Measurement Uncertainty – Part 1 of 2

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Why analyze measurement uncertainty?

Types of Measurement Uncertainty

Blunders:

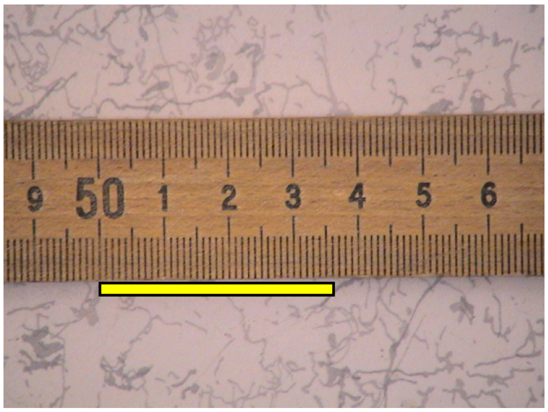
Systematic (or fixed) uncertainty:

Random uncertainty:

Measurement Uncertainty – Part 1 of 2

Estimation of Uncertainty

<u>Single Sample Experiments (aka Type B uncertainty)</u>



What is the least count for this ruler?

What is the measurement uncertainty using this ruler?

What is the length of the yellow box?

What is the least count for this stopwatch?

What is the measurement uncertainty using this stopwatch?

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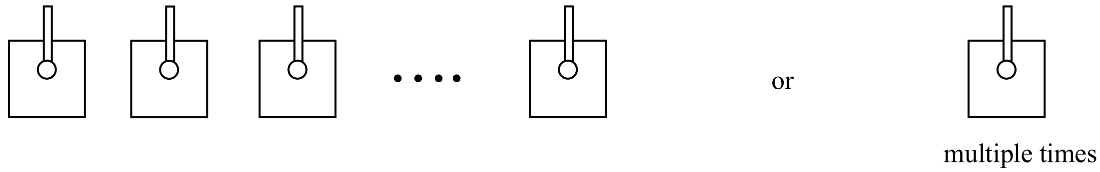
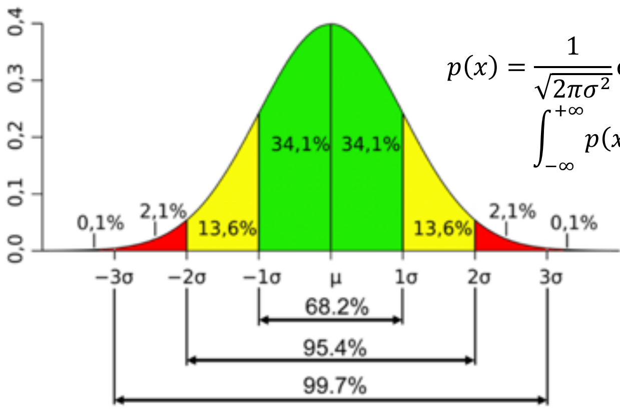
<u>Multiple Sample Experiments (aka Type A uncertainty)</u>



The Normal (aka Gaussian) Probability Distribution:



$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\int_{-\infty}^{+\infty} p(x)dx = 1$$

Sample (Arithmetic) Mean ($\bar{x}$):

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Sample Variance (s^2) and Sample Standard Deviation (s):

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

$\mu = 100 \qquad \sigma^2 = 400 \qquad (\sigma = 20)$

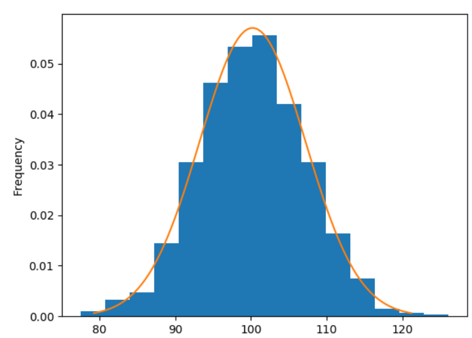
#	x_i
1	99.36
2	121.02
3	131.73
4	119.56
5	94.31
6	114.74
7	78.33

$\bar{x} = 108.44 \qquad s^2 = 342.05 \ (s = 18.49)$

Standard Error of the Sample Mean ($s_{\bar{x}}$):

