

Boltzmann equation and hydrodynamics beyond the Navier-Stokes level

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Abstract. We begin with a general discussion on relationships between the Boltzmann equation and hydrodynamics beyond the Navier-Stokes level. In particular, we discuss briefly some known results related to Burnett hydrodynamics of stationary slow gas flows (small Mach numbers) and the problem of truncation of asymptotic expansions in connection with the fact that Burnett equations are ill-posed. It is shown that the way of truncation is not unique. Then we perform a transformation of BE that clearly shows "what is missing in the Navier-Stokes equations". Burnett terms appear as a result of this transformation in a quite obvious way. Then we pass to the main problem: what precisely is wrong with classical Burnett equations (arbitrary Mach numbers)? First we deduce the criterion for hyperbolicity of these equations and show that it is not fulfilled for most typical intermolecular models (the so-called instability paradox). It is shown that well posed Burnett-type equations can be derived from the Boltzmann equation in quite systematic way, however the final set of equations is not unique. Recently published by the author Hyperbolic Burnett Equations is just one of possible answers. It is shown that the problem of the optimal choice of stable generalized Burnett-type equations can be reduced to a two-parameter family of such equations. Surprisingly, some of them are even simpler than the classical Burnett equations, whereas they have the same formal order of accuracy. These equations and choice of parameters are briefly discussed.

Keywords: Boltzmann equation, Chapman-Enskog method, Burnett equations, hyperbolicity, perturbation theory, hydrodynamics

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INTRODUCTION

The topic "Boltzmann equation and hydrodynamics" has a long and interesting history. It begins with famous works by Maxwell (1860, 1867), Boltzmann (1872, 1875), Hilbert (1910, 1912), Enskog (1912, 1917), Chapman (1916, 1917) and Burnett (1935). The book [1] of Chapman and Cowling (its first edition was published in 1939) can be considered to some extent as the end of the first period of history of this field. The main result of this period was the Chapman-Enskog method that seemed to be a systematic way of derivation of equations of hydrodynamics having formally any given order of accuracy with respect to the Knudsen number ε (mean free path divided by macroscopic length): Euler equations, Navier-Stokes equations, Burnett equations, etc.

Harold Grad was probably the first who began in 1940-50s to analyze mathematical problems of kinetic theory related to hydrodynamics and the Chapman-Enskog method. His famous review article [2] became a starting point for a lot of mathematical papers on this subject by different authors (see [3] for review of recent results). On the other hand, all known "nonlinear" mathematical results in this field are, roughly speaking, restricted to Euler and Navier-Stokes levels. What can be said about use of "higher" equations of hydrodynamics, in particular, Burnett equations? This is, of course, not the first priority question for rarefied gas dynamics. In most of cases it is sufficient to use the Navier-Stokes equations for small Knudsen numbers and the full Boltzmann equation (solved numerically by DSMC methods) otherwise. However, the question does exist and remains unclear for a long time. The present lecture can be considered as an attempt to clarify some details related to this question.

First of all we should mention a class of problems where (modified) Burnett equations are absolutely necessary even in the continuous limit $\varepsilon = 0$. This class, the stationary non-isothermal flows with low Mach numbers $Ma = O(\varepsilon)$, was introduced in 1960-70s in the series of papers (see [4] and references therein) by Kogan, Galkin and Friedlander, where the corresponding modified Burnett equations were derived and studied. The equations were derived by both Hilbert and Chapman-Enskog methods, which are, roughly speaking, equivalent in this case [4]. The authors found and discussed in detail many interesting phenomena related to new types of gas convection of order $O(\varepsilon)$.

This work had perhaps an unexpected continuation in relatively recent years. It was proved in 1995 [5] that the modified Burnett equations used in this theory contradict to the conventional heat equation in the continuous limit $\varepsilon = 0$. This result can be briefly described in the following way. Consider the stationary Boltzmann equation in its standard dimensionless form in a non-symmetric domain $\mathcal{D} \subset \mathbb{R}^3$ with diffusive (complete accommodation) boundary

conditions on non-isothermal boundaries of \mathcal{D} . Assume that the gravity is absent and pass to the limit $\varepsilon = 0$. It is clear that the limiting solution is a Maxwellian with zero bulk velocity and with some temperature $T(x)$ and density $\rho(x)$ such that $\rho T = \text{const.}$, $x \in \mathcal{D}$. One can expect that $T(x)$ satisfies the nonlinear heat equation which follows from Navier-Stokes equations in this case. The statement proved in [5] means that this cannot be true provided the Kogan-Galkin-Friedlander theory is correct. Then, in order to find $T(x)$, we need to solve the full system of their equations instead of the relatively simple heat equation.

What is wrong, the formal asymptotic theory itself or the stationary heat equation for rarefied gases? There was no hope at that time to give a mathematically rigorous answer (still it is an open problem). Therefore a numerical experiment would be the best way to answer this question. The Kyoto group of Sone and Aoki performed a thorough investigation of this problem for BGK equation by using both numerical and analytic methods. The final answer was “yes”, the theory is right and the heat equation is wrong. This was published in 1996 in the joint paper [6] by the Kyoto group and myself. My contribution to this paper was very modest, just the initial proposal to consider this question numerically and comments on some related arguments from the previous paper [5]. Sone found a nice name “Ghost effect” for this phenomenon. It reflects the fact that all convections of order $O(\varepsilon)$ disappear in the continuous limit, however the limiting distribution of temperature “remembers” that they existed.

