Superabundant-variable approach to gyrokinetic theory

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Abstract. In the context of plasma physics and astrophysics, the guiding center problem regards the perturbative expansion of the solution of the equations of motion for a charged particle in a given electromagnetic field. The perturbation expansion is based on the guiding center approximation, which may be roughly described by saying that electromagnetic effects dominate over inertial effects. The basic strategy of perturbation theory is to seek a coordinate transformation to new coordinates, called gyrokinetic, such that the new equations of motion contain, by construction, an ignorable coordinate, the gyrophase angle which describes the fast rotation around a flux line. Hamiltonian perturbation methods furnish privileged tools because the equations of motion can be derived from Hamilton variational principle, and coordinate transformations can be in terms of suitable Lie-generating functions. The usual strategy [1,2,3] is to construct the gyrokinetic transformation in terms of non-canonical variables based on non-canonical Lie-transform methods. Against this background, this paper is devoted to the development of a canonical gyrokinetic theory, adopting purely canonical Lie-transform techniques. We intend to establish a connection with the Littlejohn's scheme and point out the fundamental relationships between the Lie-generating functions.

INTRODUCTION

Although well known to the scientific community, the gyrokinetic description of single-particle dynamics in strongly magnetized plasmas (*gyrokinetic dynamics*) still presents basic difficulties and open problems. A significant case is provided by the construction of so-called canonical gyrokinetic variables for arbitrary magnetic field geometries. The adoption of canonical variables, in fact, may result of crucial importance both in theoretical problems and numerical simulations (so-called gyrokinetic particle simulations). However, such variables in the traditional approach (see for example [1]) result field-related, because they depend on the magnetic field geometry. In particular, they generally rely on the assumption of existence of nested magnetic surfaces for the magnetic field, or at least the requirement that possible chaotic perturbations of the magnetic field, which tend to destroy the magnetic surfaces, result suitably small [2, 3]. In actual applications, however, such a requirement is difficult or even impossible to be met, at it happens for non-symmetric MHD equilibria. As an example, in MHD equilibria magnetic surfaces may only exist locally namely in the neighborhood of nested magnetic surfaces. Therefore it would be highly desirable to be able to construct gyrokinetic canonical variables which result independent of the magnetic field geometry and apply also to the case of chaotic magnetic fields. Historically two methods are known for the construction of canonical gyrokinetic variables. The first is due to Gardner [4, 5] who employed mixed-variables generating functions to construct the canonical transformation from primitive canonical variables. This method, however, does not seem to warrant the complete generality in the search of the gyrokinetic canonical variables. The second approach instead is based on the Dabroux theorem [6] . This method requires the preliminary construction of a suitable set of noncanonical gyrokinetic variables, in terms of which the relevant fundamental Lagrangian differential 1-form can be expressed (which can be realized, for example, by means of noncanonical Lie-transform methods [7]). To obtain the corresponding gyrokinetic Hamiltonian differential 1-form, the so-called Darboux reduction algorithm [1] must be invoked. As a consequence, it follows that the gyrokinetic canonical coordinates result field-related - namely they depend on the particular geometry of the magnetic field flux lines. However, one can show that the proper reduction in canonical form *cannot generally be achieved*, unless suitable "regularity" assumptions are invoked, which include the requirement of local existence of magnetic surfaces or at least the assumption of suitably small "chaotic" perturbations of the magnetic field [2, 8]. These regularity conditions, however, may be locally violated in typical MHD equilibria, which are only quasi-symmetric [9] and subject also to the action of an electric field. *As a consequence, the construction of these variables can only be achieved under appropriate restrictions on the EM field.*

