# Generalized Reynolds Equation based on the Ellipsoidal Statistical Model

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**Abstract.** Rarefied gas flows in ultra-thin film slider bearings are studied in a wide range of Knudsen numbers. The generalized Reynolds equation, first derived by Fukui and Kaneko [1], [2], [3] on the basis of the linearized BGK Boltzmann equation [4] , has been extended by considering a more refined kinetic model of the collisional Boltzmann operator, that is the linearized ellipsoidal statistical (ES) model, which allows the Prandtl number to assume its proper value [5].

## THE POISEUILLE-COUETTE PROBLEM ACCORDING TO THE ES MODEL

Let us consider two plates separated by a distance $h$ and a gas flowing parallel to them in the $x$ direction due to a pressure gradient. The lower boundary (placed at $z = -h/2$) moves to the right with velocity $U$, while the upper boundary (placed at $z = h/2$) is fixed. Both boundaries are held at a constant temperature $T_0$.

If the pressure gradient is taken to be small as well as the velocity $U$, it can be assumed that the velocity distribution of the flow is nearly the same as that occurring in an equilibrium state. This means that the Boltzmann equation can be linearized about a Maxwellian $f_0$ by putting [6], [7]:

$$f = f_0(1+\tilde{h}) \tag{1}$$

where $f(x,z,\mathbf{c})$ is the distribution function for the molecular velocity $\mathbf{c}$ expressed in units of $(2RT_0)^{1/2}$ ($R$ being the gas constant), $z$ is the coordinate normal to the plates and $\tilde{h}(z,\mathbf{c})$ is the small perturbation upon the basic equilibrium state. If one assumes the linearized ES model for the collision operator [5], the Boltzmann equation reads [8]:

$$\frac{1}{2}k + c_z\frac{\partial Z}{\partial z} = \frac{1}{\theta'}\Big[\pi^{-1/2}\int_{-\infty}^{+\infty} e^{-c_{z_1}^2} Z(z,c_{z_1})\, dc_{z_1} - \lambda c_z \pi^{-1/2}\int_{-\infty}^{+\infty} e^{-c_{z_1}^2} c_{z_1} Z(z,c_{z_1})\, dc_{z_1} - Z(z,c_z)\Big] \tag{2}$$

where by definition:

$$Z(z,c_z) = \pi^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(c_x^2+c_y^2)} c_x \tilde{h}(z,\mathbf{c})\, dc_x\, dc_y\ ;$$

$$k = \frac{1}{p}\frac{\partial p}{\partial x} = \frac{1}{\rho}\frac{\partial \rho}{\partial x}$$

with $p$ and $\rho$ being the gas pressure and density, respectively, and $\theta'$ is a suitable mean free time. In Eq. (2) $\lambda$ is a constant to be chosen in such a way as to have the correct Prandtl number. $\lambda$ is equal to 0 for the BGK model ($\mathrm{Pr} = 1$) and 1 for a Maxwell gas ($\mathrm{Pr} = 2/3$).

From Eq. (2) one can obtain the momentum conservation equation:

$$\frac{k}{2} + \frac{\partial}{\partial z}P_{xz} = 0 \tag{3}$$

with

$$P_{xz}(z) = \pi^{-1/2} \int_{-\infty}^{+\infty} e^{-c_z^2} c_z Z(z, c_z)\, dc_z \tag{4}$$

The integration of Eq. (3) gives

$$P_{xz}(z) = -\frac{k}{2} z + \Pi \tag{5}$$

where $\Pi$ is an integration constant. Therefore Eq. (2) can be rewritten as

$$\frac{1}{2}k + c_z \frac{\partial Z}{\partial z} = \frac{1}{\theta'} \left[ \pi^{-1/2} \int_{-\infty}^{+\infty} e^{-c_{z_1}^2} Z(z, c_{z_1})\, dc_{z_1} + \frac{k}{2} \lambda c_z z - \lambda c_z \Pi - Z(z, c_z) \right] \tag{6}$$

