CHAPTER 4

Integral Analysis

4.1. Lagrangian and Eulerian Perspectives

There are two common ways to study a moving fluid:

(1) Look at a particular location and observe how all the fluid passing that location behaves. This is called the Eulerian point of view.
(2) Look at a particular piece of fluid and observe how it behaves as it moves from location to location. This is called the Lagrangian point of view.

For example, Let's say we want to study migrating birds. We could either:

(1) stand in a fixed spot and make measurements as birds fly by (Eulerian point of view; Figure 4.1), or
(2) tag some of the birds and make measurements as they fly along (Lagrangian point of view; Figure 4.1).

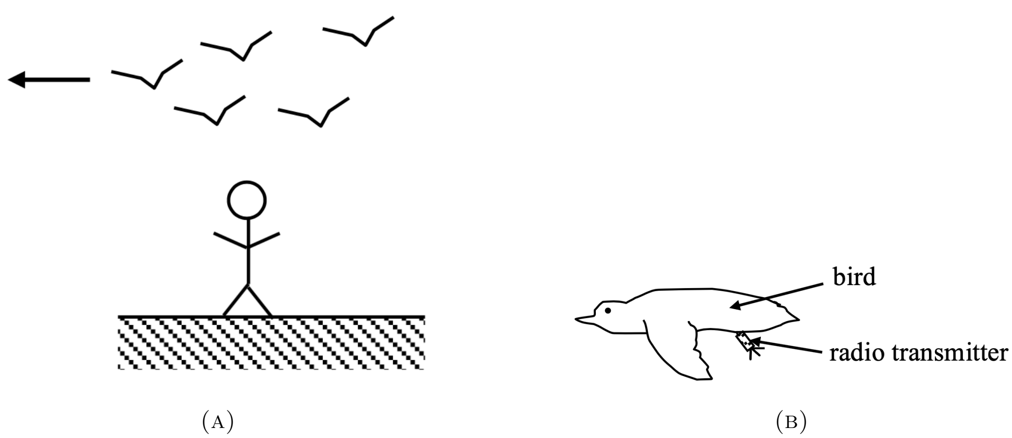


Figure 4.1. Studying birds using (A) Eulerian and (B) Lagrangian approaches.

4.1.1. Lagrangian (aka Material, Particle, Substantial) Derivative

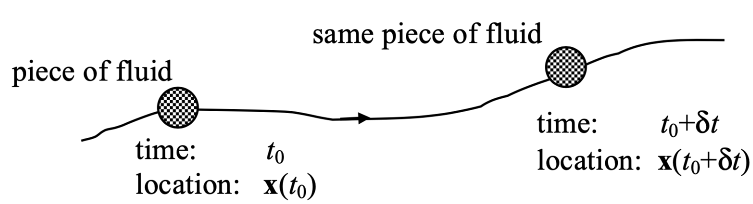


Figure 4.2. An illustration showing the path of a fluid particle.

If we follow a piece of fluid (Lagrangian viewpoint), how will some property of that particular piece of fluid change with respect to time? Let's say we're interested in looking at the time rate of change of temperature, T, that the particle observes as it moves from location to location (Figure 4.2. The particle may experience a temperature change because the temperature of the entire field of fluid may be changing with respect to time (i.e., the temperature field may be unsteady). In addition, the temperature field may have spatial gradients (different temperatures at different locations, i.e., non-uniform) so that as the particle moves from point to point it will experience a change in temperature. Thus, there are two effects that can cause a time rate of change of temperature that the particle experiences: unsteady effects, also known as local or Eulerian effects, and spatial gradient effects, also known as convective effects. We can describe this in mathematical terms by writing the temperature of the entire field as a function of time, t, and location, $\mathbf{x}$,

$$T = T(t, \mathbf{x}). \tag{4.1}$$

Note that the location of the fluid particle is a function of time: $\mathbf{x} = \mathbf{x}(t)$ so that,

$$T = T(t, \mathbf{x}(t)). \tag{4.2}$$

Taking the time derivative of the temperature, expanding the location vector into its x, y, and z components, and using the chain rule gives,

$$\left.\frac{dT}{dt}\right|_{\substack{\text{following a}\\ \text{fluid particle}}} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\underbrace{\frac{dx}{dt}}_{=u_x} + \frac{\partial T}{\partial y}\underbrace{\frac{dy}{dt}}_{=u_y} + \frac{\partial T}{\partial z}\underbrace{\frac{dz}{dt}}_{=u_z}. \tag{4.3}$$