The goal of the paper is to propose a new solution to this problem in which the gyrokinetic variables are represented by a set of *generalized canonical variables*, i.e., superabundant variables which either obey obey Hamilton equations of motion or are specified by finite-terms constraint equations. Our approach is based on a new definition of canonical transformations, here denoted as *generalized*. These transformations make use of a set of superabundant variables, which in the case of the gyrokinetic dynamics are here denoted as *generalized canonical gyrokinetic variables*. These are identified with the set $\mathbf{X}'=\left(\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}, \mathbf{v}'\right)$ and include in particular the canonical pair $\left(\phi', p_{\phi'}\right)$, being $\phi, p_{\phi'}$ respectively the gyrophase angle characterizing the fast motion of charged particles around a magnetic flux line and its canonically conjugate momentum. The remaining generalized canonical variables are represented by the guiding-center position vector $\mathbf{r}'$, its conjugate canonical momentum $\mathbf{p}_{\mathbf{r}'}$ and the drift velocity $\mathbf{v}'$. All of them can be defined independently of the magnetic flux lines geometry [2, 3] and hence without recurring to the Darboux reduction algorithm [1, 7]. Basic consequences are that, on one hand, no quasi-symmetry assumption [9] is required for the magnetic field, while, on the other hand, the magnetic moment in these variables results in all cases a single-valued function and therefore a physical observable. We intend to prove that gyrokinetic theory can be developed systematically in terms of these superabundant canonical variables by applying Lie-transform perturbative methods and in such a way that these variables result canonical in the sense here defined. As a consequence, all gyrokinetic variables can in principle be evaluated correct to arbitrary orders in the relevant expansion parameters. Key features of this approach, besides the vector form of Hamiltonian equations of motion for the conjugate variables describing the guiding center motion $(\mathbf{r}', p_{\mathbf{r}'})$, is the the identification of suitable volume-preserving superabundant canonical variables.

The paper is organized as follows: we preliminary reformulate Hamiltonian mechanics, extending it to the case of superabundant coordinates. Then, we face the gyrokinetic problem, showing how a canonical transformation can be written in terms of superabundant gyrokinetic coordinates, which are shown to obey a suitable form of Hamilton modified variational principle. The issue of the formulation of perturbative approaches in these superabundant variables is analyzed presented and basic consequences of the theory are discussed.

GENERALIZED CANONICAL TRANSFORMATIONS

As is well known, a parameter-dependent diffeomorphism $\gamma_C : \mathbf{x} \rightarrow \overline{\mathbf{x}}(\mathbf{x}, t, \alpha)$ (which defines a bijection $\Gamma = \mathbb{R}^{2g} \leftrightarrow \Gamma' = \mathbb{R}^{2g}$, with $g \in N_o$ and the independent parameters $t, \alpha \in \mathbb{R}$) conventionally defines a canonical transformation [10, 11] if it satisfies identically the symplectic condition $\underline{\underline{\mathbf{J}}} = \underline{\underline{\mathbf{M}}} \cdot \underline{\underline{\mathbf{J}}} \cdot \underline{\underline{\mathbf{M}}}^T$. Here $\underline{\underline{\mathbf{M}}} = \frac{\partial \overline{\mathbf{x}}}{\partial \mathbf{x}}$ denotes the Jacobian matrix and $\underline{\underline{\mathbf{J}}} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the symplectic (Poisson) matrix of dimension $2g \times 2g$. Traditionally, canonical tranformations concern Hamiltonian systems $\{\mathbf{x}, H(\mathbf{x}, t)\}$, which - in fact - can be natively represented in terms of canonical variables $\mathbf{x} = (\mathbf{q}, \mathbf{p})$. Indeed, as is well-known, the flow generated by the initial value problem for the corresponding Hamilton equations, i.e., $\mathbf{x}_o \leftrightarrow \mathbf{x}(t) = \chi(\mathbf{x}_o, t_o, t)$, results by construction canonical. It follows that it must exist a transformed Hamiltonian function $K(\overline{\mathbf{x}}, t)$ such that $\{\overline{\mathbf{x}}, K(\overline{\mathbf{x}}, t)\}$ is a Hamiltonian system too.

