4.2. The Reynolds Transport Theorem

Recall that we can look at the behavior of small pieces of fluid in two ways: the Eulerian perspective or the Lagrangian perspective. Often we're interested in the behavior of an entire system of fluid (many pieces of fluid) rather than just an individual piece. How do we analyze this situation? We can use Eulerian and Lagrangian approaches for analyzing a macroscopic amount of fluid but we need to first develop an important tool called the Reynolds Transport Theorem.

Why do we want to do this? It turns out that the behavior of fluids (most substances in fact) can be described in terms of a few fundamental laws. These laws include:

- Conservation of Mass,
- Newton's 2nd Law,
- the angular momentum principle,
- the First Law of Thermodynamics, and
- the Second Law of Thermodynamics.

These laws are typically easiest to apply to a particular system of fluid particles (Lagrangian perspective). However, the Lagrangian forms of the laws are typically difficult to use in practical applications since we can't easily keep track of many individual bits of fluid. It's much easier to apply the laws to a particular volume in space instead (referred to as a control volume, an Eulerian perspective). For example, tracking the behavior of individual bits of gas flowing through a rocket nozzle would be difficult. It's much easier to just look at the behavior of the gas flowing into, out of, and within the volume enclosed by the rocket nozzle. The Reynolds Transport Theorem is a tool that will allow us to convert from a system point of view (Lagrangian) to a control volume point of view (Eulerian).

Let's consider a system of fluid particles that is coincident (occupying the same region in space) as our control volume (CV) at some time, t (Figure 4.3). At some later time, $t + \delta t$, the system may have moved relative

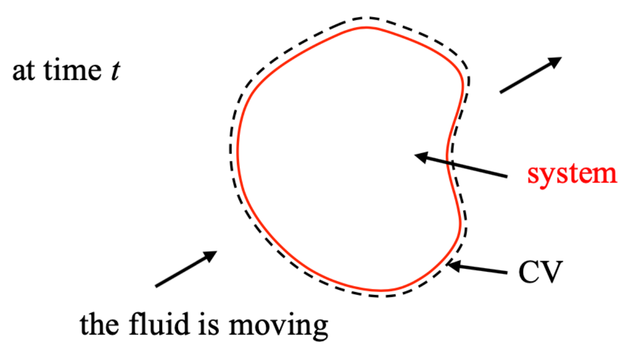


Figure 4.3. A sketch of a system and control volume of fluid that are coincident at time t.

to the CV (Figure 4.4).

Let B be some transportable property (i.e., some property that can be transported from one location to another, e.g., mass, momentum, energy) and β be the corresponding amount of B per unit mass (Figure 4.5), i.e.,

$$B_{\text{sys}} = \int_{V_{\text{sys}}} \beta \rho dV, \tag{4.14}$$

$$B_{CV} = \int_{CV} \beta \rho dV, \tag{4.15}$$

where B_{sys} and B_{CV} refer to the total amount of B in the system and control volume, respectively. Note that at time t, the total amounts of B in the system and control volume are equal since the system and CV

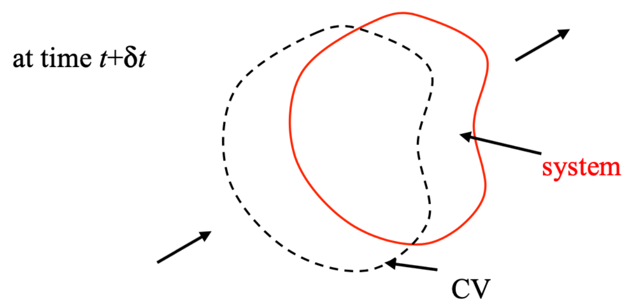


Figure 4.4. A sketch of a system and control volume of fluid at time $t + \delta t$.

are coincident,

$$B_{\text{sys}}(t) = B_{CV}(t). \tag{4.16}$$

However, at time, $t + \delta t$, the system and CV no longer occupy the same region in space so that, in general,

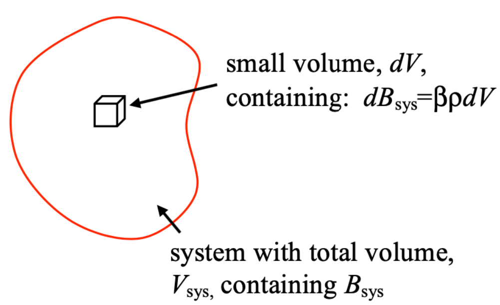


Figure 4.5. A sketch of a system of fluid illustrating a small amount the transportable quantity B, i.e., dB.