If one assumes that the bulk velocity of the gas, defined by

$$q(z) = \pi^{-1/2} \int_{-\infty}^{+\infty} e^{-c_z^2} Z(z, c_z)\, dc_z \tag{7}$$

is a known quantity, the integrodifferential Boltzmann equation (6) can be formally handled as an ordinary inhomogeneous differential equation whose solution reads:

$$\begin{aligned} Z(z, c_z) = \quad & \exp(-(z + \tfrac{h}{2}\mathrm{sgn} c_z)/(c_z \theta')) Z(-\tfrac{h}{2}\mathrm{sgn} c_z, c_z) + \int_{-\frac{h}{2} sgn c_z}^{z} \exp\left( \frac{-|z-t|}{|c_z| \theta'} \right) \\ & \times [q(t) - k\theta'/2 - \lambda c_z \Pi + \lambda k c_z t/2]/(c_z \theta')\, dt \; . \end{aligned} \tag{8}$$

The values of the $Z$ function at the boundary, $Z(-\frac{h}{2}\mathrm{sgn} c_z, c_z)$, depend on the model of boundary conditions chosen. In the following, we will consider the Maxwell boundary conditions and specialize the analysis to walls having different physical properties so that two accommodation coefficients $(\alpha_1, \alpha_2)$ must be used. In this case, the boundary conditions can be written as:

$$Z^+(h/2, c_z) = (1 - \alpha_1) Z^-(h/2, -c_z) \tag{9}$$

$$Z^+(-h/2, c_z) = \alpha_2 U + (1 - \alpha_2) Z^-(-h/2, -c_z) \tag{10}$$

where $U$ is expressed in units of $(2RT_0)^{1/2}$; $Z^-(-h/2, c_z)$, $Z^-(h/2, c_z)$ are the distribution functions of the molecules impinging upon the walls and $Z^+(-h/2, c_z)$, $Z^+(h/2, c_z)$ the distribution functions of the molecules reemerging from them.

Once the function at the boundary, $Z(-\frac{h}{2}\mathrm{sgn} c_z, c_z)$, has been evaluated following the analytical procedure reported in [8], the substitution of the integral formula (8) in the definition (7) of $q(z)$ gives the equation for the bulk velocity of the gas:

$$q^\lambda(z) = \frac{1}{2} k\theta' [1 - \psi_p(u')] + U \psi_c(u') \tag{11}$$

where the following non-dimensional variables have been introduced:

$$\delta' = h/\theta'; \qquad\qquad u' = z/\theta' \, .$$

The non-dimensional functions $\psi_p(u')$ and $\psi_c(u')$, giving the Poiseuille and Couette contributions, respectively, have been explicitly reported in [8].

Until now we have not mentioned the relation between $\theta'$ and the collision time $\theta$ defined in the BGK model [6]. In order to get the same viscosity coefficient from the BGK model and the present one, we must put

$$\theta' = \frac{(\lambda + 2)}{2} \theta \tag{12}$$

Therefore, the rarefaction parameter $\delta'$ can be rewritten in terms of the inverse Knudsen number $\delta$, appearing in the BGK solution of the Poiseuille-Couette problem, as follows

$$\delta' = \frac{2\delta}{(2+\lambda)} \tag{13}$$

Using Eq. (11), the flow rate (per unit time through unit thickness) defined by:

$$F = \rho \int_{-h/2}^{h/2} q(z)dz \tag{14}$$

can be expressed as the sum of the Poiseuille flow ($F_p$) and the Couette flow ($F_c$) as follows:

$$F^\lambda = F_p^\lambda + F_c^\lambda = -\frac{\partial p}{\partial x} h^2 Q_p^\lambda(\delta, \alpha_1, \alpha_2) + \frac{\rho U h}{2} Q_c^\lambda(\delta, \alpha_1, \alpha_2) \tag{15}$$

where

$$Q_p^\lambda(\delta, \alpha_1, \alpha_2) = -\frac{1}{\delta'} + \frac{1}{\delta'^2} \int_{-\delta'/2}^{\delta'/2} \psi_p(u')du'$$

$$Q_c^\lambda(\delta, \alpha_1, \alpha_2) = \frac{2}{\delta'} \int_{-\delta'/2}^{\delta'/2} \psi_c(u')du'$$

are the non-dimensional volume flow rates.

## GENERALIZED REYNOLDS EQUATION BASED ON THE ES MODEL

The analysis developed in the previous section can be applied to the slider bearing problem in lubrication theory. The basic geometry of the two-dimensional gas film is outlined in Fig. 1.