The paper [6] became a starting point for many interesting publications of the Kyoto group (mainly Sone, Aoki and Takata) on similar “ghost effect” in quite different problems of rarefied gas dynamics (see the book [7] for review). It is important, however, to stress that this is a long line of research that began with pioneering works by Kogan, Galkin and Friedlander in 1960-70s.

This particular class of problems is an example that shows that (a) Burnett terms are sometimes important even in the continuous limit $\varepsilon = 0$ and (b) the well-known problem of boundary conditions [8, 9] for Burnett equations is solvable at least for specific flows of rarefied gas. It seems to me that the difficulty with boundary conditions for higher equations of hydrodynamics is, to some extent, overestimated in literature. The corresponding conditions of any order can be in principle derived from asymptotic analysis of solutions to the Boltzmann equation in the Knudsen layer by using more or less standard methods [8, 9]. Another difficulty (the instability paradox, see below) seems to be more important and therefore we shall discuss it in this lecture in detail.

Let us consider the general class of non-stationary flows with arbitrary Mach numbers. Then we formally need to use the full classical Burnett equations (see, for example, [10, 11]). The main difficulty is that these equations are ill-posed. This was firstly proved in 1982 for Maxwell molecules [12]. Maxwell molecules are not an exception, as we shall see below.

In more physical terms, one can say that any solution of Burnett equations (in particular, the constant equilibrium solution) is unstable with respect to short-wave perturbations (the instability paradox, in terminology of Jin-Slemrod [13]). Roughly speaking, the Burnett equations are something like the usual diffusion equation with negative diffusion coefficient. Attempts to overcome this difficulty were made in last two decades by several authors (see, in particular, papers [14, 15] on the linearized problem and papers [13, 16–18] on the nonlinear problem). A comprehensive review of different approaches and a complete list of references can be found in [16, 18]. We also mention recent talks by Levermore [19]. All the above approaches are based on a combination of the Chapman-Enskog method with moment methods and on using some higher order in ε terms from the Chapman-Enskog expansion for regularization of Burnett equations.

An alternative method considered below does not use any information “beyond the Burnett level” and is based on the following simple idea. Let us consider a general evolution equation for a vector $x(t)$

$$x_t = T(x; \varepsilon) = A(x) + \varepsilon B(x) + \varepsilon^2 C(x) + \dots,$$

where A, B, C, \dots are time-independent differentiable nonlinear operators. In our case $x(t)$ is understood as the vector of hydrodynamical variables, whereas $A(x)$, $B(x)$ and $C(x)$ represent Euler, Navier-Stokes and Burnett terms respectively. A correct remark by Slemrod is that the problem is related not to the Chapman-Enskog expansion itself, but to its truncation at the Burnett level. His approach [13, 16, 20] is, however, quite different from ours.

Let us consider more carefully the problem of truncation. An obvious observation is that the truncation depends on a choice of coordinates. A formally invertible (for $\varepsilon \rightarrow 0$) change of variables

$$y = x + \varepsilon^2 R(x) \Rightarrow x = y - \varepsilon^2 R(y) + \dots,$$

with an arbitrary time-independent differentiable operator R , leads to the equation

$$y_t = \tilde{T}(y; \varepsilon) = A(y) + \varepsilon B(y) + \varepsilon^2 \tilde{C}(y) + \dots, \quad \tilde{C}(y) = C(y) + R'_y A(y) - A'_y R(y),$$

where R'_y and A'_y are the Fréchet derivatives (linear operators) of $R(y)$ and $A(y)$ respectively. Thus, the result of truncation at the level $O(\varepsilon^2)$ depends, generally speaking, on an arbitrary operator R (or, equivalently, on a choice of coordinates). The same, of course, can be done at any order $O(\varepsilon^n)$, $n \geq 1$. Note that even the classical Navier-Stokes equations for $n = 1$ are not uniquely defined in this sense. A remaining non-trivial problem is to choose new coordinates (the operator R) in the most reasonable way. Similar ideas are widely used in the theory of dynamical systems, see, for example, the Poincaré method of normal forms [21].

The paper is organized as follows. First we transform the Boltzmann equation in such a way that makes the Navier-Stokes equations almost obvious and clearly indicates "what is missing in Navier-Stokes equations" as compared to the Boltzmann equation. Then the Burnett equations arise quite naturally as the first correction to Navier-Stokes equations. The second main point is to explain "what is wrong with the Burnett equations". We consider these equations under very general assumptions on intermolecular forces and derive a criterion for hyperbolicity of Burnett equations. The criterion roughly means that they are hyperbolic in a very narrow interval $1 \leq \text{Pr} \leq \frac{5}{4}$ of Prandtl numbers (in contrast to the realistic value $\text{Pr} \approx \frac{2}{3}$). Then we briefly discuss how to regularize the original equations by using above discussed "changes of coordinates". The first version of equations regularized by this method (hyperbolic Burnett equations) can be found in the recent paper [22], which also contains detailed discussion of the method and its applications to linear and nonlinear cases. Here we present (at the end of the paper) another version of stable Burnett-type equations, a two-parameter family of generalized Burnett equations (GBEs). These equations are the most natural replacement for the original (unstable) Burnett equations. Surprisingly, they can be even simpler than the original equations. Non-uniqueness of GBEs seems to be inevitable. The problem of optimal choice of parameters (they can be different for different problems) is briefly discussed in the final part of the paper.

