CHAPTER 2

Fluid Statics

2.1. Hydrostatic Pressure Variation

For a flow in which there are no velocity gradients, e.g., $du/dy = 0$, such as a static fluid, the shear stresses are zero. Draw a free body diagram of a (differentially) small piece of fluid, with width, height, and depth of dx, dy, and dz, respectively, under static conditions. Note that only pressure forces and weight will act on the fluid element. If we assume that the pressure, density, and gravitational acceleration at the center of the element are p, ρ, and g, respectively, then the free body diagram looks as shown in Figure 2.1 (making use of the Taylor Series approximation discussed in Chapter 1).

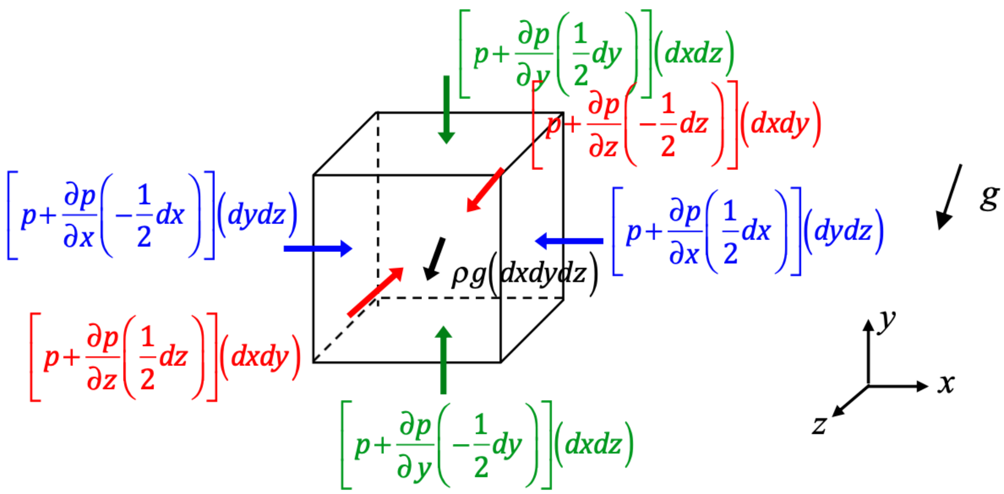


Figure 2.1. A free body diagram of a differentially-small fluid element with no shear stresses.

Summing forces in each of the three directions and noting that the fluid element is static,

$$\sum F_x = 0 = \left[p + \frac{\partial p}{\partial x}\left(-\frac{1}{2}dx\right)\right](dydz) - \left[p + \frac{\partial p}{\partial x}\left(\frac{1}{2}dx\right)\right](dydz) + \underbrace{\rho dxdydz}_{=dm} g_x, \tag{2.1}$$

$$0 = -\frac{\partial p}{\partial x}dxdydz + \rho dxdydz g_x, \tag{2.2}$$

$$0 = -\frac{\partial p}{\partial x} + \rho g_x. \tag{2.3}$$

A similar approach can be taken in the y and z directions,

$$0 = -\frac{\partial p}{\partial y} + \rho g_y, \tag{2.4}$$

$$0 = -\frac{\partial p}{\partial z} + \rho g_z. \tag{2.5}$$

The last three equations may be written more compactly in vector form as,

$$\boxed{\boldsymbol{\nabla} p = \rho \boldsymbol{g}}. \quad \text{Force balance for a static fluid particle.} \tag{2.6}$$

What this equation tells us is that for a static piece of fluid, a difference in pressure is required to balance the weight of the fluid particle.

Now consider the case where the gravitational acceleration points in the positive y direction, $g = g\hat{\boldsymbol{j}}$, so that the y-component of Eq. (2.6) is,

$$\frac{dp}{dy} = \rho g. \tag{2.7}$$

Note that the x and z components indicate that there is no change in pressure in those directions since there is no component of weight to balance, i.e., the pressure only changes in the y direction (hence the use of an ordinary derivative in Eq. (2.7) as opposed to a partial derivative since $p = p(y)$).

