

# Kinetic description of classical matter infalling in black holes

Piero Nicolini\* and Massimo Tessarotto\*

*\*Department of Mathematics and Informatics, University of Trieste, Italy*

**Abstract.** A popular aspect of black holes physics is the mathematical analogy between their laws, coming from general relativity and the laws of thermodynamics. The analogy is achieved by identifying a suitable set of observables, precisely: (a)  $E = M$  (being  $E$  the thermodynamic free energy and  $M$  the mass of the BH), (b)  $T = \alpha\kappa$  (with  $T$  the absolute temperature,  $\kappa$  the so-called surface gravity on event horizon and  $\alpha$  a suitable dimensional constant) and (c)  $S = (1/8\pi\alpha)A$  (where  $S$  is the thermodynamic entropy of the black hole and  $A$  the surface of the event horizon). However, despite numerous investigations and efforts spent on the subject, the theoretical foundations of such identifications between physical quantities belonging to apparently unrelated frameworks are not yet clear. The goal of this work is to provide the contribution to the black hole entropy, coming from matter in the black hole exterior. We propose a classical solution for the kinetic description of matter falling into a black hole, which permits to evaluate both the kinetic entropy and the entropy production rate of classical infalling matter at the event horizon. The formulation is based on a relativistic kinetic description for classical particles in the presence of an event horizon. An H-theorem is established which holds for arbitrary models of black holes and is valid also in the presence of contracting event horizons.

## INTRODUCTION

The remarkable mathematical analogy between the laws of thermodynamics and black hole (BH) physics following from classical general relativity still escapes a complete and satisfactory interpretation. In particular it is not yet clear whether this analogy is merely formal or leads to an actual identification of physical quantities belonging to apparently unrelated framework. The analogous quantities are  $E \leftrightarrow M$ ,  $T \leftrightarrow \alpha\kappa$  and  $S \leftrightarrow (1/8\pi\alpha)A$ , where  $A$  and  $\kappa$  are the area and the surface gravity of the BH, while  $\alpha$  is a constant. A immediate hint to believe in the thermodynamical nature of BH comes from the first analogy which actually regards a unique physical quantity: the total energy. However, at the classical level there are obstacles to interpret the surface gravity as the BH temperature since a perfectly absorbing medium, discrete or continuum, which is by definition unable to emit anything, cannot have a temperature different from absolute zero. A reconciliation was partially achieved by in 1975 by Hawking [1], who showed, in terms of quantum particle pairs nucleation, the existence of a thermal flux of radiation emitted from the BH with a black body spectrum at temperature  $T = \hbar\kappa/2\pi k_B$  (*Hawking BH radiation model*). The last analogy results the most intriguing, since the area  $A$  should essentially be the logarithm of the number of microscopic states compatible with the observed macroscopic state of the BH, if we identify it with the Boltzmann definition. In such a context, a complication arises when one strictly refers to the internal microstates of the BH, since for the infinite red shift they are inaccessible to an external observer. An additional difficulty with the identification  $S \leftrightarrow (1/8\pi\alpha)A$ , however, follows from the BH radiation model, since it predicts the existence of contracting BH for which the radius of the BH may actually decrease. To resolve this difficulty a modified constitutive equation for the entropy was postulated [2, 3], in order to include the contribution of the matter in the BH exterior, by setting

$$S' = S + \frac{1}{4}k\frac{c^3A}{G\hbar}, \quad (1)$$

( $S'$  denoting the so-called *Bekenstein entropy*) where  $S$  is the entropy carried by the matter outside the BH and  $S_{bh} \equiv \frac{1}{4}k\frac{c^3A}{G\hbar}$  identifies the contribution of the BH. As a consequence a generalized second law

$$\delta S' \geq 0 \quad (2)$$

was proposed [2, 3] which can be viewed as nothing more than the ordinary second law of thermodynamics applied to a system containing a BH. From this point of view one notices that, by assumption and in contrast to the first term  $S$ ,

$S_{bh}$  cannot be interpreted, in a proper sense, as a physical entropy of the BH, since, as indicated above, it may decrease in time.

