

Non-linear gyrokinetic theory of magnetoplasmas

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Abstract. A crucial issue in relativistic plasma, particularly relevant in the astrophysical context, is the description of highly magnetized plasmas based on a covariant formulation of gyrokinetic dynamics. An interesting case in question is that in which the background electric field (produced either by the same plasma or by other sources) results suitably small (or vanishing) with respect to the magnetic field, while at the same time short-wavelength EM perturbations can be present. The purpose of this work is to extend the relativistic gyrokinetic theory developed by Beklemishev *et al.* [1999-2005] to include, in particular, also the treatment of such a case. We intend to show that this requires the development of a perturbative expansion involving simultaneously both the particle 4-position vector and the corresponding 4-velocity vector. For this purpose a synchronous form of the relativistic Hamilton variational principle is adopted.

1. INTRODUCTION

This work is a part of a systematic investigation concerning the relativistic gyrokinetic theory developed by Beklemishev *et al.* [1, 2, 3] and its applications. In this paper, in particular, we intend to propose an extended formulation of covariant gyrokinetic theory specifically intended to include the effect of 4-velocity perturbations, as appropriate for the description of relativistic magnetoplasmas in the presence of suitably "small" (and regular, but otherwise arbitrary) electromagnetic and gravitational perturbations produced, via collective interactions, by the same plasma. The physical motivations of the investigation are related to the occurrence of strongly magnetized relativistic plasmas in curved space-time. In the astrophysical context these plasmas are typically related to the existence of accretion disks, plasma inflows and outflows and relativistic jets, all occurring typically close to massive object, such as neutron stars, black holes (BH) and active galactic nuclei (Mohanty [4]). These plasmas can be characterized by the presence of ultra-strong magnetic fields, possibly generated by the same plasma. Although, direct measurements of \mathbf{B} can be very difficult, for example in jets there is yet no reliable observation of the magnetic field intensity, the partial information already available for other cases indicates that in suitable circumstances the magnetic field can become extremely intense. For example, sufficiently close to the surface of pulsars the magnetic field intensity can reach values well above the so-called quantum critical field strength of $B_c = m_e^2 c^3 / e \hbar = 4.4 \times 10^{13} G$. Magnetic fields of this order or larger are said to be super strong. A similar situation might occur for BH's surrounded by strong magnetic fields, and in particular in the regions of space-time suitably close to the so-called photon-sphere [10], where collimated outflows can arise.

While such fields are typically expected to manifest together with intense electric fields (\mathbf{E}), there can be cases in which there results in particular

$$E/B \ll 1 \tag{1}$$

(*weak electric field assumption*). Here E and B are the invariant eigenvalues of the electromagnetic (EM) field tensor $F_{\mu\nu}$ (see below). This may happen, for example, in plasmas in which the magnetic field is produced primarily by strong diamagnetic currents. These, by definition, are electric currents flowing in the plasma which are driven by gradients of

the relevant fluid fields (in particular, fluid velocity, number density, temperature, magnetic field intensity, etc.) and, instead, are independent of the (local) electric field.

An example is provided by plasma outflows (or jets) in which the kinetic energy of the escaping particles is usually believed to be provided by the presence of a quasi-stationary magnetic field moving with the plasma [13]. In all such phenomena EM turbulence is expected to be ubiquitous and to affect in a significant way plasma dynamics. The region where these processes may act is typically too close to the central object (the BH or a neutron star) to be resolved observationally. Thus, one possible way to obtain some insight into this region is by studying the properties of these phenomena. In particular, the emitted radiation spectrum could be significantly modified by the effect of a finite curvature of space-time, as well as by short-wavelength EM and gravitational perturbations in the region of emission[14].

