$ (x axes).

Figure 7. The dimensionless frequency $\hat{\omega}''$ (y axes) versus parameter $r_D k$ (x axes).

Conclusion

Finally we can state that introduction of control volume by the reduced description for ensemble of particles of finite diameters leads to fluctuations (proportional to Knudsen number) of velocity moments in the volume. This fact leads to the significant reconstruction of the theory of transport processes in physical systems.

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