However, the concept of canonical transformation may result too restrictive in certain applications in classical mechanics, as for example in for the gyrokinetic problem for the reasons indicated above (i.e., the difficulty or impossibility of constructing gyrokinetic canonical variables in certain cases). These transformations, in fact, can be regarded as a particular case of what we shall denote as *generalized canonical transformation*, i.e., a smooth transformation $\gamma_G : \mathbf{x} \rightarrow \mathbf{x}_G = \mathbf{x}_G(\mathbf{x}, t, \alpha)$, of the form $\Gamma = \mathbb{R}^{2g} \rightarrow \Gamma_G = \mathbb{R}^{2g+k}$ (with $t, \alpha \in R$ independent parameters and $g \in N_o, k \in N$) for which the transformed phase-space Γ_G can have a larger dimension than the initial phase space Γ (i.e., $k > 0$) and the transformed variables $\mathbf{x}_G$ satisfy either to Hamilton or finite terms constraint equations. More precisely, assuming $\dim(\Gamma_G) = 2g + k > \dim(\Gamma) = 2g$ and let us represent $\mathbf{x}_G$ in the form $\mathbf{x}_G = (\mathbf{z}, \mathbf{u})$, with $\mathbf{z} = (z_1, ... z_{2g})$, $\mathbf{u} = (u_1, ... u_k)$, and require that $\mathbf{z}$ and $\mathbf{u}$ obey respectively the equations (for $i = 1, ..., 2g'$ and $s = 1, k$)

$$\frac{d}{dt} z_i(\mathbf{x}, t, \alpha) = \sum_{j=1,2g'} J_{ij} \cdot \frac{\partial}{\partial z_j} K(\mathbf{x}_G, t, \alpha), \tag{1}$$

$$f_s(\mathbf{z}, \mathbf{u}, t) = 0. \tag{2}$$

Here J_{ij} is the canonical Poisson tensor of rank $2g$, while $K(\mathbf{x}_G, t, \alpha)$ (the transformed Hamiltonian) and $f_s(\mathbf{z}, \mathbf{u}, t)$ (for $s = 1, k$) are real $C^{(k)}$ functions with $k \geq 2$. The components of the transformed state $\mathbf{x}_G$ will be here denoted as *superabundant canonical variables*and Eq.(1) as the corresponding equations of motion for the generalized canonical variables. It is obvious that these equations, just like the usual canonical equations, can also be set in variational form in terms of a suitably constrained formulation of modified Hamilton variational principle. To provide a straightforward example, which will be also relevant in the sequel, let us consider the case of a charged point particle subject to an EM field and defined the following generalized canonical transformation $\mathbf{x} = (\mathbf{r},p_{\mathbf{r}}) \to \mathbf{x}_G = (\mathbf{r},p_{\mathbf{r}},\mathbf{v})$.In particular, one can prove that the superabundant state $\mathbf{x}_G$ is an extremal curve of the action functional $S(\mathbf{x}_G) = \int_{t_1}^{t_2} \gamma(\mathbf{x}_G,t)$, i.e., it satisfies the *constrained* modified Hamilton principle in superabundant variables

$$\delta S(\mathbf{x}_G) = 0, \tag{3}$$

where $\gamma(\mathbf{x}_G,t) \equiv \mathscr{L}(\mathbf{x}_G,\dot{\mathbf{r}},t)dt = d\mathbf{r}\cdot p_{\mathbf{r}}) \ - dtH(\mathbf{r},p_{\mathbf{r}}),t) - [d\mathbf{r} - \mathbf{v}dt]\cdot\left[p_{\mathbf{r}}) \ - m\mathbf{v} - \frac{q}{c}\mathbf{A}\right]$, is the fundamental 1-form, $H(\mathbf{r},\ p_{\mathbf{r}},\ t) = \frac{1}{2m}\left[p_{\mathbf{r}} - \frac{q}{c}\mathbf{A}\right]^2 + q\Phi$ the Hamiltonian and $\{\mathbf{A},\Phi\}$ the EM potentials. It follows that the corresponding Euler-Lagrange equations coincide with the canonical equations $\dot{\mathbf{x}} = [\mathbf{x},H]$, plus the non-holonomic constraint $p_{\mathbf{r}} = m\mathbf{v} - \frac{q}{c}\mathbf{A}$. Therefore $\mathbf{x}_G$ defines a generalized canonical state for the Hamiltonian system.