2.1.1. Hydrostatic Pressure Variation in an Incompressible Fluid

To determine how the pressure varies in the y direction, we must solve the differential equation in Eq. (2.7),

$$dp = \rho g dy \implies \int_{p=p_0}^{p=p} dp = \int_{y=0}^{y=y} \rho g dy \implies p - p_0 = \int_0^y \rho g dy. \tag{2.8}$$

In order to solve the integral on the right-hand side of the previous equation, we must know how the density and gravitational acceleration vary with y. It's reasonable in most applications to assume that the gravitational acceleration is constant, so it can be pulled outside the integral. If we further assume that we're dealing with an incompressible fluid, then the density can also be pulled outside the integral and we're left with,

$$p - p_0 = \rho g \int_0^y dy = \rho g y, \tag{2.9}$$

$$\boxed{p = p_0 + \rho g y} \quad \text{or, alternately,} \quad \Delta p = \rho g \Delta y. \tag{2.10}$$

The previous equation is the hydrostatic pressure variation in an incompressible fluid in which gravity points in the positive y direction.

Notes:

(1) The pressure in Eq. (2.10) only changes when there are variations in elevation in the direction of gravity (the y direction). Moving perpendicular to the direction of gravity does not change the pressure.
(2) The pressure increases linearly in the direction of the gravitational acceleration. A plot of this variation is shown in Figure 2.2.

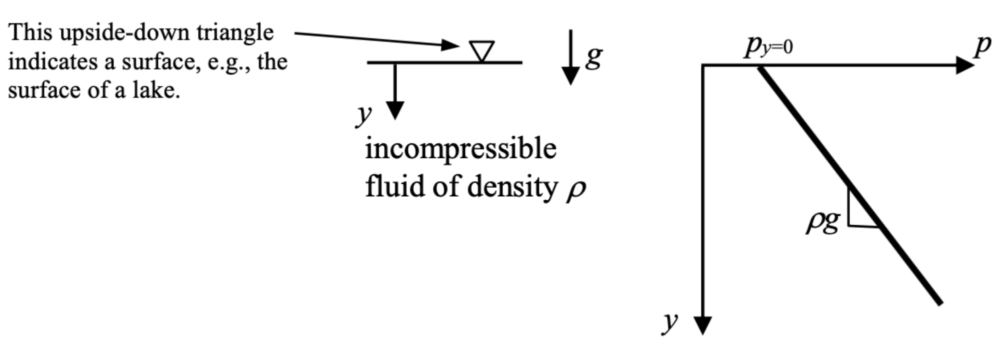


Figure 2.2. The hydrostatic pressure plotted against depth for an incompressible fluid in a constant gravity field.

(3) As mentioned previously, the reason the pressure increases with depth is because the pressure must balance the weight of all the fluid sitting above it.

(4) Since changes in pressure correspond to changes in elevation (refer to Eq. (2.10)), pressure differences are often expressed in terms of lengths, or depths of fluid. For example, the standard atmospheric pressure of 101 kPa (abs) corresponds to 760 mmHg,

$$\underbrace{101\,\text{kPa}}_{\Delta p} = \underbrace{(13\,600\,\text{kg/m}^3)}_{=\rho_{\text{Hg}}} \underbrace{(9.81\,\text{m/s}^2)}_{=g} \underbrace{(760 \times 10^{-3}\,\text{m})}_{=\Delta y}. \tag{2.11}$$

(5) When measuring pressure differences in Eq. (2.10), either both pressures must be absolute pressures or both must be gage. Don't mix gage and absolute pressures.

2.1.2. Hydrostatic Pressure Variation in a Compressible Fluid

Now consider the pressure variation in a compressible fluid. This case would be of particular interest for airplanes, rockets, and mountain climbers where large changes of elevation in the atmosphere are common. Recall from Eq. (2.7),

$$\frac{dp}{dy} = \rho g \quad (y \text{ and } g \text{ point in the same direction}). \tag{2.12}$$

For convenience, since we typically deal with elevation or altitude instead of depth when considering the hydrostatic pressure variation in compressible fluids like air, let's change our coordinate system so that y points in the direction *opposite* to gravity. Thus,

$$\frac{dp}{dy} = -\rho g \quad (y \text{ and } g \text{ point in opposite directions}). \tag{2.13}$$

If we're dealing with an ideal gas (like air), then the pressure and density are related via the Ideal Gas Law,

$$p = \rho R T \implies \rho = \frac{p}{RT}, \tag{2.14}$$

where R is the gas constant. Substituting Eq. (2.14) into Eq. (2.13) and re-arranging gives,

$$\frac{dp}{p} = -\frac{g}{RT}dy. \tag{2.15}$$

Numerous measurements have been made of the average atmospheric temperature as a function of altitude, i.e., $T = T(y)$, and can be substituted into Eq. (2.15). For example, the temperature variation with altitude from the U.S. Standard Atmosphere (standardized in 1976) is shown in Figure 2.3. In each region of the atmosphere, the temperature varies linearly and can be expressed as,

$$T = T_a - \beta y, \tag{2.16}$$

where T_a and β (known as the temperature lapse rate) are constants and y is the altitude measured from sea level. Table 2.1 lists these constants for the different parts of the atmosphere. For example, from sea level to an altitude of 11000 m, the temperature decreases by 6.5 K/km.