In order to avoid a misunderstanding we note that the above discussed difficulties with the Chapman-Enskog method are typical for the Euler limit. An alternative scaling that leads to the incompressible Navier-Stokes equations [23, 24], seems to be in complete agreement with corresponding Chapman-Enskog expansion [25].

TRANSFORMATION OF THE BOLTZMANN EQUATION

We consider the Boltzmann equation [8]

$$\mathcal{D}f = \frac{1}{\varepsilon} Q(f, f), \quad \mathcal{D} = \partial_t + v \cdot \partial_x, \quad (1)$$

for the distribution function $f(x, v, t; \varepsilon)$, where the variables $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ and $t > 0$ correspond respectively to position, velocity and time; the parameter $\varepsilon > 0$ denotes the Knudsen number. The Boltzmann collision operator reads

$$Q(f, f) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} dw d\omega |u| \sigma(|u|, \frac{u \cdot \omega}{|u|}) \left[f(v') f(w') - f(v) f(w) \right],$$

$$u = v - w, \omega \in \mathbb{S}^2, v' = \frac{1}{2}(v + w + |u| \omega), w' = \frac{1}{2}(v + w - |u| \omega), \quad (2)$$

where $\sigma(|u|, \cos \theta)$ is the differential cross-section that corresponds to the scattering angle $\theta \in [0, \pi]$ (irrelevant arguments x, t and ε of the function f are omitted in Eq.(2)).

We denote for brevity

$$\langle f, g \rangle = \langle fg \rangle = \int_{\mathbb{R}^3} dv f(v) g(v) \quad (3)$$

and introduce the so-called hydrodynamic variables (the density ρ , the bulk velocity $u \in \mathbb{R}^3$ and the temperature T)

$$\rho = \langle f \rangle, \rho u = \langle f, v \rangle, \rho T = \frac{1}{3} \langle f, |c|^2 \rangle, c = v - u. \quad (4)$$

We shall use below just a few basic properties of the collision integral:

$$(A) \quad \langle \Psi, Q(f, f) \rangle = 0, \text{ for any } f(v) \text{ if and only if } \Psi = \Psi(v) \in \text{Span}(1, v, |v|^2);$$

(B) $Q(f, f) = 0$, $f \geq 0$, if and only if $f = \exp(\alpha + \beta \cdot v - \gamma |v|^2)$, where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^3$, $\gamma \in \mathbb{R}_+$ are any constant parameters.

All considerations in this paper are quite formal and therefore we assume in advance that the cross-section $\sigma(|u|, \cos \theta)$ in Eq.(2) and the solution f of Eq.(1) satisfy all necessary restrictions such that all integrals are convergent, etc. In order to describe an asymptotic behavior of $f(x, v, t; \varepsilon)$ for small positive ε we introduce the Maxwellian distribution

$$M = (2\pi T)^{-3/2} \exp\left(-\frac{|c|^2}{2T}\right), \quad c = v - u, \quad (5)$$

and represent the solution of Eq.(1) as a sum

$$f = \rho M + \varepsilon F, \quad (6)$$

where $\rho(x, t; \varepsilon)$, $u(x, t; \varepsilon)$, $T(x, t; \varepsilon)$, correspond to the "true" hydrodynamic moments of f . Hence,

$$\langle \Psi, F \rangle = 0, \quad \langle \Psi, \mathcal{D}\rho M \rangle + \varepsilon \langle \Psi, \mathcal{D}F \rangle = 0, \quad (7)$$

for any $\Psi \in \text{Span}(1, v, |v|^2)$. This leads to the usual set of hydrodynamic conservation laws

$$\begin{aligned} \rho_t + \text{div} \rho u &= 0, \quad \frac{\partial}{\partial t} \rho u_\alpha + \frac{\partial}{\partial x_\beta} (\rho u_\alpha u_\beta + p \delta_{\alpha\beta} + \varepsilon \pi_{\alpha\beta}) = 0, \\ \frac{\partial}{\partial t} \rho (|u|^2 + 3T) + \text{div} \rho u (|u|^2 + 5T) + 2\varepsilon \frac{\partial}{\partial x_\alpha} (\pi_{\alpha\beta} u_\beta + q_\alpha) &= 0, \end{aligned} \quad (8)$$

where

$$p = \rho T, \quad \pi_{\alpha\beta} = \langle F, c_\alpha c_\beta \rangle, \quad q_\alpha = \frac{1}{2} \langle c_\alpha |c|^2, F \rangle, \quad \alpha, \beta = 1, 2, 3. \quad (9)$$

The usual rule of summation over repeating indexes is assumed here and below.