GENERALIZED HAMILTONIAN GYROKINETIC THEORY

In this section we intend to show that the previous Hamiltonian system, under a suitable assumption of "strong" EM field, can be represented by means of a appropriate set of superabundant canonical variables which are gyrokinetic, i.e. the new Hamiltonian results independent of the gyrophase ϕ when expressed in terms of them. The other basic feature is that the corresponding canonical coordinates can always be chosen to be field-independent. To construct the new gyrokinetic variables we shall follow a two-step approach. The first step consists in constructing a particular set of superabundant gyrokinetic variables $\mathbf{x}' \equiv (\mathbf{r}',p_{\mathbf{r}'},\phi',p_{\phi'})$. For definiteness let us require initially that the Hamiltonian function takes the form $H(\mathbf{r},\ p_{\mathbf{r}},\ t) = \frac{1}{2m}\left[p_{\mathbf{r}} - \frac{q}{\varepsilon c}\mathbf{A}\right]^2 + \frac{q}{\varepsilon}\Phi$, where ε is a real infinitesimal dimensionless parameter, assuming that the EM potentials Φ, $\mathbf{A}$ are analytic functions of ε and hence can be represented in the form $\Phi = \sum_{i=-1}^{\infty}\varepsilon^i\Phi_i(\mathbf{r},t)$, $\mathbf{A} = \sum_{i=-1}^{\infty}\varepsilon^i\mathbf{A}_i(\mathbf{r},t)$.Here we shall assume that the characteristic scale length L, satisfying the requirement of the Larmor radius ordering[13], be defined as the minimum of the gradient-scale lengths for the perturbations of the EM potentials $\Phi_i,\mathbf{A}_i$ as well for all relevant vector fields constructed in terms of them, namely in particular $L \le \min\left(\frac{1}{|\mathbf{A}_i|}\left|\frac{\partial|\mathbf{A}_i|}{\partial\mathbf{r}}\right|\right)^{-1}$, $\min\left(\frac{1}{|\Phi_i|}\left|\frac{\partial\Phi_i}{\partial\mathbf{r}}\right|\right)^{-1}$ for $i = -1,+\infty$. Moreover, denoting where $\mathbf{b}(\mathbf{r},t)$= $\mathbf{B}(\mathbf{r},t)/B(\mathbf{r},t)$, the magnitudes of the particle velocity $|\mathbf{v}|$ and of the electric drift velocity $\mathbf{v}_E = c\mathbf{E}\times\mathbf{b}/B$ are assumed of order $O(\varepsilon^0)$, $|\mathbf{v}| \sim |\mathbf{v}'_E|\,O(\varepsilon^0)$ and consequently the parallel electric field is ordered as $\mathbf{b}\cdot\mathbf{E} \sim O(\varepsilon^0)$ (*condition of small parallel electric field*). In validity of these hypotheses the construction of standard gyrokinetic variables is well known and has been achieved by several authors (see for example [7, 1]). In this case, for the presence of slowly varying EM fields, the Lagrangian expressed in terms of gyrokinetic variables (gyrokinetic Lagrangian) reads

$$\mathscr{L}'(\mathbf{y}',\dot{\mathbf{r}}',\dot{\phi}',t) = \dot{\mathbf{r}}'\cdot\frac{q}{\varepsilon c}\mathbf{A}^*(\mathbf{r}',u',\mu',t) - \left(\frac{\dot{\phi}'}{\Omega'}+1\right)\mu'B' - \frac{m}{2}\mathbf{v}'^2 - \frac{q}{\varepsilon}\Phi^*(\mathbf{r}',u',\mu',t), \tag{4}$$

where $\mathbf{y}' \equiv (\mathbf{r}',u',\mu',\phi')$ and the *guiding-center velocity* $\mathbf{v}'$ correct to order $o(\varepsilon)$ is defined by

$$\mathbf{v}' = u'\mathbf{b}' + \mathbf{v}'_{ED}(\mathbf{r}',t) + \varepsilon\mathbf{v}'_D(\mathbf{r}',u',\mu',t) \tag{5}$$

where $\mathbf{v}'_E$ and $\varepsilon\mathbf{v}'_D$ are respectively the electric drift and the diamagnetic drift velocities, both evaluated at the guiding center position (hereon primes denote quantities evaluated at the guiding center position $\mathbf{r}'$), the latter defined by

$$\varepsilon\widehat{\mathbf{v}}'_D = \varepsilon\frac{\mathbf{b}'}{\Omega'}\times\left\{\frac{\mu'}{m}\nabla'B' + (u'\mathbf{b}'+\mathbf{v}'_E)\cdot(u'\nabla'\mathbf{b}'+\nabla\mathbf{v}'_E)\right\}[1+o(\varepsilon)]\,. \tag{6}$$