Table 2.1. Pressure (p_a), temperature (T_a), and temperature lapse rate (β), starting at different altitudes from the (1976) U.S. Standard Atmosphere.

y (m)	**p_a (Pa (abs))**	**T_a (K)**	**β (K/m)**
0	101325	288.15	0.0065
11000	22632.1	216.65	0.0
20000	5474.89	216.65	-0.001
32000	868.019	228.65	-0.0028
47000	110.906	270.65	0.0
51000	66.9389	270.65	0.0028
71000	3.95642	214.65	0.002

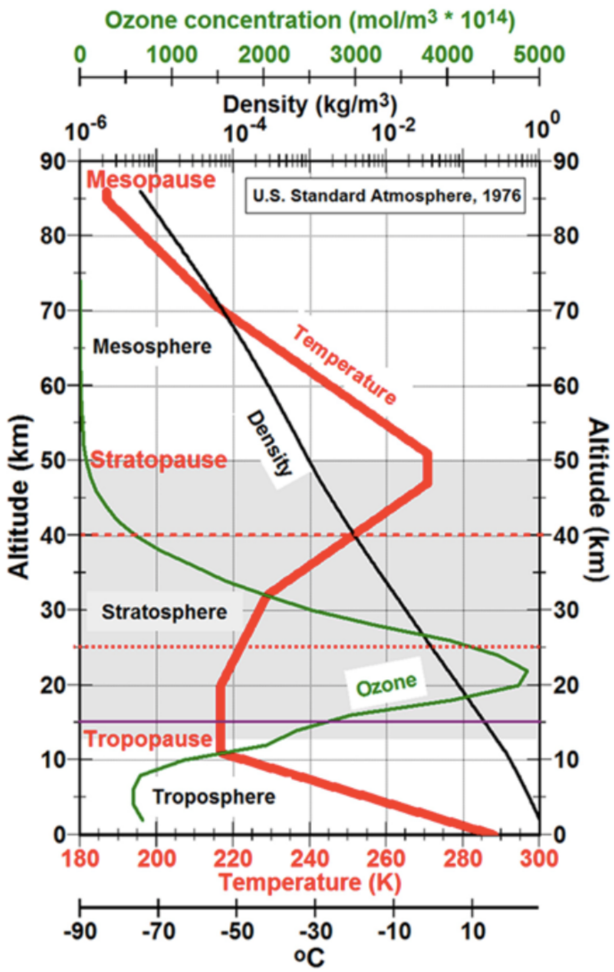


Figure 2.3. The (1976) U.S. Standard Atmosphere.

If we substitute Eq. (2.16) into Eq. (2.15) and solve the differential equation, we get,

$$\frac{dp}{p} = -\frac{g}{R(T_a - \beta y)}dy \implies \int_{p_0}^{p} \frac{dp}{p} = -\frac{g}{R}\int_0^y \frac{dy}{T_a - \beta y} \implies \ln\left(\frac{p}{p_a}\right) = \frac{g}{\beta R}\ln\left(\frac{T_a - \beta y}{T_a}\right), \tag{2.17}$$

$$\boxed{\frac{p}{p_a} = \left(1 - \frac{\beta y}{T_a}\right)^{\frac{g}{\beta R}}}. \tag{2.18}$$

Notes:

(1) The temperature lapse rate in the troposphere is $\beta = 6.5\,\mathrm{K\,km^{-1}}$ ($\approx 3.57\,^\circ\mathrm{F}/1000\,\mathrm{ft}$). Thus, a handy rule-of-thumb when hiking in the mountains (or deep canyons) is that the temperature will decrease $6.5\,^\circ\mathrm{C}$ for every 1 km of elevation gain or approximately $3.5\,^\circ\mathrm{F}$ for every 1000 ft of elevation gain.