The equation for $F(x, v, t; \varepsilon)$ reads

$$\mathcal{D}\rho M + \varepsilon \mathcal{D}F = \rho ML(F/M) + \varepsilon Q(F, F), \quad (10)$$

where the linearized collision operator L is defined by the equality

$$Q(M, Mg) + Q(Mg, M) = MLg. \quad (11)$$

We denote by \mathbf{H} the Hilbert space with the scalar product

$$\langle g_1, g_2 \rangle_M = \langle Mg_1, g_2 \rangle \quad (12)$$

and consider L as an operator acting from \mathbf{H} to \mathbf{H} . We shall use the following basic properties of L (in addition to properties (A) and (B) of the nonlinear operator Q):

(C) $Lg = 0$ if and only if $g \in \mathbf{N}(L) = \text{Span}(1, v, |v|^2)$;

(D) L is self-adjoint and semi-negative operator, i.e.

$$\langle g_1, Lg_2 \rangle_M = \langle Lg_1, g_2 \rangle_M, \quad \langle g, Lg \rangle_M \leq 0; \quad (13)$$

(E) $\mathbf{H} = \mathbf{N}(L) \oplus \mathbf{R}(L)$, where $\mathbf{R}(L) = L\mathbf{H}$ is a range of the operator L in \mathbf{H} , \oplus denotes the orthogonal sum (with respect to the scalar product (12)).

Then the problem

$$Lg = \phi, \quad g \in \mathbf{R}(L), \quad \phi \in \mathbf{R}(L) \quad (14)$$

has a unique solution $g = L^{-1}\phi$. We extend the operator $L^{-1} : \mathbf{R}(L) \rightarrow \mathbf{R}(L)$ to the whole \mathbf{H} by introducing the operator $\tilde{L}^{-1} : \mathbf{H} \rightarrow \mathbf{R}(L)$ such that

$$\tilde{L}^{-1}\phi = \begin{cases} L^{-1}\phi, & \text{if } \phi \in \mathbf{R}(L) \\ 0, & \text{otherwise.} \end{cases} \quad (15)$$

Then Eq.(10) can be transformed to

$$F = M[\tilde{L}^{-1}(\mathcal{D} \ln \rho M) + \frac{\varepsilon}{\rho} \tilde{L}^{-1} \frac{\mathcal{D}F - Q(F, F)}{M}]. \quad (16)$$

We denote

$$\tilde{Q}(F) = Q(F, F), F_0 = M\tilde{L}^{-1}(\mathcal{D}\ln\rho M) = M\tilde{L}^{-1}(c_\alpha \cdot \frac{\partial}{\partial x_\alpha} \frac{|c|^2}{2T}), c = v - u, \quad (17)$$

the same notation $c = v - u$ for the thermal velocity is often used below. Then, omitting tildes, we obtain

$$F = F_0 + \frac{\varepsilon M}{\rho} L^{-1} \frac{\mathcal{D}F - Q(F)}{M}, \quad (18)$$

where

$$\begin{aligned} F_0 &= \left[\frac{1}{T} \frac{\partial u_\alpha}{\partial x_\beta} \phi_{\alpha\beta}(c) + \frac{1}{T^2} \frac{\partial T}{\partial x_\alpha} \phi_\alpha(c) \right] M, \\ \phi_{\alpha\beta} &= L^{-1} (c_\alpha c_\beta - \frac{|c|^2}{3} \delta_{\alpha\beta}), \phi_\alpha = L^{-1} \frac{c_\alpha}{2} (|c|^2 - 5T). \end{aligned} \quad (19)$$

The general equation of hydrodynamics (the second equation (7) with any $\Psi \in \text{Span}(1, v, |v|^2)$ independent of x and t) reads now

$$\langle \Psi, \mathcal{D}\rho M \rangle + \varepsilon \frac{\partial}{\partial x_\alpha} \langle c_\alpha \Psi, F_0 \rangle + \varepsilon^2 \frac{\partial}{\partial x_\alpha} \frac{1}{\rho} \left\langle M c_\alpha \Psi, L^{-1} \frac{\mathcal{D}F - Q(F)}{M} \right\rangle = 0. \quad (20)$$

The operator L^{-1} (15) is obviously self-adjoint in \mathbf{H} and therefore we obtain

$$\begin{aligned} \langle \Psi, \mathcal{D}\rho M \rangle + \varepsilon \frac{\partial}{\partial x_\alpha} \left\langle \Phi_\alpha(\Psi), c \cdot \frac{\partial M}{\partial x} \right\rangle + \varepsilon^2 \frac{\partial}{\partial x_\alpha} \frac{1}{\rho} \langle \Phi_\alpha(\Psi), \mathcal{D}F - Q(F) \rangle &= 0, \\ \Phi_\alpha(\Psi) &= L^{-1}(c_\alpha \Psi); \Psi = 1, v, |v|^2, \end{aligned} \quad (21)$$

where F satisfies Eq.(18). The equations (21) are obviously exact. If we neglect the third term having the order $O(\varepsilon^2)$, then Eqs. (21) reduce to the classical Navier-Stokes equations. Thus, the Navier-Stokes equations arise quite naturally (also for the stationary Boltzmann equation with $\mathcal{D} = v \cdot \partial_x$). The difficulties begin with the attempt to solve Eq.(18) for small positive ε and then to use the equations (21) with terms of order $O(\varepsilon^2)$ (Burnett equations). We consider this problem in the next Section .