$B_{\text{sys}}(t+\delta t) \neq B_{CV}(t+\delta t)$. Note that B may be changing with time so that, in general, $B_{\text{sys}}(t+\delta t) \neq B_{\text{sys}}(t)$. In Figure 4.6, B_{out} is the amount of B that has left the CV and B_{in} is the amount of B that has entered the CV.

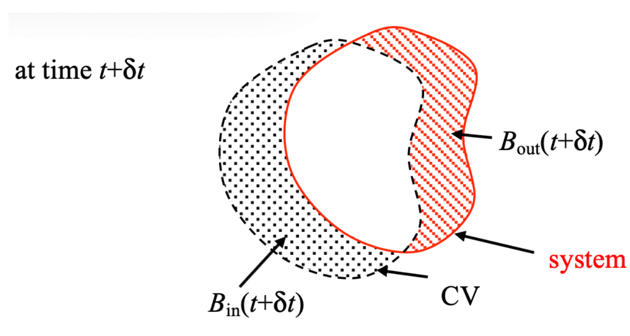


Figure 4.6. A sketch of a system and control volume showing B_{out} and B_{in} at $t + \delta t$.

Utilizing the figure shown above, we see that,

$$B_{CV}(t + \delta t) = B_{\text{sys}}(t + \delta t) - B_{\text{out}}(t + \delta t) + B_{\text{in}}(t + \delta t). \tag{4.17}$$

Subtracting $B_{\text{sys}}(t)$ from both sides and dividing through by δt gives,

$$\frac{B_{CV}(t+\delta t)-B_{\text{sys}}(t)}{\delta t}=\frac{B_{\text{sys}}(t+\delta t)-B_{\text{sys}}(t)}{\delta t}-\frac{B_{\text{out}}(t+\delta t)}{\delta t}+\frac{B_{\text{in}}(t+\delta t)}{\delta t}. \quad (4.18)$$

Now let's substitute $B_{CV}(t)=B_{\text{sys}}(t)$ on the left hand side of the equation, subtract $B_{\text{out}}(t)/\delta t$ and $B_{\text{in}}(t)/\delta t$ on the right-hand side (note that $B_{\text{out}}(t)=B_{\text{in}}(t)=0$), and then take the limit of the entire equation as $\delta t\to 0$,

$$\lim_{\delta t\to 0}\frac{B_{CV}(t+\delta t)-B_{\text{sys}}(t)}{\delta t}=\lim_{\delta t\to 0}\frac{B_{\text{sys}}(t+\delta t)-B_{\text{sys}}(t)}{\delta t} \quad (4.19)$$

$$-\lim_{\delta t\to 0}\frac{B_{\text{out}}(t+\delta t)-B_{\text{out}}(t)}{\delta t}+\lim_{\delta t\to 0}\frac{B_{\text{in}}(t+\delta t)-B_{\text{in}}(t)}{\delta t}, \quad (4.20)$$

$$\frac{dB_{CV}}{dt}=\frac{DB_{\text{sys}}}{Dt}-\frac{dB_{\text{out}}}{dt}+\frac{dB_{\text{in}}}{dt}. \quad (4.21)$$

Note that the D/Dt notation has been used to signify that the first term on the right-hand side of Eq. (4.21) represents the time rate of change as we follow a particular system of fluid (Lagrangian perspective). Rearranging the equation and substituting in for B_{CV} and B_{sys} using Eqs. (4.15) and (4.14),

$$\frac{D}{Dt}\left(\int_{V_{\text{sys}}}\beta\rho dV\right)=\frac{d}{dt}\left(\int_{V_{CV}}\beta\rho dV\right)+\frac{d(B_{\text{out}}-B_{\text{in}})}{dt}. \quad (4.22)$$

The last term on the right-hand side of Eq. (4.22) represents the net rate at which B is leaving the control volume through the control surface (CS). Let's examine this term more closely by zooming in on a small piece of the control surface and observing how much B leaves through this surface in time δt (Figure 4.7).