Note that dx/dt, dy/dt, and dz/dt are the particle velocities u_x, u_y, and u_z respectively. Writing this in a more compact form,

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z}, \tag{4.4}$$

$$= \frac{\partial T}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,T. \tag{4.5}$$

The notation, D/Dt, indicating a Lagrangian (also sometimes referred to as the material, particle, or substantial) derivative, has been used in Eq. (4.4) to indicate that we're following a particular piece of fluid. More generally, we have,

$$\underbrace{\frac{D}{Dt}(\ldots)}_{\substack{\text{Lagrangian rate of}\\ \text{change (changes as we}\\ \text{follow a fluid particle)}}} = \underbrace{\frac{\partial}{\partial t}(\ldots)}_{\substack{\text{local or Eulerian}\\ \text{rate of change (changes}\\ \text{due to unsteady effects)}}} + \underbrace{(\mathbf{u}\cdot\boldsymbol{\nabla})(\ldots)}_{\substack{\text{convective rate}\\ \text{of change (changes due to}\\ \text{a change in particle position)}}}, \tag{4.6}$$

$$= \frac{\partial}{\partial t}(\ldots) + u_x\frac{\partial}{\partial x}(\ldots) + u_y\frac{\partial}{\partial y}(\ldots) + u_z\frac{\partial}{\partial z}(\ldots). \tag{4.7}$$

where $(\ldots)$ represents any field quantity of interest.

Notes:

(1) The Lagrangian derivatives in cylindrical and spherical coordinates are,

$$\text{cylindrical:}\quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}, \tag{4.8}$$

$$\text{spherical:}\quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}. \tag{4.9}$$

(4.10)

(2) The acceleration experienced by a fluid particle is given by,

Cartesian: $$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + u_r \frac{\partial \mathbf{u}}{\partial r} + \frac{u_\theta}{r}\frac{\partial \mathbf{u}}{\partial \theta} + u_z \frac{\partial \mathbf{u}}{\partial z}, \tag{4.11}$$

cylindrical: $$\begin{cases} a_r = \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \\ a_\theta = \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} - \frac{u_r u_\theta}{r} \\ a_z = \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \end{cases} \tag{4.12}$$

spherical: $$\begin{cases} a_r = \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{1}{r}\left(u_\theta^2 + u_\phi^2\right) \\ a_\theta = \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{1}{r}\left(u_r u_\theta - u_\phi^2 \cot\theta\right) \\ a_\phi = \frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r}\left(u_r u_\phi + u_\theta u_\phi \cot\theta\right) \end{cases} \tag{4.13}$$

a. A fluid velocity field is given by:

$$\mathbf{u} = 2t\hat{\mathbf{e}}_x$$

Will a fluid particle accelerate in this flow? Why?

b. Now consider the following flow:

$$\mathbf{u} = x\hat{\mathbf{e}}_x$$

Will a fluid particle accelerate in this flow? Why?

SOLUTION:

Part (a):

The acceleration is given by:

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{=2\hat{\mathbf{e}}_x} + \underbrace{u_x}_{2t} \underbrace{\frac{\partial \mathbf{u}}{\partial x}}_{=\mathbf{0}} + \underbrace{u_y \frac{\partial \mathbf{u}}{\partial y}}_{=\mathbf{0}} + \underbrace{u_z \frac{\partial \mathbf{u}}{\partial z}}_{=\mathbf{0}}$$

Hence, for the given flow:

$\boxed{\mathbf{a} = 2\hat{\mathbf{e}}_x}$ Yes, fluid particles will accelerate due to the local (or Eulerian) derivative.