The remaining notations are standard: in particular, μ' is the magnetic moment evaluated at the guiding center position. Moreover, $\{\mathbf{A}^*,\Phi^*\}$ are the effective EM potentials, which at first order in ε read $\frac{1}{\varepsilon}\mathbf{A}^*(\mathbf{r}',u',\mu',t) = \frac{1}{\varepsilon}\mathbf{A}' + \frac{mc}{q}\mathbf{v}'\,[1+o(\varepsilon)]$, $\frac{1}{\varepsilon}\Phi^*(\mathbf{r}',u',\mu',t) = \frac{1}{\varepsilon}\Phi'\,[1+o(\varepsilon)]$. To construct a set of superabundant variables $\mathbf{x}' = (\mathbf{r}',p_{\mathbf{r}'},\phi',p_{\phi'})$, let us introduce the conjugate momenta

$$p_{\mathbf{r}'} \ \equiv \frac{q}{\varepsilon c}\mathbf{A}^* = \frac{\partial\mathscr{L}'}{\partial\left(\frac{d}{dt}\mathbf{r}'\right)} \equiv m\mathbf{v}' + \frac{q}{\varepsilon c}\mathbf{A}', \tag{7}$$

$$p_{\phi'} = \frac{\partial \mathscr{L}'}{\partial\left(\frac{d}{dt}\phi'\right)} = -\frac{1}{\Omega'}\mu' B' = -\frac{mc}{q}\mu', \tag{8}$$

in terms of which the gyrokinetic Lagrangian becomes $\mathscr{L}'(\mathbf{x}',\dot{\mathbf{r}}',\dot{\phi}',t) = \dot{\mathbf{r}}'\cdot p_{\mathbf{r}'} + \dot{\phi}' p_{\phi'} - K(\mathbf{x}',t)$, where $K(\mathbf{x}',t)$ is the Hamiltonian expressed in terms of state $\mathbf{x}'$:

$$K(\mathbf{x}',t) = -p_{\phi'}\Omega' + \frac{1}{2m}\left[p_{\mathbf{r}'} - \frac{q}{\varepsilon c}\mathbf{A}'\right]^2 + \frac{q}{\varepsilon}\Phi^*. \tag{9}$$

Manifestly, the transformation $\mathbf{x} = (\mathbf{r}, p_{\mathbf{r}}) \rightarrow \mathbf{x}' = \left(\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}\right)$ is non-canonical, even in the generalized sense previously indicated. To this purpose, we need a second step which concerns the introduction of a further transformation to a new set of superabundant gyrokinetic variables $\mathbf{X}' \equiv \left(\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}, \mathbf{v}'\right)$ where $\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}, u'$ are considered as independent variables. Thus we will consider the composed transformation

$$\gamma : \mathbf{x} = (\mathbf{r}, p_{\mathbf{r}}, \mathbf{v}) \rightarrow \mathbf{X}' = \left(\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}, \mathbf{v}'\right), \tag{10}$$

to prove that the gyrokinetic state $\mathbf{X}'$ is canonical in the generalized sense defined above, namely its components satisfy either Hamilton equations with respect to a suitable Hamiltonian function or finite-terms constraint conditions. To reach the proof, we notice first that by construction [see. equation (7)] the guiding-center vector $\mathbf{v}'$ necessarily satisfies the constraint

$$\mathbf{v}' = \frac{1}{m}\left[p_{\mathbf{r}'} - \frac{q}{\varepsilon c}\mathbf{A}'\right]. \tag{11}$$