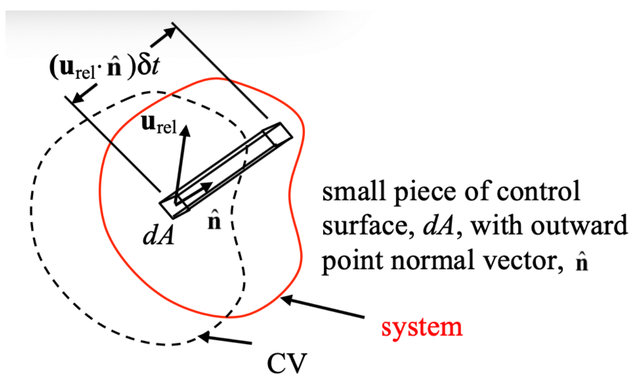


Figure 4.7. A sketch of a system and control volume of fluid at time $t+\delta t$. The parallelpiped is the small volume of fluid that has left the control volume through the area $d\mathbf{A}$ over time δt. Note that in the figure, the area $d\mathbf{A}=dA\hat{\mathbf{n}}$.

The component of the fluid velocity out of the control volume through surface, $d\mathbf{A}$, is given by,

$$u_{\text{out through }d\mathbf{A}}=\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}. \quad (4.23)$$

where $\mathbf{u}_{\text{rel}}=\mathbf{u}_{\text{sys}}-\mathbf{u}_{CS}$ is the velocity of the fluid relative to the control surface. The volume of fluid leaving through surface $d\mathbf{A}$ in time δt is then,

$$\delta V=(\mathbf{u}_{\text{rel}}\cdot d\mathbf{A})\,\delta t. \quad (4.24)$$

Thus, the volumetric flow rate, dQ, (volume per unit time) through surface $d\mathbf{A}$ is given by,

$$dQ=\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}. \quad (4.25)$$

Note that the mass flow rate, $d\dot{m}$ through the small area is,

$$d\dot{m}=\rho dQ=\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}. \quad (4.26)$$

Now use Eq. (4.25) to write the net rate at which B leaves the control volume,

$$\frac{d\left(B_{\text{out}}-B_{\text{in}}\right)}{dt}=\int_{CS}\beta\rho dQ=\int_{CS}\beta\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right). \tag{4.27}$$

Combining Eq. (4.27) with Eq. (4.22) gives,

$$\boxed{\underbrace{\frac{D}{Dt}\left(\int_{V_{\text{sys}}}\beta\rho dV\right)}_{\substack{\text{rate of increase of}\\ B\text{ within the system}}}=\underbrace{\frac{d}{dt}\left(\int_{V_{CV}}\beta\rho dV\right)}_{\substack{\text{rate of increase of}\\ B\text{ within the CV}}}+\underbrace{\int_{CS}\beta\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)}_{\substack{\text{net rate at which }B\text{ leaves}\\ \text{the CV through the CS}}}}. \tag{4.28}$$

This is the Reynolds Transport Theorem!