An important theoretical problem is therefore the prediction of such effects based on single-particle gyrokinetic dynamics. Such phenomena require their description in the framework of a consistent covariant theory taking into account the effect of space-time non-uniformity as well as the possible presence of wave-fields, in principle, both EM and gravitational. A convenient formulation is provided by the covariant gyrokinetic theory recently developed by Beklemishev *et al.* [1, 2]. As well known, object of particle gyrokinetics is the description of particle dynamics in terms of a reduced set of variables (so-called gyrokinetic). These include, in particular, the fast gyration angle around the magnetic flux lines, i.e., the gyrophase ϕ , which results ignorable. Gyrokinetic theory is therefore realized by the construction of a suitable, generally non-canonical, phase-space transformation. From a mathematical viewpoint, gyrokinetic theory provides a perturbative description of particle dynamics in terms of “simpler” equations of motion, which are characterized by a reduced number of effective degrees of freedom. Goal of the present paper is to extend the Beklemishev and Tessarotto formulation to obtain the gyrokinetic theory in curved background space-time which applies also to the special case 1 and includes the simultaneous effect of short-wavelength EM and gravitational perturbations. Starting point is the Larmor-radius ordering, requiring $\varepsilon \sim \frac{r_L}{L_{EM}} \sim \frac{r_L}{L_g} \ll 1$, and $\lambda \equiv \frac{\lambda_{EM}}{L_{EM}}, \delta \equiv \frac{\lambda_g}{L_g} \ll 1$, where r_L is the Larmor radius and L_{EM}, L_g , respectively denote the characteristic scale lengths of the background and wave EM and gravitational fields. In particular, we intend to determine new expressions for the magnetic moment $p_\phi \equiv \hat{\mu}$ and of the cyclotron frequency $\frac{d}{ds}\phi$, possibly modified by the effect of space-time curvature.

2. THE SYNCHRONOUS HYBRID HAMILTON VARIATIONAL PRINCIPLE IN CURVED SPACE

The starting point for the construction of the gyrokinetic theory in curved space-time is the proper treatment of the variational principle which determines the gyrokinetic world lines $\{r'^\alpha(s), s \in I\}$ in the presence of a non-uniform metric tensor $g_{\mu\nu}(r^\alpha)$, as well as the EM 4-vector potential A_μ . Here r'^α denotes the gyrocenter position 4-vector (in the sequel the apex will denote gyrokinetic variables or phase-functions evaluated at the gyrocenter position). Historically (see for example Einstein [5], Landau & Lifschitz [6]) the relativistic equations of motion were obtained using an asynchronous variational principle, stemming from the minimum-action principle (relativistic Hamilton principle). Accordingly, the trajectory of a point-particle is described as an extremal curve (i.e., a geodesics) in the 4-dimensional space-time. The choice of an asynchronous formulation, rather than an asynchronous one, is a natural consequence of the fact that the variational principle is a generalization of the well-known Hertz-Fermat-Maupertius variational principle (Goldstein *et al.* [8]) and that the 4-velocity of a particle, $u^\mu(s) = \frac{d}{ds}r^\mu(s)$, is a unit 4-vector, i.e., it must fulfill the constraint $u_\mu u^\mu = 1$ (*realizability condition*), where $u_\mu = g_{\mu\nu}u^\nu$. As a consequence, it follows that the line element ds along the world line $\{r^\mu(s), s \in I\}$ must satisfy the constraint $ds^2 = g_{\mu\nu}dr^\mu(s)dr^\nu(s)$, which implies that the asynchronous variation of ds results $\Delta(ds) \neq 0$, Δ denoting the operator of asynchronous variation. On the other hand, adopting the [relativistic] asynchronous principle one has to face the conceptual difficulty represented by the fact that - contrary to what happens for the non-relativistic Hamilton principle - it cannot provide a variational principle for the relativistic Hamilton equations (at least using a Legendre transformation like in non-relativistic mechanics). In reality, the difficulty can be circumvented (see for example Wheeler *et al.* [7] and Goldstein *et al.* [8]) adopting a suitable synchronous variational principle. In such a case it is immediate to show that the canonic form of the relativistic Hamilton equations is recovered, just like in non-relativistic mechanics, by means of a suitable Legendre transformation. Finally, it must be recalled that variational principles, both synchronous and asynchronous, can be expressed in arbitrary “hybrid” (i.e., generally non-canonical) variables, provided these variables are connected by a diffeomorphism to the original one [i.e., for example to be identified with the canonical variables], thus yielding a *hybrid variational principle*. These variable can also be superabundant, since the hybrid variables can be defined in