$\mu = 100 \qquad \sigma^2 = 400 \qquad (\sigma = 20)$

	Trial 1	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**	**Trial 6**
#	x_i	x_i	x_i	x_i	x_i	x_i
1	99.36	120.20	80.92	72.20	130.41	86.54
2	121.02	76.56	88.64	95.92	116.38	100.09
3	131.73	93.18	93.32	100.04	113.11	112.06
4	119.56	65.21	98.33	82.72	103.82	128.79
5	94.31	105.08	143.42	72.67	102.18	88.30
6	114.74	102.21	116.85	147.12	101.71	99.12
7	78.33	76.47	98.88	104.78	96.41	79.68
$\bar{x}$ =	**108.44**	**91.27**	**102.91**	**96.49**	**109.15**	**99.22**
s^2 =	**342.05**	**377.60**	**442.04**	**665.40**	**135.73**	**283.68**
s =	**18.49**	**19.43**	**21.02**	**25.80**	**11.65**	**16.84**



- Frequency plot of the trial mean values ($\bar{x}$).
- Superimposed normal distribution using:
  - μ = mean of the trial means
  - $\sigma = s_{\bar{x}} = 6.99$

Standard Error = standard deviation of sample means

$s_{\bar{x}} \approx \frac{s}{\sqrt{N}}$ => $s_{\bar{x}} \approx 6.99$ (based on Trial 1)

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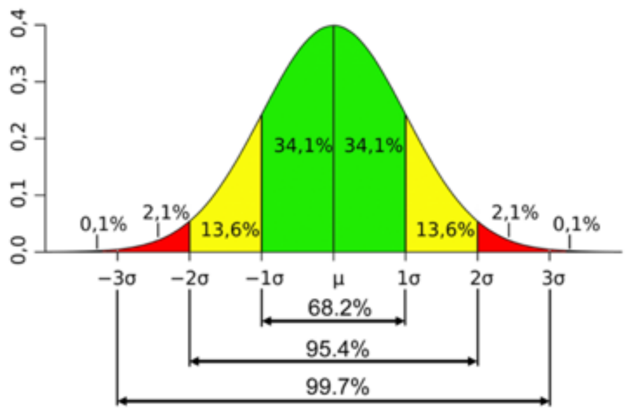
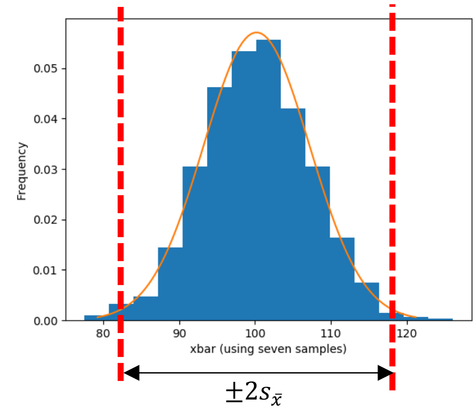
Confidence Interval (CI):





For typical engineering applications: $CI = 95\%$ ($\approx \pm 1.96\sigma$)

$$\bar{x} - 1.96 s_{\bar{x}} < \mu < \bar{x} + 1.96 s_{\bar{x}} \text{ (95\% CI)} \quad \text{or} \quad \mu = \bar{x} \pm 1.96 s_{\bar{x}} = \bar{x} \pm 1.96 \frac{s}{\sqrt{N}} \text{ (95\% CI)}$$

t-distribution factor

N	2	3	4	5	6	7	8	9	10	15	20	30	∞
$t_{95\%}$	12.71	4.30	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.14	2.09	2.04	1.96

$$\bar{x} - t_{95\%} s_{\bar{x}} < \mu < \bar{x} + t_{95\%} s_{\bar{x}} \text{ (95\% CI)} \quad \text{or} \quad \mu = \bar{x} \pm t_{95\%} s_{\bar{x}} = \bar{x} \pm t_{95\%} \frac{s}{\sqrt{N}} \text{ (95\% CI)}$$