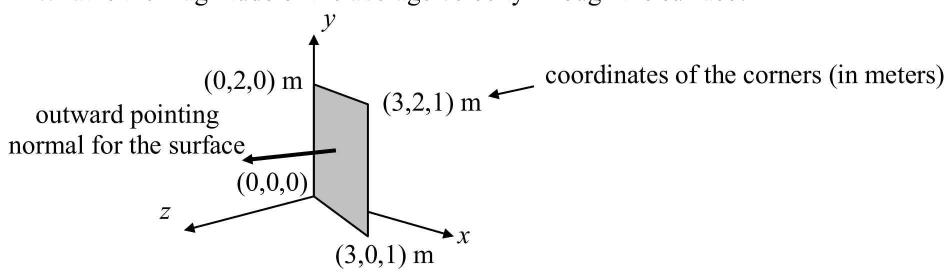
Consider a fluid flowing with the following velocity profile:

$$\mathbf{u}_{\text{fluid}} = Ax\hat{\mathbf{i}} + By\hat{\mathbf{j}} + Cxy\hat{\mathbf{k}}$$

where $A = 1\ \text{s}^{-1}$, $B = 2\ \text{s}^{-1}$, and $C = 3\ (\text{m}\cdot\text{s})^{-1}$.

a. Determine the magnitude of the volumetric flow rate through the fixed surface shown in the figure below.

b. What is the magnitude of the average velocity through the surface?



SOLUTION:

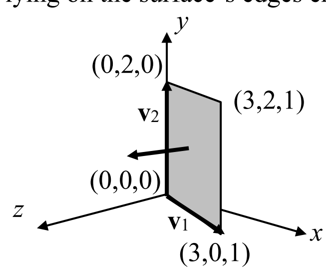
First determine the unit normal vector for the surface. The unit normal vector to the surface may be found by normalizing the cross product of the two vectors lying on the surface's edges emanating from the origin.

$$\mathbf{v}_1 = 3\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 1\hat{\mathbf{k}} \tag{1}$$

$$\mathbf{v}_2 = 0\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \tag{2}$$

$$\hat{\mathbf{n}} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\left|\mathbf{v}_1 \times \mathbf{v}_2\right|} = \frac{(3,0,1)\times(0,2,0)}{\left|(3,0,1)\times(0,2,0)\right|} = \frac{(-2,0,6)}{\left|(-2,0,6)\right|} \tag{3}$$

$$\therefore \hat{\mathbf{n}} = \tfrac{1}{\sqrt{10}}(-1,0,3) \tag{4}$$



Now find the volumetric flow rate, Q, through the surface. Note that the fluid velocity varies in all three directions so the flow rate must be found through integration. Also note that $\mathbf{u}_{\text{rel}} = \mathbf{u}_{\text{fluid}} - \mathbf{u}_{\text{surface}} = \mathbf{u}_{\text{fluid}}$ since $\mathbf{u}_{\text{surface}} = \mathbf{0}$.

$$Q = \int_A \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \int_{s=0}^{s=\sqrt{10}\ \text{m}} \int_{y=0}^{y=2\ \text{m}} \mathbf{u}_{\text{rel}} \cdot \underbrace{\hat{\mathbf{n}}\,dyds}_{=d\mathbf{A}} \quad \text{where } s \text{ is the distance in the } \mathbf{v}_1 \text{ direction} \tag{5}$$

$$Q = \int_{s=0}^{s=\sqrt{10}\ \text{m}} \int_{y=0}^{y=2\ \text{m}} \left(Ax\hat{\mathbf{i}} + By\hat{\mathbf{j}} + Cxy\hat{\mathbf{k}}\right)\cdot \tfrac{1}{\sqrt{10}}\left(-\hat{\mathbf{i}} + 3\hat{\mathbf{k}}\right) dyds = \tfrac{1}{\sqrt{10}} \int_{s=0}^{s=\sqrt{10}\ \text{m}} \int_{y=0}^{y=2\ \text{m}} \left(-Ax + 3Cxy\right) dyds \tag{6}$$

$$Q = \tfrac{1}{\sqrt{10}} \int_{s=0}^{s=\sqrt{10}\ \text{m}} \left[-Axy + \tfrac{3}{2}Cxy^2\right]_0^{2\ \text{m}} ds = \tfrac{1}{\sqrt{10}} \int_{s=0}^{s=\sqrt{10}\ \text{m}} \left[(-2\ \text{m})Ax + \left(6\ \text{m}^2\right)Cx\right] ds \tag{7}$$

Now relate s to x.