This approach however is unconvincing since the precise definition and underlying statistical basis both for  $S$  and  $S_{bh}$  remain obscure. Thus a fundamental problem still appears their precise estimates based on suitable microscopic models. Since the evaluation of  $S_{bh}$  requires the knowledge of the internal structure of the event horizon (excluding for causality the BH interior), the issue can be resolved only in the context of a consistent formulation of quantum theory of gravitation [4, 5]. This can be based, for example, on string theory [6] and can be conveniently tested in the framework of semiclassical gravity [7, 8]. Regarding, instead the entropy produced by external matter  $S$ , its evaluation depends on the nature, not yet known, of the BH. However, even if one regards the BH as a purely classical object surrounded by a suitably large number of classical particles its estimate should be achievable in the context of classical statistical mechanics.

In statistical mechanics the “disorder” characterizing a physical system, classical or quantal, endowed by a large number of permissible microstates, is sometimes conventionally measured in terms of the so-called Boltzmann entropy  $S_B = K \ln W$ . Here  $K$  is the Boltzmann constant while  $W$  is a suitable real number to be identified with the total number of microscopic complexions compatible with the macroscopic state of the system, a number which generally depends on the specific micromodel of the system. Therefore, paradoxically, the concept of Boltzmann entropy does not rely on a true statistical description of physical systems, but only on the classification of the internal microstates (quantal or classical). As is well known [10],  $S_B$  can be axiomatically defined, demanding (i) that it results a monotonic increasing function of  $W$  and (ii) that it satisfies the entropy additivity law  $S_B(W_1 W_2) = S_B(W_1) + S_B(W_2)$ . Boltzmann entropy plays a crucial role in thermodynamics where (i) and (ii) have their corresponding laws in the entropy nondecreasing monotonicity and additivity. Since in statistical mechanics of finite system it is impossible to satisfy both laws exactly, the definition of  $S_B$  is actually conditioned by the requirement of considering systems with  $W \gg 1$  (large physical systems).

An alternate definition of entropy in statistical mechanics is the one given by the Gibbs entropy, in turn related to the concept of Shannon information entropy. In contrast to the Boltzmann entropy, this is based on a statistical description of physical systems and is defined in terms of the probability distribution of the observable microstates of the system. In many cases it is sufficient for this purpose to formulate a kinetic description, and a corresponding kinetic entropy, both based on the one-particle kinetic distribution function. In particular, this is the case of classical many-particle systems, consisting of weakly interacting ultra relativistic point particles, such as those which may characterize the distribution of matter in the immediate vicinity of the BH exterior.

These issue have motivated a recent research effort on the subject by the present authors [9]. The primary goal has been, in particular, to provide an explicit expression for the contribution  $S$ , which characterizes Bekenstein law (1), to be evaluated in terms of a suitable kinetic entropy, and to estimate the corresponding entropy production rate due to infalling matter at the BH event horizon. In addition we intend to establish an H-theorem for the kinetic entropy which holds, in principle, for a classical BH characterized by event horizons of arbitrary shape and size and even in the presence of BH implosions or slow contractions. This is obtained in the framework where the classical description of outside matter and space is a good approximation to the underlying physics.