(2) Large changes in elevation are required to make appreciable changes in pressure in a gas, such as air. For example, estimate the altitude change required to drop the pressure by 1% using the U.S. Standard Atmosphere in the troposphere,

$$\frac{p}{p_a} = 0.99 = \left(1 - \frac{\beta y}{T_a}\right)^{\frac{g}{\beta R}} \implies (0.99)^{\frac{\beta R}{g}} = 1 - \frac{\beta y}{g} \implies y = \frac{T_a}{\beta}\left[1 - (0.99)^{\frac{\beta R}{g}}\right], \tag{2.19}$$

which gives $y = 85\,\text{m}$ using the troposphere values for the constants. Thus, it's reasonable to assume that unless large elevation differences occur, the pressure does not vary with elevation in the atmosphere, or any gas for that matter. However, even small elevation differences in liquids do result in appreciable pressure changes since the density of liquids is much larger than the density of gases.

What is the pressure at the bottom of the Marianas Trench (11,000 m = 36,201 ft = 6.9 mi)?

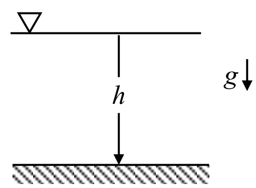
SOLUTION:

The pressure at the bottom of the Marianas Trench, assuming salt water to be incompressible, is:

$$p_{\text{bottom}} = p_{\text{top}} + \rho_{\text{saltH20}} g h$$

where

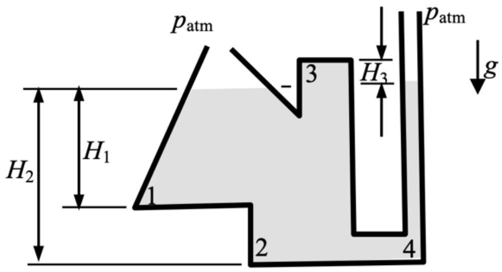
p_{top} = 101 kPa (abs)
ρ_{saltH20} = 1025 kg/m^3
g = 9.81 m/s^2
h = 11000 m



Hence, the pressure at the bottom is:

$\boxed{p_{\text{bottom}} = 110 \text{ MPa} = 1100 \text{ atm!}}$

Determine the pressure at points 1, 2, 3, and 4.



SOLUTION:
Recall that the shape of the container doesn't matter when calculating hydrostatic pressure. It's only the depth of the fluid that matters.

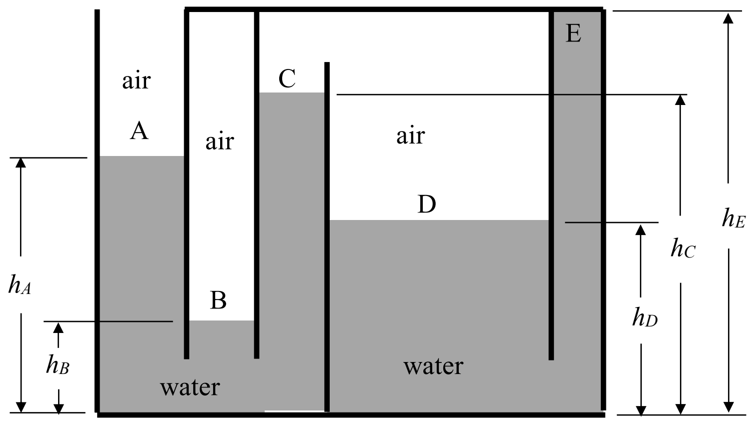
$p_1 = p_{atm} + \rho g H_1,$ (1)

$p_2 = p_{atm} + \rho g H_2,$ (2)

$p_3 = p_{atm} - \rho g H_3,$ (3)

$p_4 = p_{atm} + \rho g H_2$ (Point 4 is at the same depth as point 2.) (4)

Determine the gage pressure at points B, C, D, and E in the system shown below.



h_A = 6 m
h_B = 2 m
h_C = 7 m
h_D = 5 m
h_E = 10 m

SOLUTION:

First determine the pressure at point B,

$$\boxed{p_B = p_A + \rho g(h_A - h_B)}. \tag{1}$$

Note that the pressure at A is $p_A = p_{\text{atm}}$.