such a way to span, in principle, phase-spaces of arbitrary dimension. Thus, they can be described in the framework of the so-called *constrained dynamics* (Dirac [9]), in turn relying on the well-known *method of Lagrange multipliers*. In particular, as a special choice, the hybrid variables can also be identified with appropriate gyrokinetic variables. In this section we intend to obtain a synchronous variational principle of this type, in which the hybrid variables are provided by a suitable set of superabundant non-canonical variables. For definiteness let us introduce the action $S(\mathbf{y}) \equiv \int_1^2 \gamma(\mathbf{y}) + dF$, where \mathbf{y} denotes the set of variables $\mathbf{y} \equiv (r^\mu, u^\mu, \lambda)$, γ a suitable differential 1-form and $F(\mathbf{y}, s)$ an arbitrary gauge function [i.e., a real function which is at least $C^{(2)}$ in its existence domain]. In particular, let us impose that the fundamental differential 1-form γ takes the value

$$\gamma(\mathbf{y}) = g_{\mu\nu} [qA^\nu + u^\nu] dr^\mu + \lambda [g_{\mu\nu} u^\mu u^\nu - 1] \quad (2)$$

Here the notation is standard [1, 2]. Thus, A^ν are the counter-variant components of the EM 4-potential and $q = Ze/m_0c$ is the normalized charge. and, moreover, r^ν and u^ν coincide with the countervariant components of the 4-vectors \mathbf{r} (4-position) and \mathbf{u} (4-velocity). It is immediate to show that phase-space trajectory of the point particle with 4-position r^ν and 4-velocity u^ν results, by construction, an extremal curve of the variational functional $S(\mathbf{y})$. Another equivalent possibility is to represent the differential 1-form in terms of

$$\gamma(r^\mu, u_\mu, \lambda) = [qA_\mu + u_\mu] dr^\mu - \frac{1}{2} [g^{\mu\nu} u_\mu u_\nu - 1], \quad (3)$$

where the Lagrange multiplier λ has been set equal to its extremal value, $\lambda = -1/2$. Finally, let us require, in case (2), that S be defined in the synchronous functional class of real functions

$$\begin{aligned} \{\mathbf{y}\} &\equiv \left\{ \mathbf{x}(s) \equiv (r^\mu(s), u^\mu(s), \lambda(s)) : \mathbf{x}(s) \in C^{(2)}(I), \right. \\ &\left. \mathbf{x}(s) \in \mathbb{R}^9, I \equiv [s_1, s_2] \subseteq \mathbb{R}, \mathbf{x}(s_i) = \mathbf{x}_1 (i = 1, 2) \right\}, \end{aligned} \quad (4)$$

while for (3), λ is considered prescribed ($\lambda = -1/2$) It is immediate to prove that *in both cases* (2) and (3) the variational principle

$$\delta S = 0 \quad (5)$$

delivers the correct relativistic equations. We remark that the synchronous variations $\delta r^\mu, \delta u^\mu, \delta \lambda$ must be all considered independent. Indeed, the physical realizability condition is satisfied *only* by the extremal curve. Hence, in a proper sense, this represents an example of unconstrained variational principle.

3. EXTENDED GYRODYNAMICS

To obtain the new extended formulation of gyrokinetic theory, the variational functional $S(\mathbf{y})$ must be expressed in terms of gyrokinetic variables, which requires a suitable *extended* phase space transformation to a new set of variables y^i , including in particular the gyrophase ϕ which describes the fast gyration motion of particles around magnetic flux lines. Usually such a transformation is obtained by means of perturbation theory in ε , to be considered infinitesimal and defined as the ratio between the Larmor radius and the smallest characteristic scale length of the magnetoplasma. In particular this requires that the EM 4-potential A^μ and the metric tensor $g_{\mu\nu}$ are assumed to be expressed in terms of the representations

$$A^\mu = \frac{1}{\varepsilon} A^\mu(r^\alpha) + \xi a^\mu(r^\alpha/\xi), \quad (6)$$

$$g_{\mu\nu} = G_{\mu\nu}(r^\alpha, \varepsilon) + \xi g_{\mu\nu}^1(r^\alpha/\xi) \quad (7)$$

(*Larmor-radius ordering*). Here $\frac{1}{\varepsilon} A^\mu(r^\alpha, \varepsilon)$ and $G_{\mu\nu}(r^\alpha, \varepsilon)$, denoting the equilibrium terms, are assumed to be $C^{(\infty)}$ in all variables, while $\xi a^\mu(r^\alpha/\xi)$ and $\xi g_{\mu\nu}^1(r^\alpha/\xi)$, define the short-wavelength (and high-frequency) perturbations.