## KINETIC DESCRIPTION OF INFALLING MATTER

The basic assumption is that the matter falling into the BH is formed by a system  $S_N$  of  $N$  classical point particles moving in a classical spacetime and described by Hamiltonian dynamics, while the event horizon can be treated as a classical absorbing porous wall. We adopt for this purpose a covariant kinetic formalism taking into account the presence of an event horizon. The evolution of such a system is well known and results uniquely determined by the classical equations of motion, defined with respect to an arbitrary observer  $O$ . To this purpose let us choose  $O$ , without loss of generality, in a region where space time is (asymptotically) flat, endowing with the proper time  $\tau$ , with  $\tau$  assumed to span the set  $I \subseteq R$  (observer’s time axis). Each particle is described by the canonical state  $\mathbf{x}$ , which in the case of point particles spans the 8-dimensional phase space  $\Gamma$ , where  $\mathbf{x} = (r^\mu, p_\mu)$ . Moreover, its evolution is prescribed in terms of a suitable relativistic Hamiltonian  $H = H(\mathbf{x})$  so that the canonical state  $\mathbf{x} = (r^\mu, p_\mu)$  results parameterized in terms of the world line arc length  $s$  (see [11]). As a consequence, requiring that  $s = s(\tau)$  results a strictly monotonic function it follows that, the particle state can be also parameterized in terms of the observer’s time  $\tau$ . To obtain the a kinetic description for such a system we require, as usual,  $N \gg 1$ . In addition, it is assumed that interactions between point particles of  $S_N$  take place only via a mean-field Hamiltonian  $H(\mathbf{x})$  and hence that  $S_N$  can

be identified with a weakly interacting relativistic gas. For  $S_N$  we introduce the kinetic distribution function for the observer  $O$ ,  $\rho_G(\mathbf{x})$ , defined as follows

$$\rho_G(\mathbf{x}) \equiv \rho(\mathbf{x}) \delta(s - s(\tau)) \delta(\sqrt{u_\mu u^\mu} - 1) \quad (3)$$

where  $\rho(\mathbf{x})$  is the conventional kinetic distribution function in the 8-dimensional phase space, to be assumed suitably smooth and summable in  $\Gamma$  (*Assumption  $\alpha$  of regularity*). Notice that the Dirac deltas here introduced must be intended as *physical realizability equations*. In particular the condition placed on the arc length  $s$  implies that the particle of the system is parameterized with respect to  $s(\tau)$ , i.e., it results functionally dependent on the proper time of the observer; instead the constraints placed on 4-velocity implies that  $u^\mu$  is a tangent vector to a timelike geodesic. The *event horizon* of a classical BH is defined by the surface  $r_H$  specified by the equation

$$R(x) = r_H \quad (4)$$

where  $x$  denotes a point of the space time manifold, while  $R(x)$  reduces to the radial coordinate in the spherically symmetric case.

According to a classical point of view, let us now assume that the particles are "captured" by the BH (i.e., for example, they effectively disappear for the observer since their signals are red shifted in such a way that they cannot be anymore detected [12]) when they reach a suitable surface  $\gamma$  (*capture surface*) of equation

$$R_\varepsilon(x) = r_\varepsilon. \quad (5)$$

Here  $r_\varepsilon = (1 + \varepsilon)r_H$ , while  $\varepsilon > 0$  depends on the detector and the 4-momentum of the particle. The presence of the BH event horizon is taken into account by defining suitable boundary conditions for the kinetic distribution function on the hypersurface  $\gamma$ , to be treated as an effective absorbing porous wall. For this purpose, we distinguish between incoming and outgoing distributions on  $\gamma$ ,  $\rho_G^-(\mathbf{x})$  and  $\rho_G^+(\mathbf{x})$  corresponding respectively to  $n_\alpha u^\alpha \leq 0$  and  $n_\alpha u^\alpha > 0$ , where  $n_\alpha$  is a locally radial outward 4-vector. Therefore, the boundary conditions on  $\gamma$  are specified as follows

$$\rho_G^-(\mathbf{x}) \equiv \rho(\mathbf{x}) \delta(s - s(\tau)) \delta(\sqrt{u_\mu u^\mu} - 1) \quad (6)$$

$$\rho_G^+(\mathbf{x}) \equiv 0. \quad (7)$$

It follows that it is possible to represent the kinetic distribution function in the whole space time manifold in the form

$$\rho_G(\mathbf{x}) = \rho_G^-(\mathbf{x}) + \rho_G^+(\mathbf{x}) \quad (8)$$

where

$$\begin{aligned} \rho_G^\pm(\mathbf{x}) &\equiv \rho(\mathbf{x}) \delta(s - s(\tau)) \delta(\sqrt{u_\mu u^\mu} - 1) \times \\ &\times \Theta^\pm(R_\varepsilon(x) - r_\varepsilon(s(\tau))) \end{aligned} \quad (9)$$

with  $\Theta^\pm$  respectively denoting the strong and the weak Heaviside functions

$$\Theta^-(a) = \begin{cases} 1 & \text{for } a \geq 0 \\ 0 & \text{for } a < 0. \end{cases} \quad (10)$$

and

$$\Theta^+(a) = \begin{cases} 1 & \text{for } a > 0 \\ 0 & \text{for } a \leq 0. \end{cases} \quad (11)$$