$$s = \tfrac{\sqrt{10}}{3}x \Rightarrow ds = \tfrac{\sqrt{10}}{3}dx \tag{8}$$

Substitute Eqn. (8) into Eqn. (7) and solve.

$$Q = \tfrac{1}{\sqrt{10}} \int_{x=0}^{x=3\ \text{m}} \left[(-2\ \text{m})Ax + \left(6\ \text{m}^2\right)Cx\right] \tfrac{\sqrt{10}}{3} dx = \tfrac{1}{3}\left[(-1\ \text{m})Ax^2 + \left(3\ \text{m}^2\right)Cx^2\right]_0^{3\ \text{m}} \tag{9}$$

$$Q = \left(-3\ \text{m}^3\right)A + \left(9\ \text{m}^4\right)C \tag{10}$$

$$\boxed{\therefore Q = 24\ \text{m}^3/\text{s}} \tag{11}$$

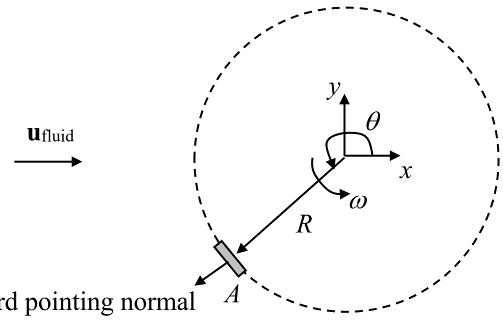
The average velocity through the surface is:

$$\bar{u} = \frac{Q}{A} = \frac{24\ \text{m}^3/\text{s}}{(2\ \text{m})\left(\sqrt{10}\ \text{m}\right)} \Rightarrow \boxed{\therefore \bar{u} = \tfrac{12}{\sqrt{10}}\ \text{m/s}} \tag{12}$$

Consider a fluid velocity field given by:

$$\mathbf{u}_{\text{fluid}} = C\hat{\mathbf{i}}$$

where C is a constant. Determine the volumetric flow rate through a surface of area A that is attached to an arm of radius R that rotates with constant angular speed ω in the x-y plane as shown in the figure below. Express your result in terms of (a subset of) C, R, ω, A, and θ.



SOLUTION:

The unit normal vector for the surface is:

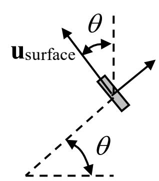
$$\hat{\mathbf{n}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}} \tag{1}$$

The flow rate through the surface, Q, is:

$$Q = \int_A \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \mathbf{u}_{\text{rel}} \cdot A\hat{\mathbf{n}} \tag{2}$$

Note that since the fluid velocity is uniform, integration over the area isn't required.

The velocity of the fluid relative to the surface, $\mathbf{u}_{\text{rel}}$, is:



$$\mathbf{u}_{\text{rel}} = \mathbf{u}_{\text{fluid}} - \mathbf{u}_{\text{surface}} = C\hat{\mathbf{i}} - \omega R\left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}\right) \tag{3}$$

$$\therefore \mathbf{u}_{\text{rel}} = \left(C + \omega R\sin\theta\right)\hat{\mathbf{i}} - \omega R\cos\theta\hat{\mathbf{j}} \tag{4}$$

Substitute Eqns. (4) and (1) into Eqn. (2) and simplify.

$$Q = \left[\left(C + \omega R\sin\theta\right)\hat{\mathbf{i}} - \omega R\cos\theta\hat{\mathbf{j}}\right] \cdot A\left[\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right] \tag{5}$$

$$Q = A\left[\left(C + \omega R\sin\theta\right)\cos\theta - \omega R\sin\theta\cos\theta\right] \tag{6}$$

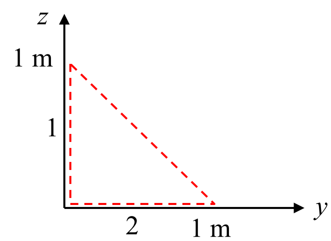
$$\boxed{\therefore Q = AC\cos\theta}$$

The velocity field in the region shown is given by:

$$\mathbf{u} = az\hat{\mathbf{j}} + b\hat{\mathbf{k}}$$

where $a = 10\ \text{s}^{-1}$ and $b = 5$ m/s. For depth w into the page, an element of area 1 may be represented by $wdz(-\mathbf{j})$ and an element of area 2 by $wdy(-\mathbf{k})$. (Note that both are drawn *outward* from the control volume, hence the minus signs.)