In particular, ξ is an infinitesimal parameter, assumed generally independent of ε , and in particular such that $\varepsilon \ll \lambda$, which characterizes the tensors $\xi a^\mu(r^\nu/\xi)$ and $\xi g_{\mu\nu}^1(r^\nu/\xi)$. Hence, the equilibrium quantities can be Taylor expanded near r'^α

$$\begin{aligned}\frac{A^\mu}{\varepsilon} &= \frac{A'^\mu}{\varepsilon} + A_1^\mu + o(\varepsilon), \\ G_{\mu\nu} &= G'_{\mu\nu} + \varepsilon G_{\mu\nu}^1 + o(\varepsilon),\end{aligned}$$

where $A'^\mu = A^\mu(r'^\alpha)$, $G'_{\mu\nu} = g_{\mu\nu}(r'^\alpha)$, $A_1^\mu = r_1^\alpha \partial'_\alpha A'^\mu$ and $G_{\mu\nu}^1 \equiv r_1^\alpha \partial'_\alpha G'_{\mu\nu}$. The perturbations of relevant fields depend generally on the gyrophase angle ϕ' through $\varepsilon r_i^\alpha(y^i)$. The perturbative scheme converges, at least in an asymptotic sense, only if one assumes the validity of a suitable ordering conditions (Larmor radius ordering), implying that EM contributions must dominate, in a suitable sense, over inertial ones in the variational functional. Formally this corresponds to replace the charge e by e/ε , leaving the rest mass m_0 unchanged. In this work, extending the approach developed by Beklemishev *et al.* [1, 2, 3], the gyrokinetic transformation is taken of the general form $(r^\alpha, u^\beta) \leftrightarrow y^i \equiv (r'^\alpha, u'^\alpha)$, where r'^α and u'^α identify respectively the so-called gyrocenter 4-vector and a transformed 4-velocity, to be determined respectively in terms of the two power series

$$r^\mu = r'^\mu + \sum_{s=1}^{\infty} \varepsilon^s r_s^\mu \quad (9)$$

$$u^\mu = u'^\mu \oplus \sum_{s=1}^{\infty} \xi^s v_s^\mu(\mathbf{y}) \quad (10)$$

(*extended gyrokinetic transformation*). Here $r_i^\mu(y^i)$ and $v_i^\alpha(y^i)$ ($i = 0, 1, \dots$) are respectively the 4-position and 4-velocity perturbations, while the operator \oplus must be suitably defined as corresponds to the 4-velocity addition law [11]. In particular, the perturbations $r_i^\mu(y^i)$ and $v_i^\alpha(y^i)$ are defined in such a way that both are pure oscillatory with respect to the gyrophase ϕ' . As usual, u'^α can be characterized in terms of suitable independent variables, one of which includes the gyrophase ϕ' . This is obtained by projecting u'^α along the four independent directions defining the so-called fundamental tetrad unit 4-vectors e_a^μ (with $a = 0, \dots, 3$), hereon also denoted $(l'^\mu, l''^\mu, l^\mu, \tau^\mu)$. In particular, the latter can always be identified with the EM fundamental tetrad, i.e., the basis vectors of the EM field tensor admitting, in this case to be evaluated at the gyrocenter position, $F'_{\mu\nu} \equiv F_{\mu\nu}(x'^\alpha)$, with eigenvalues B and E . In particular, the space-like vectors e_2^μ, e_{a3}^μ are assumed to satisfy the eigenvalue equations

$$\begin{aligned}F'_{\mu\nu} e_2^\nu &= B e_{3\mu}, \\ F'_{\mu\nu} e_3^\nu &= -B e_{2\mu}.\end{aligned}$$