$\mu = 100 \qquad \sigma^2 = 400 \qquad (\sigma = 20)$

#	x_i
1	99.36
2	121.02
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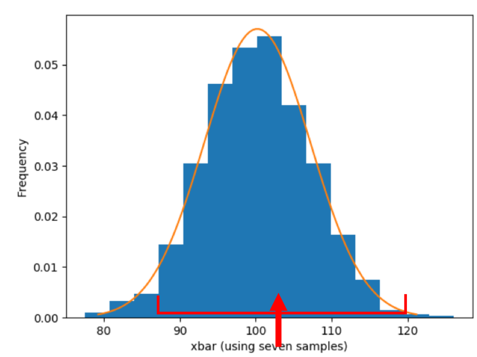


$\bar{x} = 108.44 \qquad s^2 = 342.05 \ (s = 18.49)$
$t_{95\%} = 2.45$

$\mu = 108.44 \pm 2.45\left(\frac{18.49}{\sqrt{7}}\right) = 108.44 \pm 17.12$ (95% CI)
$\mu = [91.32, 125.56]$ (95% CI)

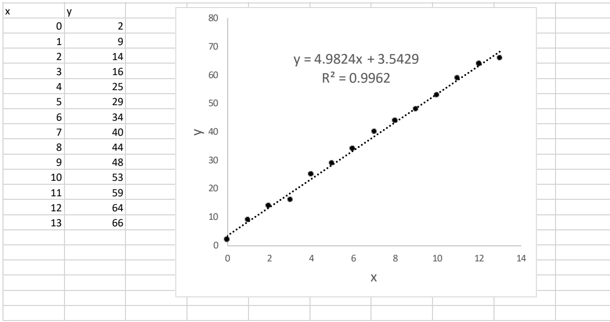
What happens as the number of measurements increases?

What happens as the sample standard deviation decreases?

What's the difference between the sample standard deviation and the standard error of the sample mean?

Measurement Uncertainty – Part 1 of 2

Uncertainty of Least Squares Fits



What is the uncertainty on the (least squares) curve fit slope (m) and intercept (b)?

Answer:

$m \pm 1.96 s_m$, where s_m is the standard error for m

$b \pm 1.96 s_b$, where s_b is the standard error for b

How to find the standard error in the fitting parameters?

In Excel: Use the LINEST command, e.g., LINEST(B2:B15, A2:A15, TRUE, TRUE)

Excel output:

4.98241758	3.54285714
0.08845073	0.676518
0.9962324	1.33411149
3173.05351	12
5647.57033	21.3582418

Format:

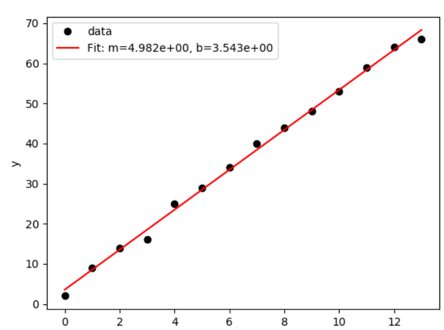
fit for m	fit for b
std error for m, s_m	std error for b, s_b
coeff. of det., R^2	std. error for y, s_y
F statistic	degrees of freedom
regression sum of squares	residual sum of squares

In Python:

```
# Least squares fit, including standard error estimates.
# Disclaimer:  I'm new to python programming.  There may be a better way to do this.
import numpy as np
import scipy.optimize as opt
import matplotlib.pyplot as plt

# The data to fit.
x = np.array([0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13])
y = np.array([2, 9, 14, 16, 25, 29, 34, 40, 44, 48, 53, 59, 64, 66])

# The equation to fit the data to.
def func(x, m, b):
    return m*x + b    # equation of a line, can specify any function here

# Guess initial estimates for the parameters to seed the curve fit.
p0 = [1, 1]   # [m, b] These are just reasonable guesses.
popt, pcov = opt.curve_fit(func, x, y, p0)

# Print the fitting parameter values.
print("m = %.3e" % popt[0])
print("b = %.3e" % popt[1])

# Print the uncertainties on the fit parameters.
perr = np.sqrt(np.diag(pcov))
print("standard error for m = %.3e" % perr[0])
print("standard error for b = %.3e" % perr[1])

# Plot the data.
plt.plot(x, y, color="black", marker="o", linestyle="", label="data")

# Plot the fitting line.
y_model = func(x, *popt)
plt.plot(x, y_model, color="red", linestyle="solid", label="Fit: m=%.3e, b=%.3e\
" % tuple(popt))
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()
```

Python output:

```
m = 4.982e+00
b = 3.543e+00
standard error for m = 8.845e-02
standard error for b = 6.765e-01
```