In the sequel we shall introduce the hypothesis that the distribution  $\rho_G^-(\mathbf{x})$  has a compact support (*Assumption  $\beta$* ).

It is important to stress that in the definition of the boundary conditions no detailed physical model is actually introduced for the particle loss mechanism, since all particles are assumed to be captured on the same hypersurface  $\gamma$ , independent of their mass, charge and state. This provides a classical loss model for the BH event horizon.

Let us now consider the evolution of the kinetic distribution function  $\rho_G(\mathbf{x})$  in external domains, i.e. outside the event horizon. Assuming that binary collisions are negligible, or can be described by means of a mean field, and provided that the phase space volume element is conserved, it follows the collisionless Boltzmann equation, or the Vlasov equation in the case of charged particles [20],

$$\frac{ds}{d\tau} \left\{ \frac{dr^\mu}{ds} \frac{\partial \hat{\rho}(\mathbf{x})}{\partial r^\mu} + \frac{dp_\mu}{ds} \frac{\partial \hat{\rho}(\mathbf{x})}{\partial p_\mu} \right\} = 0 \quad (12)$$

with summation understood over repeated indexes, while  $\hat{\rho}(\mathbf{x})$  denotes  $\rho_G(\mathbf{x})$  evaluated at  $r^0 = r^0(s(\tau))$  and  $p_0 = m \left| \frac{\partial r_0(s)}{\partial s} \right|_{s=s(\tau)}$ . This equation resumes the conservation of the probability in the relativistic phase space in the domain external to the event horizon. Invoking the Hamiltonian dynamics for the system of particles, the kinetic equation takes the conservative form

$$\frac{ds}{d\tau} [\hat{\rho}(\mathbf{x}), H]_{\mathbf{x}} = 0. \quad (13)$$

## BH STATISTICAL ENTROPY AND H-THEOREM

Let us now introduce the appropriate definition of kinetic entropy  $S(\rho)$  in the context of relativistic kinetic theory. We intend to prove that in the presence of the BH event horizon it satisfies an  $H$  theorem.

The concept of entropy in relativistic kinetic theory can be formulated by direct extension of customary definition given in nonrelativistic setting [14, 15, 16]. For this purpose we introduce the notion of kinetic entropy, measured with respect to an observer endowed with proper time  $\tau$ , as follows

$$S(\rho) = -P \int_{\Gamma} d\mathbf{x}(s) \delta(s - s(\tau)) \delta(\sqrt{u_\mu u^\mu} - 1) \rho(\mathbf{x}) \ln \rho(\mathbf{x}), \quad (14)$$

where  $\rho(\mathbf{x})$  is strictly positive and, in the 8-dimensional integral, the state vector  $\mathbf{x}$  is parameterized with respect to  $s$ , with  $s$  denoting an arbitrary arc length. Here  $P$  is the principal value of the integral introduced in order to exclude from the integration domain the subset in which the distribution function vanishes. Hence  $S(\rho)$  can also be written:

$$S(\rho) = -P \int_{\Gamma} d\mathbf{x}(s) \delta(s - s(\tau)) \rho_1(\mathbf{x}) \ln \rho(\mathbf{x}), \quad (15)$$

where  $\rho_1(\mathbf{x})$  now reads

$$\rho_1(\mathbf{x}) = \Theta(r(s) - r_\epsilon(s)) \delta(\sqrt{u_\mu u^\mu} - 1) \rho(\mathbf{x}(s)). \quad (16)$$