Here $e_{a\mu} = g'_{\mu\nu} e_a^\nu$ are the covariant components of e_a^μ and the labels a assume a tensor meaning in the tangent space. Thus we shall raise and lower them in terms of $e_\mu^a = \eta^{ab} e_{b\mu}$, by means of the Minkowskian tensor $\eta = \text{diag}(1, -1, -1, -1)$. It follows to leading order in ε

$$u'^\mu = a^\mu \cos \phi' + b^\mu \sin \phi' + \bar{u}^\mu \quad (11)$$

which can be taken as a definition for the gyrophase ϕ . Here $\bar{g} \equiv \langle g \rangle_\phi = \frac{1}{2\pi} \int_0^{2\pi} g(y) d\phi'$ denotes the ϕ' -average of a function $g(y)$ and a^μ and b^μ can be identified, up to a constant factor, with the two space-like 4-vectors, e_2^μ and e_3^μ (which together with \bar{u}^μ are evaluated at the gyrocenter position x'^α), thus letting $a^\mu = w e_2^\mu$ and $b^\mu = w e_3^\mu$. The components a^μ , b^μ and \bar{u}^μ are not completely arbitrary, but satisfy the constraint $g_{\mu\nu} u^\mu u^\nu = 1$ for all ϕ' . Thus, we shall demand the mutual orthogonality

$$a^\mu b_\mu = a^\mu \bar{u}_\mu = b^\mu \bar{u}_\mu = 0, \quad (12)$$

while their moduli are expressed via a single real scalar w

$$a^\mu a_\mu = b^\mu b_\mu = 1 - g_{\mu\nu} \bar{u}^\mu \bar{u}^\nu = -w^2. \quad (13)$$

We stress that only to the leading order $\bar{u}_\mu = g'_{\mu\nu}\bar{u}^\mu$ where $g'_{\mu\nu} \equiv g_{\mu\nu}(x'^\alpha)$ (and similarly $a_\mu = g'_{\mu\nu}a^\nu$, $b_\mu = g'_{\mu\nu}b^\nu$), while in general $\bar{u}_\mu = g'_{\mu\nu}\bar{u}^\nu \neq g_{\mu\nu}\bar{u}^\nu$. From orthogonality relations (12) one finds the contravariant representation:

$$\bar{u}^\mu = u^0 e_0^\mu + u^\parallel e_1^\mu. \quad (14)$$

Here $u^0 = u_0$, and u^\parallel is the "parallel" component $u^\parallel = -u_\parallel$, with u_0 and u_\parallel denoting the covariant components of \bar{u}^μ , and $\mathbf{e}_0, \mathbf{e}_1$ are the remaining time-like and space-like unit vectors of the tetrad. From Eq. (13) one obtains $u_0^2 = 1 + u_\parallel^2 + w^2$, which permits to express u_0 by means of the independent functions u_\parallel, w . With the above positions for the transformed 4-velocity it can be shown that $r_i^\mu(y^i)$ are purely oscillatory arbitrary 4-vectors, namely the ϕ' -averages of $r_i^\mu(y^i)$ are zero.

4. GYROKINETIC EQUATIONS IN CURVED SPACE

The extended gyrokinetic transformation (9),(10) can be carried out in principle at any order in ε and ξ . Here, to illustrate some interesting features of the approach we intend to formulate the leading-order perturbative theory with respect to the parameter ε (i.e., of order ε^1). In this case the variational functional $S(\mathbf{y})$ when expressed in terms of the gyrokinetic variables $(\phi', u_0, u_\parallel, \hat{\mu}, r'^\mu, \lambda)$ can be shown to take the form

$$S_g(\mathbf{y}') \equiv \int \gamma_g(\phi', u_0, u_\parallel, \hat{\mu}, r'^\mu, \lambda), \quad (15)$$