Part (b):

The acceleration is given by:

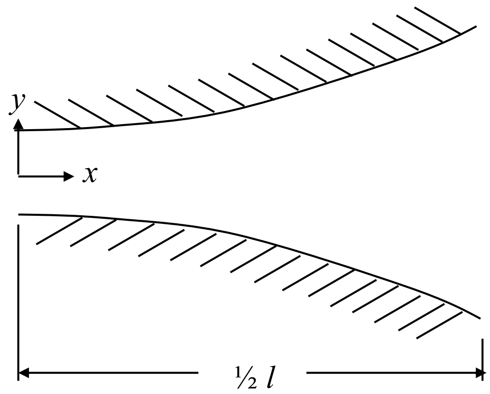
$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{=\mathbf{0}} + \underbrace{u_x}_{x} \underbrace{\frac{\partial \mathbf{u}}{\partial x}}_{=\hat{\mathbf{e}}_x} + \underbrace{u_y \frac{\partial \mathbf{u}}{\partial y}}_{=\mathbf{0}} + \underbrace{u_z \frac{\partial \mathbf{u}}{\partial z}}_{=\mathbf{0}}$$

Hence, for the given flow:

$\boxed{\mathbf{a} = x\hat{\mathbf{e}}_x}$ Yes, fluid particles will accelerate due to the convective derivative.

For the diffuser shown below, determine:

a. the acceleration of a fluid particle for any x and t, and
b. the value of c (other than c=0) for which the acceleration is zero for any x at t=2 s. Assume V_0=10 ft/s and l=5 ft.
c. Explain how the acceleration can be zero if the flow rate is increasing with time.



$$\mathbf{u} = V_0(1-e^{-ct})(1-\frac{x}{l})\hat{\mathbf{i}}$$

where V_0, c, and l are constants

SOLUTION:

The acceleration of a fluid particle is the Lagrangian derivative of the velocity.

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + u\frac{\partial \mathbf{u}}{\partial x} \quad (1)$$

Substitute the given velocity field.

$$\mathbf{a} = \frac{\partial}{\partial t}\left[V_0\left(1-e^{-ct}\right)\left(1-\frac{x}{l}\right)\hat{\mathbf{i}}\right] + \left[V_0\left(1-e^{-ct}\right)\left(1-\frac{x}{l}\right)\right]\frac{\partial}{\partial x}\left[V_0\left(1-e^{-ct}\right)\left(1-\frac{x}{l}\right)\hat{\mathbf{i}}\right]$$

$$= V_0 c e^{-ct}\left(1-\frac{x}{l}\right)\hat{\mathbf{i}} + \left[V_0\left(1-e^{-ct}\right)\left(1-\frac{x}{l}\right)\right]\left[-\frac{V_0}{l}\left(1-e^{-ct}\right)\hat{\mathbf{i}}\right]$$

$$\boxed{\therefore \mathbf{a} = V_0\left(1-\frac{x}{l}\right)\left[ce^{-ct} - \frac{V_0}{l}\left(1-e^{-ct}\right)^2\right]\hat{\mathbf{i}}} \quad (2)$$

At t = 2 s:

$$\mathbf{a}(x, t=2) = \mathbf{0} = V_0\left(1-\frac{x}{l}\right)\left[ce^{-2c} - \frac{V_0}{l}\left(1-e^{-2c}\right)^2\right]\hat{\mathbf{i}}$$

$$ce^{-2c} = \frac{V_0}{l}\left(1-e^{-2c}\right)^2 \quad (3)$$

Solve numerically for c when V_0 = 10 ft/s and l = 5 ft.

$\Rightarrow \boxed{c = 0.124 \text{ s}^{-1}}$

The acceleration of a fluid particle can be zero even though the flow rate is increasing because the local acceleration ($\partial u/\partial t$) exactly balances the convective deceleration ($u\partial u/\partial x$).

A fluid velocity field is given by,

$$\mathbf{u} = (cy^2)\hat{\mathbf{i}} + (cx^2)\hat{\mathbf{j}},$$

where c is a constant. Determine

a. the components of the acceleration and
b. the points in the flow field where the acceleration is zero.