1. Find an expression for $\mathbf{u}\cdot d\mathbf{A}_1$.
2. Evaluate $\int_{A_1} \mathbf{u}\cdot d\mathbf{A}_1$.
3. Find an expression for $\mathbf{u}\cdot d\mathbf{A}_2$.
4. Find an expression for $\mathbf{u}$ ($\mathbf{u}\cdot d\mathbf{A}_2$).
5. Evaluate $\int_{A_2} \mathbf{u}\left(\mathbf{u}\cdot d\mathbf{A}_2\right)$.



SOLUTION:

$$\mathbf{u}\cdot d\mathbf{A}_1 = \left(az\hat{\mathbf{j}} + b\hat{\mathbf{k}}\right)\cdot wdz\left(-\hat{\mathbf{j}}\right) = -awzdz \tag{1}$$

$$\int_{A_1} \mathbf{u}\cdot d\mathbf{A}_1 = \int_{z=0}^{z=1\text{ m}} -awzdz = -aw\tfrac{1}{2}z^2\Big|_0^{1\text{ m}} = -\left(10\ \text{s}^{-1}\right)(w)\left(\tfrac{1}{2}\ \text{m}^2\right) = \left(-5\frac{\text{m}^2}{\text{s}}\right)w \tag{2}$$

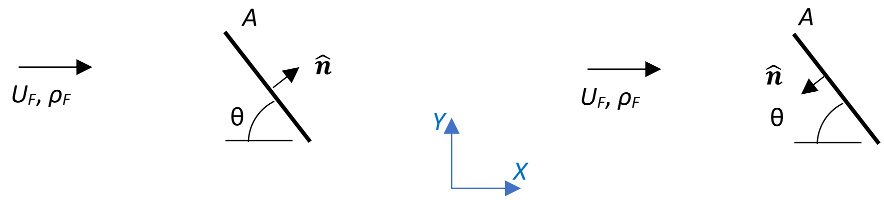
$$\mathbf{u}\cdot d\mathbf{A}_2 = \left(az\hat{\mathbf{j}} + b\hat{\mathbf{k}}\right)\cdot wdy\left(-\hat{\mathbf{k}}\right) = -bwdy \tag{3}$$

$$\mathbf{u}\left(\mathbf{u}\cdot d\mathbf{A}_2\right) = \left(az\hat{\mathbf{j}} + b\hat{\mathbf{k}}\right)\left(-bwdy\right) \tag{4}$$

$$\int_{A_2} \mathbf{u}\left(\mathbf{u}\cdot d\mathbf{A}_2\right) = \int_{y=0}^{y=1\text{ m}} \left(az\hat{\mathbf{j}} + b\hat{\mathbf{k}}\right)\left(-bwdy\right) = \hat{\mathbf{j}}bwaz\,y\Big|_0^{1\text{ m}} - \hat{\mathbf{k}}b^2w\,y\Big|_0^{1\text{ m}} = \left(-25\tfrac{\text{m}^3}{\text{s}^2}\right)w\hat{\mathbf{k}} \tag{5}$$

Note that $z = 0$ along $d\mathbf{A}_2$.

Determine the mass flow rate through the fixed surfaces shown below. Assume the velocities are uniform over the area.