BURNETT EQUATIONS

The equation (18) can be written as

$$F = F_0 + \varepsilon F_1 + O(\varepsilon^2), F_1 = \frac{1}{\rho} M L^{-1} \frac{\mathcal{D}F_0 - Q(F_0)}{M}, \quad (22)$$

where F_0 is given in Eq. (19). The Burnett equations are the equations (8) with

$$\begin{aligned} \pi_{\alpha\beta} &= \pi_{\alpha\beta}^{NS} + \varepsilon \pi_{\alpha\beta}^B, q_\alpha = q_\alpha^{NS} + \varepsilon q_\alpha^B, \\ \pi_{\alpha\beta}^{NS} &= \langle F_0, c_\alpha c_\beta \rangle, q_\alpha^{NS} = \frac{1}{2} \langle F_0, c_\alpha |c|^2 \rangle, \\ \pi_{\alpha\beta}^B &= \langle F_1, c_\alpha c_\beta \rangle, q_\alpha^B = \frac{1}{2} \langle F_1, c_\alpha |c|^2 \rangle, \end{aligned} \quad (23)$$

plus the rule of calculation of $\mathcal{D}F_0$ in Eq.(22) (see below). We can represent the Burnett terms in the equivalent form:

$$\pi_{\alpha\beta}^B = \frac{1}{\rho} \langle \phi_{\alpha\beta}, \mathcal{D}F_0 - Q(F_0) \rangle, q_\alpha^B = \frac{1}{\rho} \langle \phi_\alpha, \mathcal{D}F_0 - Q(F_0) \rangle \quad (24)$$

in the notations (19). We note that

$$\begin{aligned} \mathcal{D}_0 \rho &= -\rho \text{div} u, \mathcal{D}_0 u = -\frac{1}{\rho} \nabla p + O(\varepsilon), \\ \mathcal{D}_0 T &= -\frac{2}{3} T \text{div} u + O(\varepsilon), \mathcal{D}_0 = \partial_t + u \cdot \partial_x, \end{aligned} \quad (25)$$

and this explains the rule of calculation of the derivative

$$\mathcal{D}F_0 = \mathcal{D}_0F_0 + c \cdot \frac{\partial F_0}{\partial x}, \quad \mathcal{D}_0 = \partial_t + u \cdot \partial_x, \quad (26)$$

in Eqs.(22), (24): we should use the Euler formulas (25) and neglect all terms $O(\varepsilon)$. This completes the definition of the Burnett equations.

One can easily verify that Eqs.(24) can be transformed to the following form:

$$\begin{aligned} \pi_{\alpha\beta}^B &= \mathcal{D}_0P_{\alpha\beta} + \frac{1}{\rho} \left[\frac{\partial}{\partial x_\gamma} \langle c_\gamma \phi_{\alpha\beta}, F_0 \rangle - \Delta_{\alpha\beta} \right], \\ q_\alpha^B &= \mathcal{D}_0Q_\alpha + \frac{1}{\rho} \left[\frac{\partial}{\partial x_\beta} \langle c_\beta \phi_\alpha, F_0 \rangle - \Delta_\alpha \right], \end{aligned} \quad (27)$$

where

$$\begin{aligned} P_{\alpha\beta} &= \frac{1}{\rho} \langle F_0, \phi_{\alpha\beta} \rangle, \quad \Delta_\alpha = \langle F_0, \mathcal{D}\phi_\alpha \rangle + \langle \phi_\alpha, Q(F_0) \rangle, \\ Q_\alpha &= \frac{1}{\rho} \langle F_0, \phi_\alpha \rangle, \quad \Delta_{\alpha\beta} = \langle F_0, \mathcal{D}\phi_{\alpha\beta} \rangle + \langle \phi_{\alpha\beta}, Q(F_0) \rangle. \end{aligned} \quad (28)$$

Eqs.(27) are more convenient for our goals, since the reminder terms $\Delta_{\alpha\beta}$ and Δ_α are bilinear forms with respect to first derivatives of ρ , T and u . Therefore they disappear in the linearized (near the constant solution) equation and do not influence higher derivatives in the nonlinear equations. Therefore we concentrate on the "main" terms in Eqs.(27).

The straightforward calculation of the integrals with F_0 given in Eq.(19) yields

$$\begin{aligned} P_{\alpha\beta} &= \frac{A}{\rho} \overline{\frac{\partial u_\alpha}{\partial x_\beta}}, \quad Q_\alpha = \frac{B}{\rho} \frac{\partial T}{\partial x_\alpha}, \quad \frac{\partial}{\partial x_\gamma} \langle c_\gamma \phi_{\alpha\beta}, F_0 \rangle = \overline{\frac{\partial}{\partial x_\alpha} C \frac{\partial T}{\partial x_\beta}}, \\ \frac{\partial}{\partial x_\beta} \langle c_\beta \phi_\alpha, F_0 \rangle &= \frac{\partial}{\partial x_\beta} C T \overline{\frac{\partial u_\alpha}{\partial x_\beta}}, \quad \overline{a_{\alpha\beta}} = \frac{1}{2} (a_{\alpha\beta} + a_{\beta\alpha} - \frac{2}{3} \delta_{\alpha\beta} Tr a), \end{aligned} \quad (29)$$

where

$$\begin{aligned} A &= A(T) = \frac{1}{5T} \langle \phi_{\alpha\beta}, \phi_{\alpha\beta} \rangle_M, \quad B = B(T) = \frac{1}{3T^2} \langle \phi_\alpha, \phi_\alpha \rangle_M, \\ C &= C(T) = \frac{1}{5T^2} \langle c_\alpha \phi_\beta, \phi_{\alpha\beta} \rangle_M. \end{aligned} \quad (30)$$