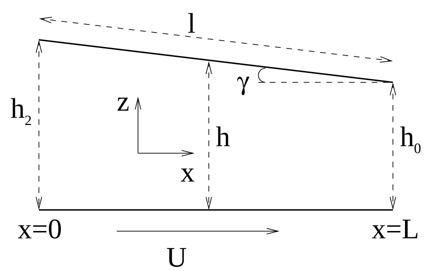


**FIGURE 1.** Geometry of a slider bearing.

Unlike the configuration considered in the previous section, the upper plate is slightly inclined at a small angle $\gamma$. Since the pitch angle $\gamma$ is typically less than $1^o$, the pressure $p$ in the gas is taken to be constant in the vertical direction, while at the left ($x = 0$) and right ($x = L$) boundaries it is fixed at ambient pressure $p_o$. Assuming that the heat generation in the gas is very small, so that an isothermal process can be considered, the generalized Reynolds equation, modified to take into account the gas rarefaction effects, can be derived applying the mass flow conservation equation across the film thickness with the flow rate $F^\lambda$ given by Eq. (15) [9]:

$$\frac{d}{dx}\left(\frac{dp}{dx} h^2 Q_p^\lambda(\delta, \alpha_1, \alpha_2) - \frac{\rho U h}{2} Q_c^\lambda(\delta, \alpha_1, \alpha_2)\right) = 0 \tag{16}$$

In terms of the following dimensionless quantities

$$X = \frac{x}{l}\,; \quad P = \frac{p}{p_o}\,; \quad H = \frac{h}{h_o},$$

the non-dimensional generalized Reynolds equation reads:

$$\frac{d}{dX}\left(\tilde{Q}_p^\lambda(\delta_o PH, \alpha_1, \alpha_2) PH^3 \frac{dP}{dX} - Q_c^\lambda(\delta_o PH, \alpha_1, \alpha_2)\Lambda PH\right) = 0 \tag{17}$$

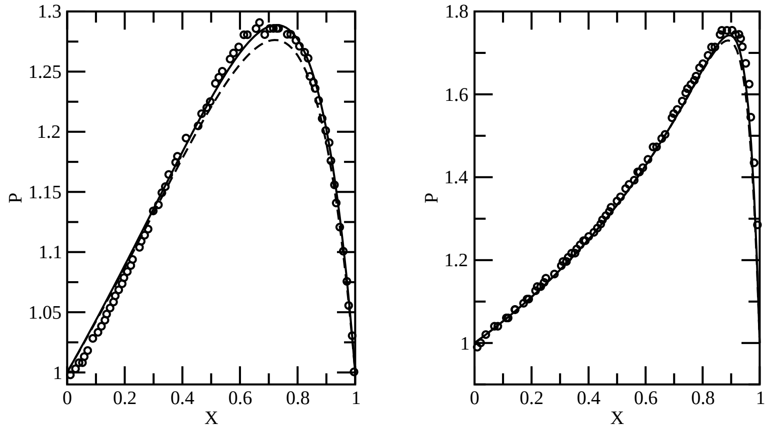


**FIGURE 2.** Pressure profile versus X. Comparison between the Reynolds-BGK results (solid line), the Reynolds-ES results (dashed line) and DSMC data (Alexander et al. 1994) (open circles). The parameters are: $\delta_o = 0.7$, $\Lambda = 61.6$, $\alpha_1 = \alpha_2 = 1$. (left); $\delta_o = 0.2$, $\Lambda = 1264$, $\alpha_1 = \alpha_2 = 1$. (right).

where $\tilde{Q}_p^\lambda$ is the Poiseuille relative flow rate: $\tilde{Q}_p^\lambda = Q_p^\lambda / Q_{con}$, with $Q_{con} = \delta/6$ being the continuum flow limit; $\Lambda$ is the bearing number defined as

$$\Lambda = \frac{6\mu U l}{p_o h_o^2}$$

with $\mu$ being the dynamic viscosity of the gas. Furthermore, the rarefaction parameter $\delta$ has been expressed as: $\delta = \delta_o PH$, with $\delta_o$ being the characteristic inverse Knudsen number defined by the minimum film thickness, $h_o$, and the ambient pressure $p_o$:

$$\delta_o = \frac{p_o h_o}{\mu\sqrt{2RT_o}}.$$

If the continuum limit ($\delta \to \infty$) is taken for any fixed $\alpha_1 = \alpha_2$, then the limiting solution for the Poiseuille and Couette flow rates is given by