In fact, in this approximation the gyrokinetic variables $\left(\mathbf{r}', u', p_{\phi'}, \phi'\right)$ satisfy the Euler–Lagrange equations,

$$\begin{aligned} &-\frac{d}{dt}p_{\mathbf{r}'} + m\left(u'\nabla'\mathbf{b}' + \nabla'\mathbf{v}'_E + \frac{q}{\varepsilon mc}\nabla'\mathbf{A}'\right)\cdot\dot{\mathbf{r}}' + \\ &+ p_{\phi'}\nabla'\Omega' - m\left(u'\nabla'\mathbf{b}' + \nabla'\mathbf{v}'_E\right)\cdot\left[u'\mathbf{b}' + \mathbf{v}'_E\right] - \frac{q}{\varepsilon}\nabla'\Phi^* = \mathbf{0}, \end{aligned} \tag{12}$$

from which one can prove that it follows $\dot{\mathbf{r}}' = \mathbf{v}'$. As a consequence, there result the following generalized Hamiltonian equations with respect to $K(\mathbf{x}',t)$ [see equation (9)]

$$\frac{d}{dt}\mathbf{r}' = \frac{\partial}{\partial p_{\mathbf{r}'}}K(\mathbf{x}',t) = \frac{1}{m}\left[p_{\mathbf{r}'} - \frac{q}{\varepsilon c}\mathbf{A}'\right], \tag{13}$$

$$\frac{d}{dt}p_{\mathbf{r}'} = -\frac{\partial}{\partial \mathbf{r}'}K(\mathbf{x}',t). \tag{14}$$

$$\frac{d}{dt}p'_{\phi} = -\frac{\partial}{\partial \phi'}K(\mathbf{x}',t) = 0, \tag{15}$$

$$\frac{d}{dt}\phi' = \frac{\partial}{\partial p'_{\phi}}K(\mathbf{x}',t). \tag{16}$$

We stress that the partial derivatives in equations (14), (13), (15) and (16) are defined with respect to the superabundant variables $\mathbf{X}'$ and hence together with equation (11) they are generalized canonical equations in the sense indicated above. A key aspect of the present approach is that such equations are necessarily variational *with respect to the functional class of the superabundant variables* $\{\mathbf{X}'\}$ [this property should not be confused with the customary variational property of Hamiltonian systems]. In fact, it is easy to show that they follow from the variational principle (3) *in which the superabundant variables* $\mathbf{X}' = \left(\mathbf{r}',\ p_{\mathbf{r}'},\ \phi',\ p_{\phi'},\ \mathbf{v}'\right)$ *are considered independent and the extremal value of* $\mathbf{v}'$ *coincides with equation (5)*. In particular, one can prove that, to leading order in the asymptotic parameter ε, this is provided in terms of the constrained Hamilton principle determined by the fundamental $1-$form $\Gamma(\mathbf{X}',t)$

$$\begin{aligned} \Gamma(\mathbf{X}',t) \quad = \quad & \mathscr{L}'(\mathbf{X}',\dot{\mathbf{r}}',\dot{\phi}',t)dt = d\mathbf{r}'\cdot p_{\mathbf{r}'} + d\phi' p_{\phi'} - dtK(\mathbf{x}',t) - \\ & - \left[d\mathbf{r}' - dt\mathbf{v}'\right]\cdot\left[p_{\mathbf{r}'} - m\mathbf{v}' - \frac{q}{c\varepsilon}\mathbf{A}'\right], \end{aligned} \tag{17}$$

where $\mathscr{L}'(\mathbf{X}',\dot{\mathbf{r}}',\dot{\phi}',t)$ and $K(\mathbf{x}',t)$ are respectively the gyrokinetic Lagrangian and the Hamiltonian defined by (9). It is immediate to prove that the Euler–Lagrange equations corresponding to (17) coincide with the previous generalized