Then we use the identity

$$\mathcal{D}_0 \frac{\partial}{\partial x_\beta} = \frac{\partial}{\partial x_\beta} \mathcal{D}_0 - \frac{\partial u_\gamma}{\partial x_\beta} \frac{\partial}{\partial x_\gamma} \quad (31)$$

and the above described rule for calculating $\mathcal{D}_0\rho$, \mathcal{D}_0u , \mathcal{D}_0T by Eqs.(25). Thus we obtain

$$\begin{aligned} \mathcal{D}_0P_{\alpha\beta} &= -\frac{A}{\rho} \left[\overline{\frac{\partial}{\partial x_\alpha} \frac{1}{\rho} \frac{\partial p}{\partial x_\beta}} + \overline{\frac{\partial u_\alpha}{\partial x_\gamma} \frac{\partial u_\gamma}{\partial x_\beta}} - \alpha(T) (div u) \overline{\frac{\partial u_\alpha}{\partial x_\beta}} \right], \\ \mathcal{D}_0Q_\alpha &= -\frac{B}{\rho} \left[\frac{2}{3} \frac{\partial}{\partial x_\alpha} T div u + \frac{\partial u_\beta}{\partial x_\alpha} \frac{\partial T}{\partial x_\beta} - \beta(T) (div u) \frac{\partial T}{\partial x_\alpha} \right], \\ \alpha(T) &= 1 - \frac{2}{3} \frac{TA'(T)}{A(T)}, \quad \beta(T) = 1 - \frac{2}{3} \frac{TB'(T)}{B(T)}. \end{aligned} \quad (32)$$

Hence,

$$\pi_{\alpha\beta}^B = -\frac{1}{\rho} \left[\overline{\Pi_{\alpha\beta}^{(1)}} + \overline{\Pi_{\alpha\beta}^{(2)}} + \Delta_{\alpha\beta} \right], \quad q_\alpha^B = -\frac{1}{\rho} \left[S_\alpha^{(1)} + S_\alpha^{(2)} + \Delta_\alpha \right], \quad (33)$$

where

$$\begin{aligned}
\Pi_{\alpha\beta}^{(1)} &= A \frac{\partial}{\partial x_\alpha} \frac{1}{\rho} \frac{\partial p}{\partial x_\beta} - \frac{\partial}{\partial x_\alpha} C \frac{\partial T}{\partial x_\beta}, \quad p = \rho T, \\
\Pi_{\alpha\beta}^{(2)} &= A \left[\frac{\partial u_\alpha}{\partial x_\gamma} \frac{\partial u_\gamma}{\partial x_\beta} - \alpha(T) \frac{\partial u_\alpha}{\partial x_\beta} \operatorname{div} u \right], \\
S_\alpha^{(1)} &= \frac{2B}{3} \frac{\partial}{\partial x_\alpha} T \operatorname{div} u - \frac{\partial}{\partial x_\beta} C T \frac{\partial u_\alpha}{\partial x_\beta}, \\
S_\alpha^{(2)} &= B \frac{\partial T}{\partial x_\beta} \left[\frac{\partial u_\beta}{\partial x_\alpha} - \beta(T) \delta_{\alpha\beta} \operatorname{div} u \right],
\end{aligned} \tag{34}$$

other notations are given in Eqs.(28), (30), (32).

These formulas are sufficient to explain why Burnett equations are ill-posed. Considering just the terms with higher derivatives we transform Eqs.(8), (23), (33) to

$$\rho_t = \dots, \quad \frac{\partial u_\alpha}{\partial t} = \frac{\varepsilon^2}{\rho^2} \frac{\partial \Pi_{\alpha\beta}^{(1)}}{\partial x_\beta} + \dots, \quad T_t = \frac{2}{3} \frac{\varepsilon^2}{\rho^2} \operatorname{div} S^{(1)} + \dots, \tag{35}$$

where dots denote terms that do not contain third derivatives. Then, after simple calculations, we obtain

$$\begin{aligned}
\rho_t &= \dots, \quad u_t = \frac{2}{3} \frac{\varepsilon^2}{\rho^2} \left[\frac{AT}{\rho} \Delta(\nabla \rho) + (A - C) \Delta(\nabla T) \right] + \dots, \\
T_t &= \left(\frac{2}{3} \right)^2 \frac{\varepsilon^2}{\rho^2} T (B - C) \Delta \operatorname{div} u + \dots
\end{aligned} \tag{36}$$

It is sufficient to consider 1d solutions

$$\rho = \rho(x_1, t), \quad u = \{u_1(x_1, t), 0, 0\}, \quad T = T(x_1, t),$$

then the matrix M of the coefficients for third derivatives reads

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{AT}{\rho} & 0 & A - C \\ 0 & \frac{2(B - C)T}{3} & 0 \end{pmatrix}. \tag{37}$$