$$\tilde{Q}_p^\lambda \to 1, \qquad Q_c^\lambda \to 1 \tag{18}$$

so that Eq. (17) reduces to the classical Reynolds equation used in standard hydrodynamic lubrication theory [1], [2], [3]. Writing the non-dimensional film thickness $H$ in terms of the longitudinal coordinate $X$,

$$H = \frac{h_2}{h_o} - \frac{l}{L}\left(\frac{h_2}{h_o} - 1\right) X \tag{19}$$

such that

$$\frac{dP}{dX} = -\frac{l}{L}\left(\frac{h_2}{h_o} - 1\right)\frac{dP}{dH}$$

Eq. (17) can be analytically integrated to give:

$$\frac{l}{L}\left(\frac{h_2}{h_o} - 1\right)\tilde{Q}_p^\lambda(\delta_o PH, \alpha_1, \alpha_2) PH^3 \frac{dP}{dH} + Q_c^\lambda(\delta_o PH, \alpha_1, \alpha_2)\Lambda PH = K_1 \tag{20}$$

where $K_1$ is a constant of integration. The substitution of

$$PH = \zeta \tag{21}$$

in Eq. (20) gives:

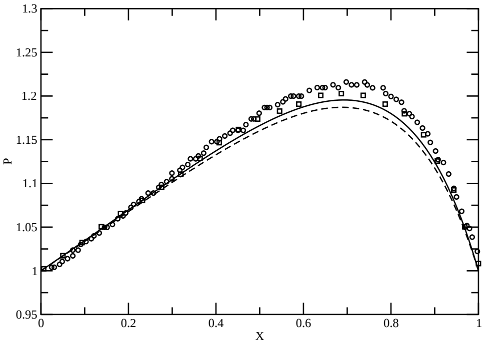


**FIGURE 3.** Pressure profile versus X. Comparison between the Reynolds-BGK results (solid line), the Reynolds-ES results (dashed line), DSMC data (Alexander et al. 1994) (open circles) and IP data (Jiang et al. 2005) (open squares). The parameters are: $\delta_o = 0.7$, $\Lambda = 61.6$, $\alpha_1 = \alpha_2 = 0.7$.

$$\frac{d\zeta}{dH} = \frac{\zeta}{H} - \frac{[Q_c^{\lambda}(\delta_o\zeta,\alpha_1,\alpha_2)\Lambda\zeta - K_1]}{l/L(h_2/h_o - 1)\tilde{Q}_p^{\lambda}(\delta_o\zeta,\alpha_1,\alpha_2)H\zeta} \tag{22}$$

The boundary conditions to be matched to Eq. (22) are given by

$$\zeta = h_2/h_o \quad \text{at} \quad H = h_2/h_o$$

$$\zeta = 1 \quad \text{at} \quad H = 1$$

Eq. (22) has been solved numerically using the relaxation methods. To apply this numerical scheme, the differential equations have to be replaced by finite-difference equations on a point mesh. The solution of the resulting set of equations is determined by starting with a guess and improving it iteratively using Newton's method. The Poiseuille and Couette flow rate coefficients, $Q_p^{\lambda}(\delta,\alpha_1,\alpha_2)$ and $Q_c^{\lambda}(\delta,\alpha_1,\alpha_2)$, respectively, have been evaluated by means of the numerical method described in [8].

## RESULTS AND DISCUSSION

Once $\zeta(H)$ has been numerically evaluated on a grid that spans the domain of interest, Eqs. (19) and (21) give the pressure field in the gas film as a function of the longitudinal coordinate $X$.

A comparison between the Reynolds equation solutions, obtained using the ES and BGK models, and the numerical findings obtained from DSMC (Direct Simulation Monte Carlo) simulations [10] and IP (Information Preservation) method [11] in the case of Maxwell's boundary conditions on two physically identical walls, is shown in Figures 2 and 3. The parameters describing the gas film geometric configuration were fixed at the following values: $h_2/h_o = 2$, $L/h_o = 100$.