In the sequel  $S(\rho)$  will be denoted as *BH classical entropy*. Differentiating with respect to  $\tau$  and introducing the invariant volume element  $d^3\mathbf{r}d^3\mathbf{p}$ , the entropy production rate results manifestly proportional to the area  $A$  of the event horizon and reads

$$\frac{dS(\rho)}{d\tau} \equiv \frac{dS_1}{d\tau} + \frac{dS_2}{d\tau} = -P \int_{\Gamma^-} d^3\mathbf{r}d^3\mathbf{p} F_{rr_\epsilon} [\delta(r - r_\epsilon) \hat{\rho} \ln \hat{\rho}], \quad (17)$$

where  $F_{rr_\epsilon}$  is the characteristic integrating factor

$$F_{rr_\epsilon} \equiv \frac{ds(\tau)}{d\tau} \left( \frac{dr}{ds} - \frac{dr_\epsilon}{ds} \right). \quad (18)$$

Indeed, the r.h.s represents the entropy flux across the event horizon while  $\frac{dS_1}{d\tau}$  and  $\frac{dS_2}{d\tau}$  are the contributions to the entropy production rate which depend, respectively, on the velocity of the infalling matter and of the event horizon:

$$\frac{dS_1}{d\tau} = P \int_{\Gamma^-} d^3\mathbf{r}d^3\mathbf{p} \frac{ds(\tau)}{d\tau} \frac{dr}{ds} \delta(r - r_\epsilon) \hat{\rho} \ln \hat{\rho}, \quad (19)$$

$$\frac{dS_2}{d\tau} \equiv P \int_{\Gamma^-} d^3\mathbf{r}d^3\mathbf{p} \frac{ds(\tau)}{d\tau} \frac{dr_\epsilon}{ds} \delta(r - r_\epsilon) \hat{\rho} \ln \hat{\rho}. \quad (20)$$

As a consequence,  $\frac{dS_1}{d\tau}$  and  $\frac{dS_2}{d\tau}$  can be interpreted as the contributions to the BH entropy production rate carried respectively by the infalling matter and the event horizon. Here,  $\Gamma^-$  is the sub-domain of phase space corresponding to the particle falling into the BH. Hence, it follows that in the above integrals,  $\frac{dr}{ds}$  results by definition negative. Instead,  $\frac{dr_\epsilon}{ds}$  has not a definite sign and it can be negative in the case of contracting event horizons. We can also write the above expression in terms of the kinetic probability density evaluated at the hypersurface  $\sqrt{u_\mu u^\mu} = 1$ , defined as  $\hat{f}(\mathbf{x}) \equiv \hat{\rho}/N$ .

It follows  $\hat{\rho} \ln \hat{\rho} \equiv N \hat{f} \ln N \hat{f}$ . At this point we adopt a customary procedure in statistical mechanics [17] invoking the inequality

$$N \hat{f} \ln N \hat{f} \geq N \hat{f} - 1 \quad (21)$$

and notice that in the sub-domain be in which  $F_{rr_\epsilon} \geq 0$  there results by definition  $\hat{\rho} = 0$ . Hence it follows that in  $\Gamma^-$

$$F_{rr_\epsilon} < 0 \quad (22)$$

where by construction  $\frac{ds(\tau)}{d\tau} > 0$ . This result holds independent of the value of  $\frac{dr_\epsilon}{ds}$ . Next, due to Assumption  $\alpha$  and  $\beta$  on the surface  $\gamma$  the kinetic distribution function results necessarily non-zero only in a bounded subset of phase space. Therefore, if  $\delta$  is an arbitrary infinitesimal, there exists a bounded subset  $\Omega \subset \Gamma^-$  such that  $N \hat{f}$  results infinitesimal (of order  $\delta$ ) in the complementary set  $\Gamma^- \setminus \Omega$  (which includes the set of improper points of  $\Gamma^-$ ) and moreover

$$P \int_{\Gamma^- \setminus \Omega} d^3 \mathbf{r} d^3 \mathbf{p} |F_{rr_\epsilon}| \delta(r - r_\epsilon) [N \hat{f} \ln N \hat{f}] \sim O(\delta \ln \delta). \quad (23)$$