Now determine the gage pressure at C using the known pressure at B,

$$\boxed{p_C = p_B - \rho g(h_C - h_B)} \tag{2}$$

The pressure at point D will be the same as the pressure at point C since both contact the same air and we're assuming the variations in air pressure over the small elevations in this problem are negligible,

$$\boxed{p_D = p_C.} \tag{3}$$

The pressure at point E is,

$$\boxed{p_E = p_D - \rho g(h_E - h_D)}. \tag{4}$$

Using the given data,

p_A = p_{atm} = 0 (gage)
ρ = 1000 kg/m^3
g = 9.81 m/s^2
h_A = 6 m
h_B = 2 m
h_C = 7 m
h_D = 5 m
h_E = 10 m

$\Rightarrow$
p_B = 39.2 kPa (gage)
p_C = -9.8 kPa (gage)
p_D = -9.8 kPa (gage)
p_E = -58.9 kPa (gage)

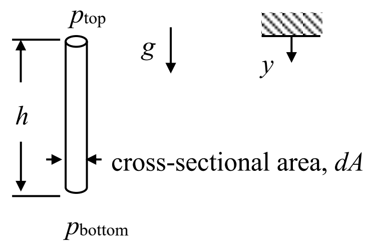
Assuming that air is incompressible, determine the height of a column of air required to give a pressure difference of 0.1 psi. Assume that the density of air is 2.38*10^{-3} slug/ft^3.

SOLUTION:

Assuming air as being incompressible:

$$p_{\text{bottom}} = p_{\text{top}} + \rho_{\text{air}} g h$$

$$\boxed{h = \frac{p_{\text{bottom}} - p_{\text{top}}}{\rho_{\text{air}} g}}$$



For:

$p_{\text{bottom}} - p_{\text{top}}$ = 0.1 psi = 14.4 lb$_f$/ft^2
ρ_{air} = 2.38*10^{-3} slug/ft^3
g = 32.2 ft/s^2

gives:

$\boxed{h = 188 \text{ ft}}$

Hence, very large elevation differences must occur to give appreciable differences in pressure when dealing with atmospheric air (or gases in general).

Another way to determine the height, h, is to perform a vertical force balance on the column.

$$\sum F_y = 0 = -p_{\text{bottom}} dA + p_{\text{top}} dA + \rho_{\text{air}} g h dA$$

$$h = \frac{p_{\text{bottom}} - p_{\text{top}}}{\rho_{\text{air}} g} \quad \text{(Same answer as above!)}$$

What is the air pressure at the top of the Burj Khalifa, which has a height of 828 m (2717 ft)? If there was a pipe containing water that extended from the top of the Burj Khalifa to the ground, what would be the gage pressure in the water at the bottom of the pipe?

SOLUTION:

Assuming constant air density,

$$p_{y=H} = p_{y=0} - \rho_{\text{air}} g H \quad (1)$$

where

$\rho = 1.225$ kg/m^3
$g = 9.81$ m/s^2
$H = 828$ m
$p_{y=0} = 101.33$ kPa (abs) (= p_{atm})

Thus, $p_{y=H} = 91.4$ kPa (abs) or $p_{y=H}/p_{y=0} = 0.902$.

If we treat air as a compressible, ideal gas and assume the air temperature varies according to the U.S. Standard Atmosphere,

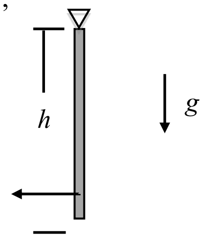
$$p_y = p_{y=0}\left(1 - \frac{\beta y}{T_{y=0}}\right)^{\frac{g}{R\beta}} \quad (2)$$

where the previous values for g, $p_{y=0}$, and H are assumed, and,

$\beta = 0.00650$ K/m
$T_{y=0} = 288$ K (= 15 degC)
$R = 286.9$ J/(kg.K)
$\Rightarrow$ $p/p_{y=0} = 0.906$, which is nearly identical to the previous calculation.

The gage pressure in a water column with a depth of 828 m is given by,

$$p_{gage} = \rho g h, \quad (3)$$



where

ρ = 1000 kg/m^3
g = 9.81 m/s^2
h = 828 m
$\Rightarrow$ $\boxed{p_{gage} = 8.12*10^6 \text{ Pa}}$ = 80.2 atm!

