Pipe Flows – The Extended Bernoulli Equation

Dogfish Head Craft Brewery

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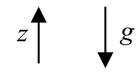
The 1st Law of Thermodynamics for a system:

$$\frac{D}{Dt}\left(E_{sys}\right) = \dot{Q}_{into\ sys} + \dot{W}_{on\ sys} \quad \text{where } E_{sys} = \int_{V_{sys}} e\rho dV \text{ and } e = u + \frac{1}{2}V^2 + gz$$



Use the Reynolds Transport Theorem to write in a control volume perspective:

$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS} e(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A}) = \dot{Q}_{into\ CV} + \dot{W}_{on\ CV}$$

Substitute for the specific total energy into the CS integral:

$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS}\left(u + \frac{1}{2}V^2 + gz\right)(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A}) = \dot{Q}_{into\ CV} + \dot{W}_{on\ CV}$$

Express the rate of pressure work separately on the right hand side:

$$\dot{W}_{on\ CV} = \dot{W}_{pressure,on\ CV} + \dot{W}_{other,on\ CV}$$

$$\dot{W}_{pressure,on\ CV} = \int_{CS} -p d\boldsymbol{A} \cdot \boldsymbol{u}_{rel} = \int_{CS} -\frac{p}{\rho}(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A})$$

Substitute and re-arrange:

$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS}\left(u + \frac{p}{\rho} + \frac{1}{2}V^2 + gz\right)(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A}) = \dot{Q}_{into\ CV} + \dot{W}_{other,on\ CV}$$

Assume steady flow with one inlet and one outlet

$$\int_{CS,out}\left(u + \frac{p}{\rho} + \frac{1}{2}V^2 + gz\right)(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A}) - \int_{CS,in}\left(u + \frac{p}{\rho} + \frac{1}{2}V^2 + gz\right)(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A}) = \dot{Q}_{into\ CV} + \dot{W}_{other,on\ CV}$$

Note that the mass flow rate at the inlet and outlet will be identical from Conservation of Mass.

Assume uniform properties at the inlet and outlet (density, internal energy, pressure, but not velocity) and that the elevation change over the inlet and outlet is negligible. Also re-write the velocity integral using an average velocity and a kinetic energy correction factor, α:

$$\alpha\frac{1}{2}\bar{V}^2\dot{m} = \int_{CS}\frac{1}{2}V^2(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A})$$

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$$\left(\frac{p}{\rho} + \alpha\frac{1}{2}\bar{V}^2 + gz\right)_{out} = \left(\frac{p}{\rho} + \alpha\frac{1}{2}\bar{V}^2 + gz\right)_{in} - \frac{(u_{out} - u_{in}) - \dot{Q}_{into\ CV}}{\dot{m}} + \frac{\dot{W}_{other,on\ CV}}{\dot{m}}$$

$$\left(\frac{p}{\rho} + \alpha\frac{1}{2}\bar{V}^2 + gz\right)_{out} = \left(\frac{p}{\rho} + \alpha\frac{1}{2}\bar{V}^2 + gz\right)_{in} - h_L + h_S$$

or

$$\boxed{\left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_{out} = \left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_{in} - H_L + H_S}$$

$$h_L \equiv \frac{(u_{out} - u_{in}) - \dot{Q}_{into\ CV}}{\dot{m}}$$

$$h_S \equiv \frac{\dot{W}_{other,on\ CV}}{\dot{m}}$$

$$H_L \equiv \frac{h_L}{g}$$

$$H_S \equiv \frac{h_S}{g} = \frac{\dot{W}_{other,on\ CV}}{\dot{m}g}$$