SOLUTION:

The acceleration of a fluid element is given by,

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + u_x \frac{\partial \mathbf{u}}{\partial x} + u_y \frac{\partial \mathbf{u}}{\partial y} \tag{1}$$

where,

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{0} \quad \text{(steady flow)}$$

$$u_x \frac{\partial \mathbf{u}}{\partial x} = \left(cy^2\right)\left(2cx\hat{\mathbf{j}}\right) = 2c^2xy^2\hat{\mathbf{j}}$$

$$u_y \frac{\partial \mathbf{u}}{\partial y} = \left(cx^2\right)\left(2cy\hat{\mathbf{i}}\right) = 2c^2x^2y\hat{\mathbf{i}}$$

$$\boxed{\therefore \mathbf{a} = 2c^2x^2y\hat{\mathbf{i}} + 2c^2xy^2\hat{\mathbf{j}}} \tag{2}$$

Set the acceleration equal to zero,

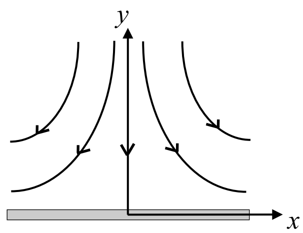
$$\mathbf{a} = \mathbf{0} = 2c^2x^2y\hat{\mathbf{i}} + 2c^2xy^2\hat{\mathbf{j}}$$

$$\boxed{\therefore \text{either } x = 0 \text{ or } y = 0} \text{ (This is locus of points where the total acceleration is zero.)} \tag{3}$$

The velocity field near a planar stagnation point (see the figure below) is given as,

$$\mathbf{u} = U_0\left(\frac{x}{L}\right)\hat{\mathbf{e}}_x - U_0\left(\frac{y}{L}\right)\hat{\mathbf{e}}_y \qquad \text{where } U_0 \text{ and } L \text{ are positive constants}$$

Determine the acceleration of a fluid particle along the line $x = 0$.



SOLUTION:

The acceleration of a fluid particle is,

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + u_x\frac{\partial \mathbf{u}}{\partial x} + u_y\frac{\partial \mathbf{u}}{\partial y}, \tag{1}$$

where,

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{0} \quad \text{(steady flow)}$$

$$u_x\frac{\partial \mathbf{u}}{\partial x} = \left[U_0\left(\frac{x}{L}\right)\right]\left[U_0\left(\frac{1}{L}\right)\hat{\mathbf{e}}_x\right] = U_0^2\left(\frac{x}{L^2}\right)\hat{\mathbf{e}}_x$$

$$u_y\frac{\partial \mathbf{u}}{\partial y} = \left[-U_0\left(\frac{y}{L}\right)\right]\left[-U_0\left(\frac{1}{L}\right)\hat{\mathbf{e}}_y\right] = U_0^2\left(\frac{y}{L^2}\right)\hat{\mathbf{e}}_y$$

$$\therefore \mathbf{a} = U_0^2\left(\frac{x}{L^2}\right)\hat{\mathbf{e}}_x + U_0^2\left(\frac{y}{L^2}\right)\hat{\mathbf{e}}_y \tag{2}$$

Along the line $x = 0$,

$$\boxed{\therefore \mathbf{a}(0, y) = U_0^2\left(\frac{y}{L^2}\right)\hat{\mathbf{e}}_y} \tag{3}$$

The market price, P (in dollars), of used cars of a certain model is found to be:

$$P = \$1000 + \left(\$0.02\,/\,\text{mile}\right)x - \left(\$2\,/\,\text{day}\right)t$$

where x is the distance in miles west of Detroit, MI and t is the time in days. If a car of this model is driven from Detroit at t=0 towards the west at a rate of 400 miles per day, determine:

a. whether the value of the car is increasing or decreasing, and
b. how much of this change is due to depreciation and how much is due to moving into a better market.