At least in a subset of  $\Omega$ ,  $N \hat{f}$  is by assumption positive and such that  $N \hat{f} > \delta$ . Therefore there results

$$0 < P \int_{\Omega} d^3 \mathbf{r} d^3 \mathbf{p} |F_{rr_\epsilon}| \delta(r - r_\epsilon) \equiv M \quad (24)$$

where  $M$  is a suitable finite constant. Thus one obtains

$$\frac{dS(\rho)}{d\tau} \geq P \int_{\Omega} d^3 \mathbf{r} d^3 \mathbf{p} |F_{rr_\epsilon}| \delta(r - r_\epsilon) [N \hat{f} - 1] + O(\delta \ln \delta) \quad (25)$$

The first term of the r.h.s of (25) can be interpreted in terms of the effective radial velocity of incoming particles

$$V_r^{eff} \equiv \frac{1}{n_0} \int_{\Omega} d^3 \mathbf{p} |F_{rr_\epsilon}| \delta(r - r_\epsilon) \hat{f}, \quad (26)$$

while  $n_0$  is the surface number density of the incoming particle distribution function. Finally we invoke the majorization

$$\begin{aligned} \frac{dS(\rho)}{d\tau} &\geq \frac{dS}{d\tau} \geq N \inf \left\{ \int_{\Omega} d^3 \mathbf{r} n_0 V_r^{eff} \right\} \\ &\quad - M + O(\delta \ln \delta) \end{aligned} \quad (27)$$

and notice that, due to the arbitrariness of  $\delta$ ,  $N$  can always be chosen sufficiently large to satisfy the inequality  $\dot{S} > 0$ . We stress that  $\inf \left\{ \int d^3 \mathbf{r} n_0 V_r^{eff} \right\}$  can be assumed strictly positive for non isolated BH's surrounded by matter while  $\delta$  is arbitrary. This yields the relativistic H-theorem:

**THEOREM - H-Theorem for the BH classical entropy**

*In validity of the kinetic equation (12), boundary conditions (6),(7) and regularity conditions defined by Assumptions  $\alpha$  and  $\beta$ , the BH classical entropy  $S(\rho)$  results uniquely defined, according to Eq.(15). Moreover, if the number  $N \gg 1$  of classical point particles of the system  $S_N$  is assumed suitably large, there results:*

$$\frac{dS(\rho)}{d\tau} \equiv \frac{dS_1}{d\tau} + \frac{dS_2}{d\tau} \geq 0, \quad (28)$$

where  $\frac{dS_1}{d\tau}$  and  $\frac{dS_2}{d\tau}$  are respectively the contributions to entropy production rate (19),(20) due to respectively to infalling matter and the event horizon. In particular there results  $\frac{dS(\rho)}{d\tau} = 0$ , if and only if  $n_0 \equiv 0$ , being  $n_0$  the number density on the boundary surface  $\gamma$ . Moreover, the inequality (28) holds even if:

$$\frac{dr_\epsilon}{ds} < 0 \quad (29)$$

(contracting event horizon).

## CONCLUSIONS

Let us briefly analyze the basic implications of this result. First we notice that the H-theorem here obtained appears of general validity even if achieved in the classical framework and under the customary requirement  $N \gg 1$  (large classical system). Indeed the result applies to BH having, in principle, arbitrary shape of the event horizon. The description adopted is purely classical both for the falling particles (charged or neutral [18, 19, 20, 21]) and for the gravitational field and is based on the relativistic collisionless Boltzmann equation and/or the Vlasov equation respectively for neutral and charged particles. A key aspect of our formalism is the definition of suitable boundary conditions for the kinetic distribution function in order to take into account the presence of the event horizon. The expressions for the entropy and entropy production rate, respectively Eqs.(14) and (17), can be used for specific applications to astrophysical problems. Finally interesting features of the derivation are that the entropy production rate results proportional to the area of the event horizon and that the formalism is independent of the detailed model adopted for the BH. In particular also the possible presence of an imploding star (contracting event horizon) is permitted.

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