Its non-zero eigenvalues are

$$\lambda_{\pm} = \pm \left[\frac{2}{3} T (B - C) (A - C) \right]^{1/2}.$$

Hence, the hyperbolicity condition (under obvious assumption $T > 0$) reads

$$(B - C)(A - C) \geq 0. \tag{38}$$

It is easy to verify that this condition is not fulfilled in most typical molecular models for the Boltzmann equation. In order to do this we use temporary notations

$$\Psi_{\alpha\beta}(c) = c_\alpha c_\beta - \frac{|c|^2}{3} \delta_{\alpha\beta}, \quad \Psi_\alpha(c) = \frac{c_\alpha}{2} (|c|^2 - 5T) \tag{39}$$

and represent the Navier-Stokes terms in Eqs.(23) by equalities

$$\pi_{\alpha\beta}^{NS} = -2\mu(T) \frac{\partial u_\alpha}{\partial x_\beta}, \quad q_\alpha^{NS} = -\lambda(T) \frac{\partial T}{\partial x_\alpha}, \tag{40}$$

where $\mu(T)$ and $\lambda(T)$ denote respectively the coefficients of viscosity and heat conductivity. It follows from Eq.(19) that

$$\mu(T) = -\frac{1}{10T} \langle \phi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle_M, \lambda(T) = -\frac{1}{3T^2} \langle \phi_\alpha, \Psi_\alpha \rangle_M. \quad (41)$$

The usual approximation [1] for functions $\phi_{\alpha\beta}(c)$ and $\phi_\alpha(c)$ (19) is given by

$$\phi_{\alpha\beta} \approx \gamma(T) \Psi_{\alpha\beta}(c), \phi_\alpha \approx \delta(T) \Psi_\alpha(c), \quad (42)$$

then

$$\gamma(T) = -\frac{\mu(T)}{T}, \delta(T) = -\frac{2\lambda(T)}{5T} \quad (43)$$

since

$$\langle \Psi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle_M = 10T^2, \langle \Psi_\alpha, \Psi_\alpha \rangle_M = \frac{15}{2}T^3. \quad (44)$$

The approximation (42) is exact for Maxwell molecules, moreover $\lambda(T) = 15\mu(T)/4$ in that case. By using Eqs.(42), (43) we evaluate the integrals (30) and obtain

$$A = 2\frac{\mu^2}{T}, B = \frac{2\lambda^2}{5T}, C = \frac{4\lambda\mu}{5T}. \quad (45)$$

Therefore

$$A - C = \frac{4\lambda\mu}{5T} \left(\frac{5\mu}{2\lambda} - 1 \right), B - C = \frac{2\lambda^2}{5T} \left(1 - \frac{\mu}{\lambda} \right).$$

The ratio

$$Pr = \frac{5\mu(T)}{2\lambda(T)} \quad (46)$$

is called in fluid mechanics the Prandtl number (for monoatomic gases) [10]. It is well known that the approximate equality $Pr \simeq 2/3$ (exact for Maxwell molecules) holds for all typical molecular models (hard spheres, etc.). On the other hand, the hyperbolicity condition (38) can be approximately (under the assumption (42)) written as

$$1 \leq Pr \leq 5/4. \quad (47)$$

The realistic value $Pr = 2/3$ obviously violates this condition. Therefore the Burnett equations are probably ill-posed for all typical molecular models, though our proof is rigorous just for Maxwell molecules (all above formulas are exact in this case). We note that $Pr = 1$ for BGK model, however this model is too unrealistic.

It should be stressed that we need something more than hyperbolicity for any reasonable modification of Burnett equations. It is clear that any constant solution $\{\rho = \rho_0 > 0, u = 0, T = T_0 > 0\}$ must be stable with respect to small perturbations. Such stable modifications are discussed in the next section.

GENERALIZED BURNETT EQUATIONS

In order to regularize the Burnett equations we use the approach described briefly in Introduction, i.e. “changes of coordinates”. The first successful attempt to do this led to a version of hyperbolic Burnett equations published in [22]. These equations are, however, more complicated than original Burnett equations and do not have the standard form of equations of hydrodynamics. We present below another version of stable Burnett equations which is much more convenient for applications. Details of derivation of the Generalized Burnett Equations (GBEs) and all necessary proofs can be found in [26].

GBEs are equations for *auxiliary* variables for which we keep the initial notations $\rho(x, t)$, $u(x, t)$ and $T(x, t)$. The *true* hydrodynamical variables $(\rho^{tr}, u^{tr}, T^{tr})$ are expressed through (ρ, u, T) by equalities

$$\rho^{tr} = \rho, u^{tr} = u, T^{tr} = T - \frac{\varepsilon^2}{\rho} \operatorname{div} \frac{S}{\rho}, \quad (48)$$

where

$$S = a(T) \nabla \log \rho + b(T) \nabla \log T. \quad (49)$$

The functions a and b are expressed through Burnett coefficients A, B, C (30) and two parameters $\theta_{1,2}$ (a reason for such notation is clarified below):

$$\begin{aligned} a(T) &= \frac{2}{3}[-TA(T) + \theta_1 g(T)], \\ b(T) &= \frac{2}{3}T[C(T) - A(T)] + \theta_2 g(T), \\ g &= \frac{T}{3}(5A + 2B - 4C). \end{aligned} \quad (50)$$