Looking at the pictures, one sees that the pressure distribution in the gas film increases with increasing $\Lambda$. Furthermore, at fixed bearing number, the pressure field reduces by increasing the fraction of gas molecules specularly reflected by the walls. Figures 2 and 3 show that the present Reynolds equation solutions, obtained using the ES and BGK models, are in good agreement with the DSMC data presented by Alexander et al. (1994) and the IP results reported by Jiang et al. (2005). It is worth noting that, in Fig. 3, the results of the IP method given by Jiang et al. (2005) are closer to the Reynolds equation numerical solutions than the DSMC data obtained previously by Alexander et al. (1994). Furthermore, the solution of the Reynolds equation based on the ES model slightly underestimates the pressure profiles given by the DSMC and IP simulations suggesting that in isothermal conditions and at low Mach numbers the corrections introduced by a more refined kinetic model of the collisional Boltzmann operator are extremely small.

In order to investigate the effects of the rarefaction parameter $\delta_o$ and the accommodation coefficients on the basic lubrication characteristics, Figure 4 shows the pressure profiles, obtained through the Reynolds equation based on the ES model, in the near-free molecular flow and near-continuum flow limits for different values of $\alpha_1$ and $\alpha_2$. The picture reveals that, for small $\delta_o$, if one keeps the accommodation coefficient of the slider ($\alpha_1$) fixed and varies the

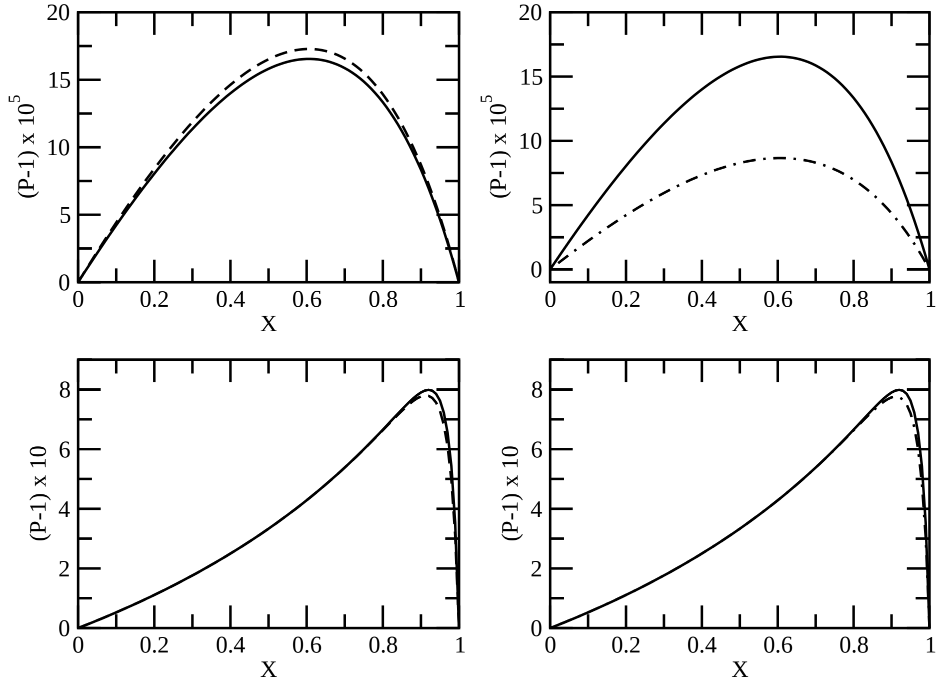


**FIGURE 4.** Pressure profiles, from the Reynolds-ES equation, versus X for $\Lambda = 100$. The line styles indicate $\alpha_1 = 0.5$ $\alpha_2 = 0.8$ (dashed), $\alpha_1 = 0.8$ $\alpha_2 = 0.8$ (solid), $\alpha_1 = 0.8$ $\alpha_2 = 0.5$ (dot dashed). The inverse Knudsen number $\delta_o$ is $10^{-3}$ (top panels) and 10 (bottom panels).

other one ($\alpha_2$), the pressure distribution in the gas film, at fixed bearing number, increases with increasing $\alpha_2$, as it always happens in the continuum region, while at fixed $\alpha_2$, the pressure distribution decreases by increasing $\alpha_1$. Such kind of inverted pressure profiles, which appear in studying the slider air bearing problem in the free-molecular flow regime, are triggered by the Couette contribution to the lubrication flow rate [12], irrespective of which of the two kinetic models (BGK or ES) have been considered to derive the Reynolds equation.

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