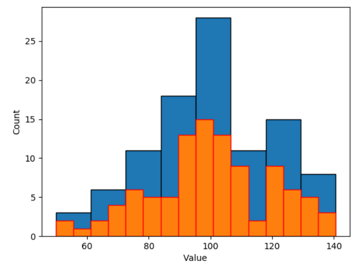
Measurement Uncertainty – Part 1 of 2

Notes:

1. What is the difference between histograms, frequency distributions, and probability distributions?

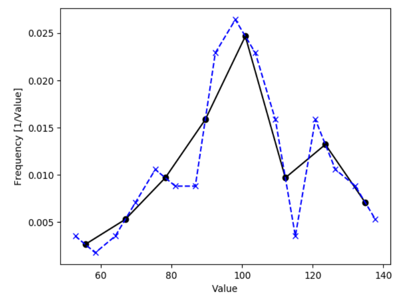
Histograms

- Plots the <u>number</u> of samples within a specified size bin.
- Plot changes depending on the bin size.
- Area under the curve isn't equal to one.



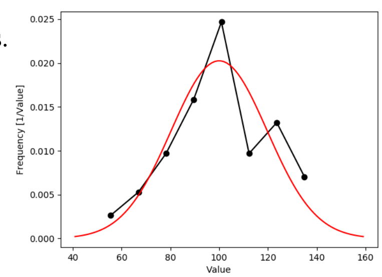
Frequency distribution

- The <u>fraction</u> of samples in the range $(x, x + \Delta x)$ is the area under the curve in that range, i.e.,
  fraction of samples in the range $(x, x + \Delta x) = \int_{x}^{x+\Delta x} f(x)dx$
- Plot is insensitive to the bin size.
- Area under the curve is equal to one, i.e., $\int_{-\infty}^{+\infty} f(x)dx = 1$.
- $f(x_i, x_i + \Delta x_i) = \frac{1}{\Delta x_i}\frac{n(x_i, x_i + \Delta x_i)}{N}$



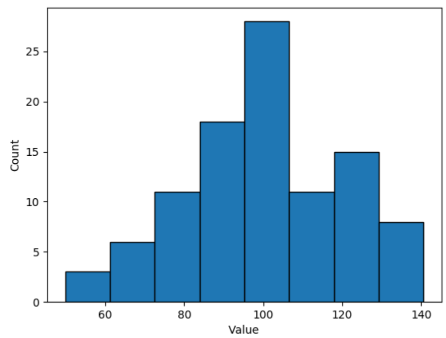
Probability distribution

- A frequency distribution, but with an infinite number of samples.

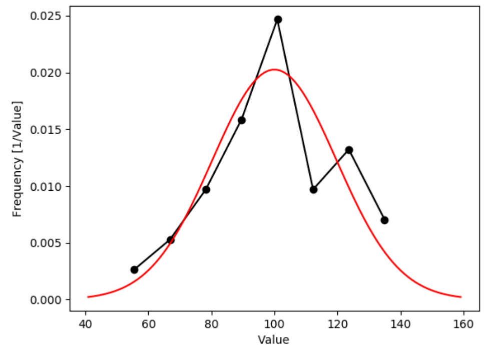


Measurement Uncertainty – Part 1 of 2

2. How do you know if your data is normally distributed?
   a. Qualitative comparison of frequency distribution to a normal probability distribution with the same mean and standard deviation [not recommended]