Hamiltonian equations of motion defined by (11), (14), (13), (15), 16) and hence provide a variational formulation for them. It is important to realize that, in the subset of phase-space in which resonant effects are negligible [12], gyrokinetic theory can be formulated in a systematic way in terms of superabundant canonical variables by perturbative methods, i.e., evaluating, in principle at any order in the perturbative parameters, all relevant gyrokinetic variables, including the guiding-center velocity $\mathbf{v}'$. The additional freedom, involved in this approach, and provided by the superabundant variables can be used to satisfy, in principle, appropriate dynamical constrains and further simplify the form of the gyrokinetic equations of motion. In particular, the gyrokinetic Lagrangian $\mathscr{L}'(\mathbf{X}',\dot{\mathbf{r}}',\dot{\phi}',t)$ can be constructed, accurate to arbitrary order in the relevant expansion parameters, directly from the Lagrangian of a charged point particle, written in the superabundant variables $\mathbf{x}_G = (\mathbf{r}, p_{\mathbf{r}}, \mathbf{v})$ [see equation (3)]. This can be achieved by means of an appropriate gyrokinetic transformation in which all variables $\mathbf{x}_G$ (including in principle also the variable t) are transformed simultaneously and therefore can be assumed of the form

$$\left\{ \begin{array}{c} \mathbf{r} = \mathbf{r}' + \varepsilon \mathbf{r}_1 + ... \\ p_{\mathbf{r}} = p'_{\mathbf{r}'} + \varepsilon p_{\mathbf{r}_1} + ... \\ \mathbf{v} = \mathbf{v}' + \varepsilon \mathbf{v}_1 + ... \end{array} \right. \tag{18}$$

where the perturbations $\varepsilon\mathbf{r}_1, \varepsilon p_{\mathbf{r}_1}, \varepsilon\mathbf{v}_1$ are defined in such a way to generate a set of superabundant gyrokinetic canonical variables, to be denoted for simplicity with the same symbol $\mathbf{X}' = \left(\mathbf{r}',\, p_{\mathbf{r}'},\, \phi',\, p_{\phi'},\, \mathbf{v}'\right)$. In addition, $(\mathbf{r}',\, p_{\mathbf{r}'},\, \mathbf{v}')$ identify the guiding center variables and, in particular, $\mathbf{v}'$ is suitably defined so that the extremal values $(\mathbf{r}',\, p_{\mathbf{r}'},\, \mathbf{v}')$, determined by the variational principle, satisfy identically the relevant constraint equation (11). This can be obtaines, for example, by means of standard Lie-transform methods. Thus, it is possible to prove that by proper definition of the guiding-center velocity $\mathbf{v}'$, the remaining variables $\left(\mathbf{r}',\, p_{\mathbf{r}'},\, \phi',\, p_{\phi'}\right)$ satisfy the Hamilton equations with respect to a suitable gyrophsae-independent Hamiltonian function $K(\mathbf{X}',t)$ (gyrokientic Hamiltonian), defined by equation (9). Therefore, the superabundant state $\mathbf{X}'$ results by construction generalized canonical and at the same time gyrokinetic. We stress, that the precise expressions for $\mathbf{v}'$ and the gyrokinetic Hamiltonian $K(\mathbf{X}',t)$ depend on the accuracy of the perturbative calculations. In particular, it is obvious that the perturbative theory can be developed, in principle, at arbitrary orders both in ε and λ for all gyrokinetic variables, by applying the present Lie-transform approach. However, as well known [12], an exact gyrokinetic Hamiltonian (and Lagrangian) cannot be achieved, since the perturbative series (18) generally does not uniformly converge in the phase space, due to resonant effects [12]. It is important to realize that the guiding-center velocity $\mathbf{v}'$ can actually be parametrized in terms of the remaining variables, i.e., letting $\mathbf{v}' = \mathbf{v}'\left(\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}\right)$, where $\mathbf{x}' = \left(\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}\right)$ with $\mathbf{v}'\left(\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}\right)$ to be suitably prescribed. As a consequence, it follows that the superabundant canonical variables $\mathbf{x}' = \left(\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}\right)$ *are obviously volume-preserving*, i.e., fulfill the exact Liouville theorem$\left|\frac{\mathbf{x}'(t)}{\mathbf{x}'(t_o)}\right| = 1$ (*exact Liouville theorem*). On the other hand, since $p_{\phi'}$ is by construction an adiabatic invariant [for example we may assume that it results, at least of second order in ε, i.e. such that $\frac{d}{dt}$ ln $p_{\phi'} \sim O(\varepsilon^2)$] while the gyrophase ϕ' is ignorable in the same approximation, it follows that also the following approximate Liouville theorem holds $\frac{\partial\left(\mathbf{z}'(t)\right)}{\partial\left(\mathbf{z}'(t_o)\right)} = 1 + O(\varepsilon^2)$ (*approximate Liouville theorem*).Therefore also the state $\mathbf{z}' = (\mathbf{r}',\, p_{\mathbf{r}'})$ results area-preserving at least in an asymptotic sense.