SOLUTION:

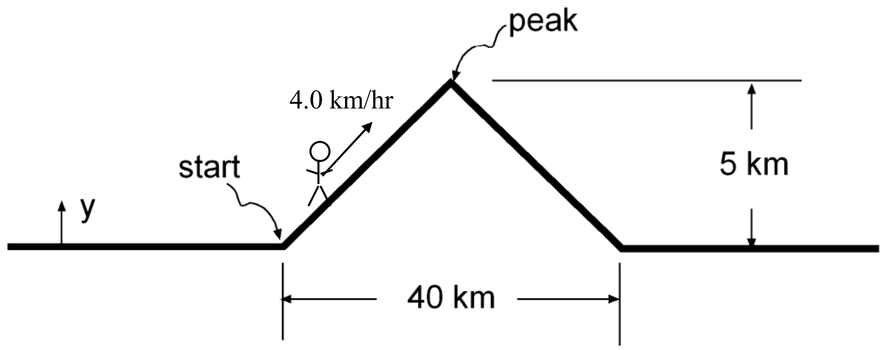
To determine if the value of the car is decreasing, take the Lagrangian derivative of the market price.

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} = \left(-\$2/\text{day}\right) + \underbrace{\left(400\text{ miles/day}\right)\left(\$0.02/\text{mile}\right)}_{=\$8/\text{day}} \quad \text{(where } u \text{ is the speed of the car)} \tag{1}$$

$$\boxed{\therefore \frac{DP}{Dt} = \$6/\text{day} \quad \text{Hence, the value of the car is increasing.}}$$

The car depreciates at a rate of -\$2/day (this is $\partial P/\partial t$). The change in the car's value increases at a rate of \$8/day due to moving into a different market (this is $u\partial P/\partial x$).

You are climbing up the side of Triangle Mountain, so named because the sides are relatively straight, making the cross-section of the mountain look like a triangle. The mountain is 5 km high and has a base of 40 km, as shown.



You are worried about how hot you will get on your trip and the rate at which the temperature will change with time. The temperature decreases with altitude at a rate of 5 °C/km. Also, the temperature changes in time as the sun heats the ground. The temperature (in °C) as you climb the mountain is given by:

$$T(y,t) = 25\ ^\circ\text{C} - \left(0.005\ ^\circ\text{C/m}\right) y + \left(5\ ^\circ\text{C}\right)\sin\left[\frac{2\pi(t-t_o)}{24\ \text{hrs}}\right]$$

where t is measured in hours from midnight, $t_o = 9$ hrs, and y is the altitude measured in meters from the base of the mountain.

You start ascending the mountain at 6:00 A.M. and travel at a speed of 4.0 km/hr up the mountain side.

a. Derive an expression for the time derivative of temperature you experience as you climb up the mountain.
b. Calculate the rate of change in temperature at the moment you reach the mountain peak (in °C/hr).

SOLUTION:

Write the Lagrangian time derivative, keeping in mind that the two variables of interest are t and y:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_y \frac{\partial T}{\partial y} \tag{1}$$

where

$$\frac{\partial T}{\partial t} = 5\ ^\circ\text{C} \cdot \frac{2\pi}{24\ \text{hrs}} \cos\left[\frac{2\pi(t-t_0)}{24\ \text{hrs}}\right], \tag{2}$$

$$u_y = \left(4.0\ \text{km/hr}\right)\left[\frac{5\ \text{km}}{\sqrt{(20\ \text{km})^2 + (5\ \text{km})^2}}\right] = 0.97\ \text{km/hr}, \tag{3}$$

$$\frac{\partial T}{\partial y} = -0.005\ ^\circ\text{C/m}. \tag{4}$$

Substitute and simplify.

$$\boxed{\frac{DT}{Dt} = \frac{5\pi\ ^\circ\text{C}}{12\ \text{hrs}} \cos\left[\frac{2\pi(t-t_0)}{24\ \text{hrs}}\right] - \left(4.85\ ^\circ\text{C/hr}\right)} \tag{5}$$

You reach the peak in 5.15 hrs (= 5 km / (0.97 km/hr)), which means you reach the peak at 11:09 A.M ($t = 11.15$ hrs). Evaluating Eqn. (5) using $t_0 = 9$ hrs gives, at the moment you reach the peak:

$$\boxed{DT/Dt = -3.74\ ^\circ\text{C/hr}} \tag{6}$$