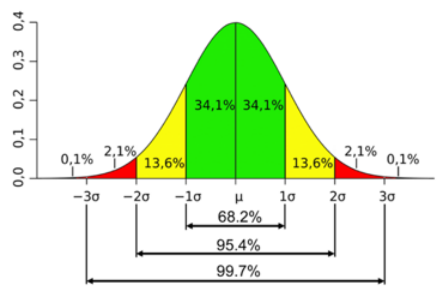


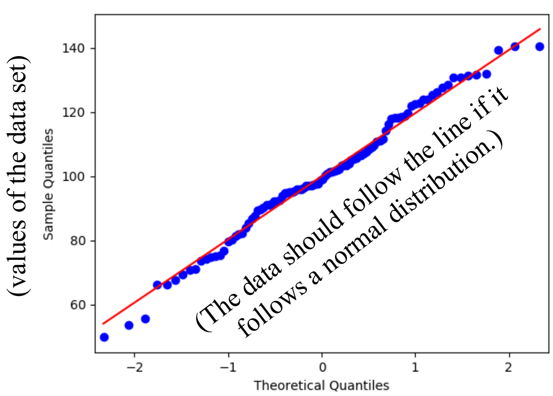
histogram



frequency plot / normal probability distribution

   b. Normal Q-Q plot (Normal Quantile-Quantile plot) [qualitative – provides a visual check]
      Quantiles (aka percentiles) are the data values below which a certain proportion of the data fall:
      0.1% of the data have values less than $\mu - 3\sigma$
      2.2% of the data have values less than $\mu - 2\sigma$
      15.8% of the data have values less than $\mu - 1\sigma$
      50% of the data have values less than μ
      84.2% of the data have values less than $\mu + 1\sigma$
      97.8% of the data have values less than $\mu + 2\sigma$
      99.9% of the data have values less than $\mu + 3\sigma$





(standard deviations from the mean)

Does the data follow a normal distribution?

Where does the data deviate most from a normal distribution?

   c. Anderson-Darling Goodness-of-Fit Test (quantitative – recommended)
      - Details not presented here – out of scope for this course.
      - Tests if the sample comes from a specified probability distribution, e.g., a normal distribution.
      - Numerically compares the sample distribution to the specified probability distribution.
      - Requires a more advanced knowledge of statistics to apply and understand.
      - There are other, similar quantitative tests of normality, e.g., the Shapiro-Wilks test, the Kolmogorov-Smirnov test, and the Skewness-Kurtosis test.

d. Example python code to make these various plots and perform the Anderson-Darling test.

```
# Tests for data normality.
# Disclaimer:  I barely know how to program in python.  You can probably do better.

import statsmodels.api as sm
import scipy.stats as stats
import numpy as np
import pylab as plt

# Load the data set from a file (single data entry per line)
my_data = np.loadtxt("data.txt")

# Report some statistics about the data
mean = np.mean(my_data)
stdev = np.std(my_data)
print("# of data entries=", len(my_data))
print("mean = ", mean)
print("std. dev = ", stdev)

# First plot a histogram of the data.
Nbins = 8
plt.figure(1)
plt.hist(my_data, density=False, bins=Nbins, linewidth=1, edgecolor='black')
plt.ylabel('Count')
plt.xlabel('Value')

# Plot a frequency distribution of the data.
counts, bin_edges = np.histogram(my_data, Nbins, density=True)
# Plot the counts at the center of the bins.
bin_centers = np.empty([len(bin_edges)-1])
for i in range(len(bin_edges)-1):
  bin_centers[i] = (bin_edges[i]+bin_edges[i+1])/2
plt.figure(2)
plt.plot(bin_centers, counts, color='black', marker='o', linestyle='solid')
plt.ylabel('Frequency [1/Value]')
plt.xlabel('Value')

# Include a plot of a normal distribution on top of the frequency distribution.
x = np.linspace(mean-3*stdev, mean+3*stdev,100)
plt.plot(x, stats.norm.pdf(x, mean, stdev), color='red', linestyle='solid')

# Create a QQ plot.
sm.qqplot(my_data, line='s')

# Check data for normality using the Anderson-Darling test.
statistic, significance_values, critical_values = stats.anderson(my_data,'norm')
print("statistic = ", statistic)
print("critical_values = ", critical_values)
print("significance_values = ", significance_values)

for i in range(len(critical_values)):
  if significance_values[i] < 0.05:
    print("The data is NOT consistent with a normal distribution for the critical value of ",
critical_values[i])
  else:
    print("The data IS consistent with a normal distribution for the critical value of ",
critical_values[i])

plt.show()
```

Python output (plots shown previously):

```
# of data entries= 100
mean =  99.98454800000003
std. dev =  19.693678789517616
statistic =  0.36885781711968946
critical_values =  [15.  10.   5.   2.5  1. ]
significance_values =  [0.555 0.632 0.759 0.885 1.053]
The data IS consistent with a normal distribution for the critical value of  15.0
The data IS consistent with a normal distribution for the critical value of  10.0
The data IS consistent with a normal distribution for the critical value of  5.0
The data IS consistent with a normal distribution for the critical value of  2.5
The data IS consistent with a normal distribution for the critical value of  1.0
```

References

"Evaluation of measurement data – Guide to the Expression of Uncertainty in Measurement", https://www.bipm.org/utils/common/documents/jcgm/JCGM_100_2008_E.pdf.

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