Thus, the only “unusual” variable is $T(x, t)$ which does not coincide with true absolute temperature. Equations for (ρ, u, T) have the standard form of equations of hydrodynamics

$$\begin{aligned} \rho_t + \operatorname{div} \rho u &= 0, \quad \rho \mathcal{D}_0 u_\alpha + \frac{\partial p}{\partial x_\alpha} + \varepsilon \frac{\partial}{\partial x_\beta} \Pi_{\alpha\beta} = 0, \\ \frac{3}{2} \rho \mathcal{D}_0 T + p \operatorname{div} u + \varepsilon (\Pi_{\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} + \operatorname{div} Q) &= 0, \quad p = \rho T, \quad \mathcal{D}_0 = \partial_t + u \partial_x \end{aligned} \quad (51)$$

with the new fluxes Π and Q given by equalities

$$\begin{aligned} \Pi_{\alpha\beta} &= \pi_{\alpha\beta}^{NS} + \varepsilon \pi_{\alpha\beta}^B - \varepsilon \delta_{\alpha\beta} \operatorname{div} \frac{S}{\rho}, \\ Q_\alpha &= q_\alpha^{NS} + \varepsilon q_\alpha^B + \frac{\varepsilon}{\rho} \left\{ 3S_\beta \frac{\partial u_\alpha}{\partial x_\beta} + \left(\frac{3}{2}a + b \right) \frac{\partial}{\partial x_\alpha} \operatorname{div} u + (\operatorname{div} u) \left[(a'T - 2a) \frac{\partial}{\partial x_\alpha} \log \rho + (b'T - 2b) \frac{\partial}{\partial x_\alpha} \log T \right] \right\}, \end{aligned} \quad (52)$$

where primes denote differentiation with respect to T . The first two terms in both equalities are usual Navier-Stokes and Burnett fluxes discussed in detail in previous sections. Eqs.(51) can be obviously written in the form of conservation laws (8) with Π and Q instead of π and q respectively. Note, however, that

$$\Pi_{\alpha\alpha} = -3\varepsilon \operatorname{div} \frac{S}{\rho} \neq 0. \quad (53)$$

It can be shown that, under very general assumptions on intermolecular potential, there exists a non-empty region on the plane of parameters (θ_1, θ_2) , where Eqs.(51) are stable. Exact necessary and sufficient conditions of stability can be found in [26]. The reason for the notation (50) becomes clear if we consider the matrix of third derivatives for Eqs.(51). Then, instead of Eqs.(36), we obtain (the continuity equation is omitted):

$$\begin{aligned} u_t &= \frac{\varepsilon^2}{\rho^2} g \left[\frac{2\theta_1}{3\rho} \triangle (\nabla \rho) + \frac{\theta_2}{T} \triangle (\nabla T) \right] + \dots, \\ T_t &= \frac{2\varepsilon^2}{3\rho^2} g \theta_3 \triangle \operatorname{div} u + \dots, \quad \theta_3 = 1 - (\theta_1 + \theta_2). \end{aligned} \quad (54)$$

Therefore we denote the generalized Burnett equations (51), (52) (in the notation of Eqs. (48)-(50)) by a symbol GBEs $(\theta_1, \theta_2, \theta_3)$, having in mind that only two of three parameters $\{\theta_i, i = 1, 2, 3\}$ are independent. It follows from Eqs.(54) that GBEs $(\theta_1, \theta_2, \theta_3)$ are well-posed (hyperbolic) if $\theta_2 \theta_3 \geq 0$. Conditions of stability depend, generally speaking, on the sign of $g(T)$ [26]. The approximation (45) (exact for Maxwell molecules) leads to estimate

$$g(T) \approx \frac{37}{12} \mu^2(T). \quad (55)$$

Hence, for practical applications we can assume that $g(T) > 0$ for all $T > 0$. Then the simplest region of stability of GBEs $(\theta_1, \theta_2, \theta_3)$ is given by inequalities

$$\theta_{1,2,3} \geq 0, \quad \theta_1 + \theta_2 + \theta_3 = 1. \quad (56)$$

Thus, we can choose any two of three real numbers (generally speaking, they can be also functions of T), satisfying the conditions (56) and obtain the above described stable GBEs $(\theta_1, \theta_2, \theta_3)$. It is easy to verify that new fluxes (52) contain, roughly speaking, the same tensor and vector terms as the original Burnett fluxes (with other coefficients). Hence, GBEs $(\theta_1, \theta_2, \theta_3)$ are not more complicated than the original equations. Moreover they can be simpler, in particular, in the following three cases: GBEs $(1, 0, 0)$, GBEs $(0, 1, 0)$, GBEs $(0, 0, 1)$ (note that the stability conditions (56) are satisfied). In other words, we can reduce a number of third derivatives. It is clear that the problem of boundary conditions should be considered separately for each of the three cases. We hope to return to this problem in another paper.

We also mention that the original Burnett equations can be understood as a particular case of GBEs $(\theta_1, \theta_2, \theta_3)$ with parameters

$$\theta_1 = \frac{TA(T)}{g(T)}, \theta_2 = \frac{2T[C(T) - A(T)]}{3g(T)}, \theta_3 = 1 - (\theta_1 + \theta_2), \quad (57)$$

which lie (for typical molecular models) in the unstable region of the plane (θ_1, θ_2) . A specific choice of parameters can be different for different problems. For example, it is known that Navier-Stokes equations give the correct (in the Hilbert class) answer for $1d$ stationary heat transfer problems in the case of Maxwell molecules. This property can be preserved if we choose $\theta_{1,2}$ in Eqs.(50) in such a way that $a = b$. It can be done without violation of conditions (56).

Thus, the whole class of stable GBEs $(\theta_1, \theta_2, \theta_3)$ seems to be a natural replacement for the classical (unstable) Burnett equations, though the problem of optimal choice of parameters for specific cases and the problem of corresponding boundary conditions need further investigation.

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