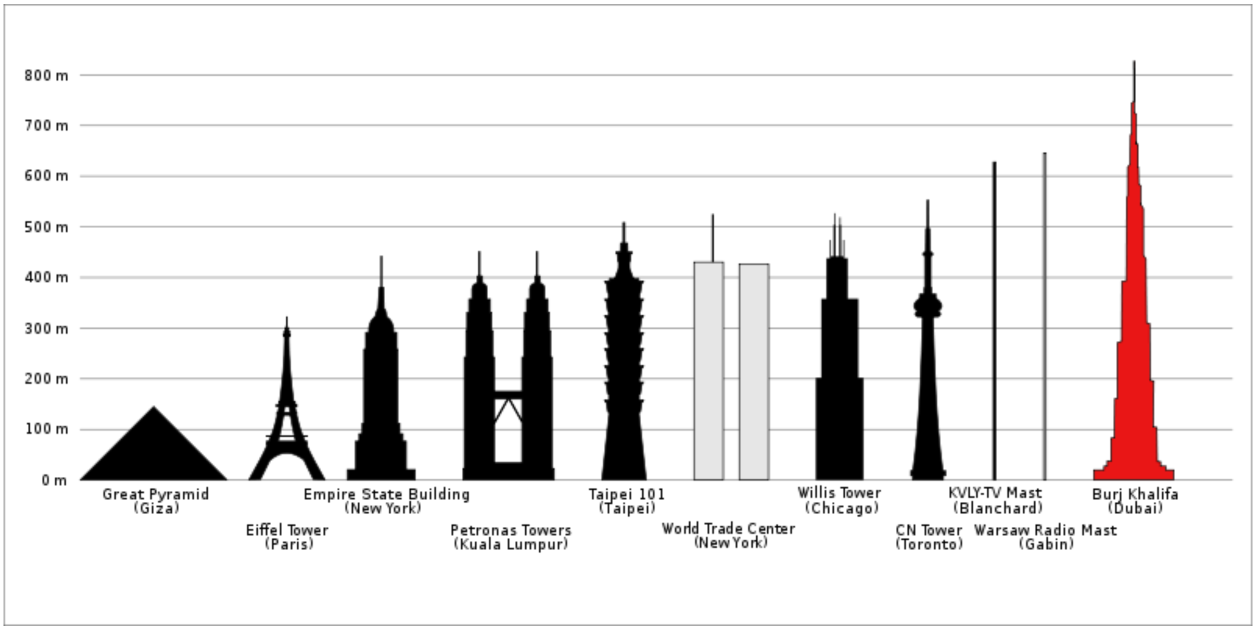
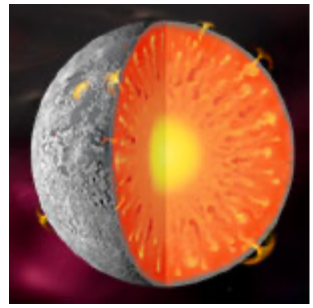


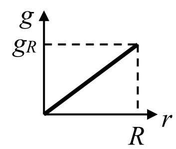
Image from Wikipedia (2012 Jan 10; http://en.wikipedia.org/wiki/File:BurjKhalifaHeight.svg)

It is often conjectured that the Earth was, at one time, comprised of molten material. If the acceleration due to gravity within this fluid sphere (with a radius of 6440 km) varied linearly with distance, r, from the Earth's center, the acceleration due to gravity at r = 6440 km was 9.81 m/s^2, and the density of the fluid was uniformly 5600 kg/m^3, determine the gage pressure at the center of this fluid Earth.



SOLUTION:

Since the acceleration due to gravity, g, varies linearly with r:



$$g = cr \quad (1)$$

where c is a constant. Since $g(r = R = 6440 \text{ km}) = g_R = 9.81$ m/s^2:

$$c = \frac{g_R}{R} = \frac{9.81 \text{ m/s}^2}{6440*10^3 \text{ m}} = 1.523*10^{-6} \text{ s}^{-2} \quad (2)$$

From the hydrostatic pressure distribution (neglecting the curvature of the Earth):

$$\frac{dp}{dr} = -\rho g \quad (3)$$

Substitute Eqn. (1) and solve the resulting differential equation.

$$\frac{dp}{dr} = -\rho cr \Rightarrow \int_{p=p_0}^{p=0} dp = -\rho c \int_{r=0}^{r=R} r dr \quad (4)$$

$$\boxed{\therefore p_0 = \tfrac{1}{2}\rho c R^2} \quad (5)$$

Using the given data:

ρ = 5600 kg/m^3
c = 1.523*10^{-6} s^{-2}
R = 6.440*10^6 m
$\boxed{p_0 = 1.769*10^{11} \text{ Pa} = 1.769*10^{6} \text{ atm}}$