where

$$\begin{aligned} \gamma_g(\phi', u_0, u_\parallel, \hat{\mu}, r'^\mu, \lambda) = & \left\{ \left(\frac{q}{\varepsilon} A'_\mu + u_\parallel l_\mu + u_0 \tau_\mu \right) dr^\mu + \right. \\ & \left. + \hat{\mu} d\phi + \lambda \left[u_0^2 - u_\parallel^2 - 2qB\hat{\mu} - 1 \right] ds \right\} \end{aligned} \quad (16)$$

is the fundamental gyrokinetic differential 1-form. In particular, in case (3) there results $\lambda = -1/2$ in Eq.(16). Here the notation is standard and in particular $\hat{\mu}$ is the relativistic magnetic moment [1, 2, 3]. Next, we wish to investigate the consequences of the new synchronous variational principle introduced in Sec.2, in which the . We have proven that the Lagrange multiplier λ must necessarily take the value $\lambda = -\frac{1}{2}$ in both cases (2) and (3), thanks to the variational principle (5). The same result manifestly holds also for the gyrokinetic functional $S_g(\mathbf{y}')$, hence

$$\lambda = -\frac{1}{2} \frac{\tau_\mu}{u_0} (wl'^\mu \cos(\phi) + wl''^\mu \sin(\phi) + u_\parallel l^\mu + u_0 \tau^\mu) = -\frac{1}{2}. \quad (17)$$

Therefore, the Euler-Lagrange equations for the variables $(\phi', u_0, u_\parallel, \hat{\mu}, r'^\mu)$ achieve the simple expressions

$$d\hat{\mu} = 0 \Rightarrow \hat{\mu} = \text{const}. \quad (18)$$

$$\tau_\mu dr'^\mu - u_0 ds = 0 \Rightarrow u_0 = \frac{dr'^\mu}{ds} \tau_\mu, \quad (19)$$

$$l_\mu dr'^\mu + u_\parallel ds = 0 \Rightarrow u_\parallel = -\frac{dr'^\mu}{ds} l_\mu, \quad (20)$$

$$d\phi' + qBds = 0 \Rightarrow \frac{d\phi'}{ds} = -\Omega, \quad (21)$$

$$\frac{q}{\varepsilon} F'_{\mu\nu} dr'^\nu - l_\mu du_\parallel - \tau_\mu du_0 - 2\chi q \hat{\mu} H_{,\mu} = 0, \quad (22)$$

$$u_0^2 - u_\parallel^2 - 2qB\hat{\mu} - 1 = 0, \quad (23)$$

where $\Omega = qB$ is the particle gyrofrequency (or cyclotron frequency) defined with respect to the rest mass. It is interesting to point out that Eq.(21) is formally identical to the non-relativistic one. Hence, at this order *no correction due to space-time curvature* appears in this equation. The remaining equations have a straightforward meaning. Thus, Eq.(18) implies the conservation of the relativistic magnetic moment $\hat{\mu}$, which is therefore an adiabatic invariant. Actually the result is expected to hold at any order in ε and λ , although the perturbative theory locally may not converge due to phase-space resonances. Furthermore, Eqs.(21) and (22) deliver the equations for the guiding-center 4-velocity, while (23) implies that the physical realizability condition is satisfied also by the guiding-center velocity. We stress that these equations, while equivalent to those obtained earlier [1], have a simpler structure. This follows from the adoption of the synchronous variational principle here adopted which makes explicit the inner structure and relationship existing between the gyrokinetic variables.

5. CONCLUSIONS

In this paper we have formulated a covariant gyrokinetic approach for single-particle dynamics, which applies to curved space-time. The theory goes beyond the domain of validity of the treatment previously considered by Beklemishev and Tessarotto [1] and includes the case in which the electric field can locally vanish or result much smaller than the magnetic field. As an illustration of the theory the perturbative theory has been developed to leading order in ε . For this purpose we have adopted a synchronous variational principle based on constrained dynamics. The resulting Euler-Lagrange equations satisfy the required constraint demanded by the physical realizability condition for the 4-velocity and displays the inner relationships between the gyrokinetic variables. We have found that to leading-order no effect of curvature of space-time appears on the gyrofrequency at order ε^1 . This information is relevant for plasma diagnostics in astrophysical plasmas.

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