FINAL REMARKS

All previous treatments of Hamiltonian gyrokinetic theory appeared in the literature (see for example [1, 2, 4, 5, 7, 8, 14, 15], concern the construction of canonical gyrokinetic variables, but in all such cases these variables (in particular the coordinates identifying the guiding center position vector) are field-related, i.e., they depend on the magnetic field geometry. Thus the explicit determination of these variables, particularly for non-symmetric and/or chaotic magnetic fields, results, however, difficult or impossible [3], since flux coordinates are difficult or impossible to construct in such cases. The crucial feature of the present approach is that gyrokinetic theory can be cast in canonical form without recurring to field-related coordinates, by making use of superabundant canonical variables obeying the constrained dynamics defined by a variational principle of the type (3). Thus, the canonical coordinates $\mathbf{r}'$ results manifestly independent of any particular magnetic field geometry, which implies that the vector $\mathbf{r}'$ can be represented in the form $\mathbf{r}' = \mathbf{r}'(\mathbf{q}',t)$, being $\mathbf{q}' = (q'_1, q'_2, q'_3)$ arbitrary, field-independent, curvilinear coordinates, such as for instance orthogonal Cartesian coordinates As a result, no restriction is placed on the magnetic field geometry for the definition of these canonical variables. A fundamental consequence is that, by construction, the magnetic moment, to be identified with $-P_{\phi'}\Omega'/B'$, can always be defined in such a way to be a single-valued function with respect with to any angle-like coordinates $\mathbf{q}' = (q_1', q_2', q_3')$ and therefore is a physical observable. For these reasons such properly defined magnetic moment does not require the existence of a single family of nested magnetic surfaces (quasi-symmetric magnetic field) [9]. Indeed, these variables can only be found using a superabundant-variable setting, as first proven in our paper. As an example, the development of nonlinear gyrokinetics for electrostatic perturbation has been given in the superabundant canonical variables $\mathbf{X}'$. The test case considered deals with small amplitude EM perturbations with characteristic frequencies and scale length much smaller than those of the background fields. On the other hand Lagrangian mechanics in superabundant variables is well-known. Therefore its adoption in the context of gyrokinetic theory does not exhibit conceptual limitations. Another significant aspect of this approach to be enlightened is that the equations of motion for the dynamical variables describing the guiding center motion, i.e., $(\mathbf{r}', p_{\mathbf{r}'}, \mathbf{v}')$, as given by the Hamilton equations (13), (14) and the constraint (11), are all expressed in vector form.In particular, we have proven that various superabundant gyrokinetic variables, i.e., $\mathbf{x}' = (\mathbf{r}',\ p_{\mathbf{r}'},\ \phi',\ p_{\phi'})$ and $\mathbf{z}' = (\mathbf{r}',\ p_{\mathbf{r}'})$, result volume-preserving, either exactly or in an approximate sense. Finally, it must be stressed that the superabundant variables here discussed (in particular $\mathbf{z}'$) result potentially convenient for numerical simulations of particle dynamics, particularly in chaotic magnetic fields. In fact, even if the dimension of the phase phase-space spanned by $\mathbf{z}'$ result identical with that of free particles, the corresponding gyrokinetic equations of motion result much simples because they do not contain any fast dependence in terms of the gyrophase angle. A specific example is provided by gyrokinetic particle simulation of neoclassical transport for non-symmetric MHD equilibria in which particle orbits are numerically precalculated. In such a case the property of phase space volume preserving is crucial for the validity of the simulation.

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