SOLUTION:
The mass flow rate is,

$$\dot{m} = \rho \boldsymbol{u}_{rel} \cdot \boldsymbol{A}, \tag{1}$$

where,

$$\boldsymbol{u}_{rel} = \boldsymbol{u}_{fluid} - \boldsymbol{u}_{CS} = (U_F \hat{\boldsymbol{\imath}}) - (\boldsymbol{0}) \text{ (in both cases)}, \tag{2}$$

$$|\boldsymbol{A}| = A \text{ (in both cases)}, \tag{3}$$

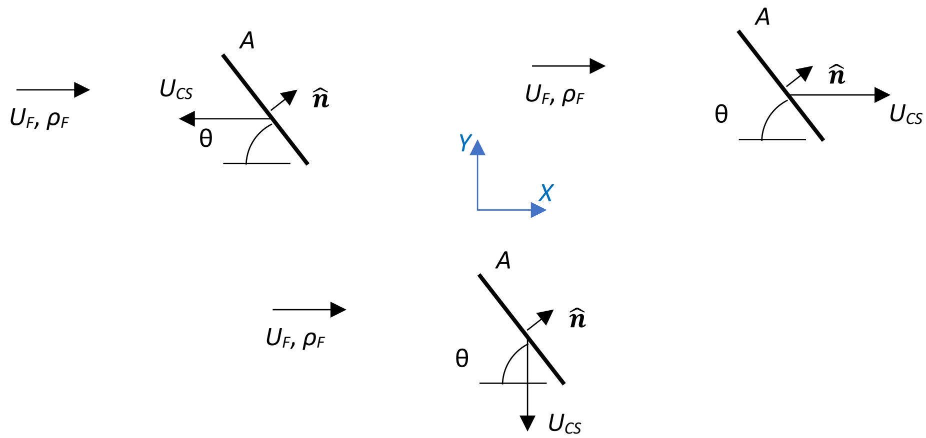
$$\hat{\boldsymbol{n}}_1 = \sin\theta\, \hat{\boldsymbol{\imath}} + \cos\theta\, \hat{\boldsymbol{\jmath}} \text{ (left)}, \tag{4}$$

$$\hat{\boldsymbol{n}}_2 = -\sin\theta\, \hat{\boldsymbol{\imath}} - \cos\theta\, \hat{\boldsymbol{\jmath}} \text{ (right)}. \tag{5}$$

$$\dot{m}_1 = \rho_F U_F A \sin\theta \text{ (left)}. \tag{6}$$

$$\dot{m}_2 = -\rho_F U_F A \sin\theta \text{ (right)}. \tag{7}$$

Determine the mass flow rate through the moving surfaces shown below. Assume the velocities are uniform over the area.



SOLUTION:
The mass flow rate is,

$$\dot{m} = \rho \boldsymbol{u}_{rel} \cdot \boldsymbol{A}, \quad (1)$$

where,

$$\boldsymbol{u}_{rel,1} = \boldsymbol{u}_{fluid} - \boldsymbol{u}_{CS} = (U_F \hat{\boldsymbol{\imath}}) - (-U_{CS} \hat{\boldsymbol{\imath}}) \quad \text{(upper left)}, \quad (2)$$

$$\boldsymbol{u}_{rel,2} = \boldsymbol{u}_{fluid} - \boldsymbol{u}_{CS} = (U_F \hat{\boldsymbol{\imath}}) - (U_{CS} \hat{\boldsymbol{\imath}}) \quad \text{(upper right)}, \quad (3)$$

$$\boldsymbol{u}_{rel,3} = \boldsymbol{u}_{fluid} - \boldsymbol{u}_{CS} = (U_F \hat{\boldsymbol{\imath}}) - (-U_{CS} \hat{\boldsymbol{\jmath}}) \quad \text{(bottom)}. \quad (4)$$

and, in all cases,

$$\boldsymbol{A} = \sin\theta \, \hat{\boldsymbol{\imath}} + \cos\theta \, \hat{\boldsymbol{\jmath}}, \quad (5)$$

Substituting and performing the dot products,

$$\dot{m}_1 = \rho_F A (U_F + U_{CS}) \sin\theta, \quad (6)$$

$$\dot{m}_2 = \rho_F A (U_F - U_{CS}) \sin\theta, \quad (7)$$

$$\dot{m}_3 = \rho_F A (U_F \sin\theta + U_{CS} \cos\theta). \quad (8)$$