7.5. Modeling and Similarity

Models are often used in fluid mechanics to predict the kinematics and dynamics of full-scale (often referred to as prototype) flows. From previous discussions of dimensional analysis, we observe that we can write the governing equations and boundary conditions of our flow in dimensionless terms (Π terms). Thus, if we have two different flows, e.g., a large-scale, prototype flow and a small scale, model flow, that have identical dimensionless parameters, then the same solution, also in terms of dimensionless parameters, will hold for both. This is extremely helpful when modeling fluid systems.

When a model and the prototype have the same dimensionless parameters, we say that they are similar. We typically discuss similarity in three categories: geometric, dynamic, and kinematic.

- Geometric similarity occurs when the model is an exact geometric replica of the prototype. In other words, all of the lengths in the model are scaled by exactly the same amount as in the prototype, as shown in Figure 7.7. Note that surface roughness may even need to be scaled if it is a significant

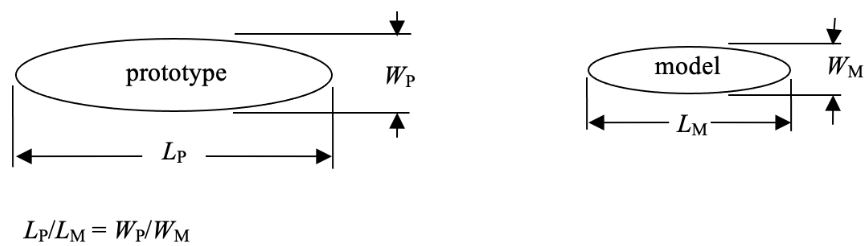


Figure 7.7. An illustration of geometric similarity between a model and prototype. All of the lengths are scaled by the same amount.

  factor in the flow.
- Dynamic similarity occurs when the ratio of forces in the model is the same as the ratio of significant forces in the prototype. For example,

$$(\text{ratio of unsteady to conv. inertial forces})_P = (\text{ratio of unsteady to conv. forces})_M \implies \mathrm{St}_P = \mathrm{St}_M, \tag{7.75}$$

$$(\text{ratio of inertial to viscous forces})_P = (\text{ratio of inertial to viscous forces})_M \implies \mathrm{Re}_P = \mathrm{Re}_M, \tag{7.76}$$

$$(\text{ratio of pressure to inertial forces})_P = (\text{ratio of pressure to inertial forces})_M \implies \mathrm{Eu}_P = \mathrm{Eu}_M, \tag{7.77}$$

$$(\text{ratio of inertial to grav. forces})_P = (\text{ratio of inertial to grav. forces})_M \implies \mathrm{Fr}_P = \mathrm{Fr}_M. \tag{7.78}$$

- Kinematic similarity occurs when the prototype and model fluid velocity fields have identical streamlines (but scaled speeds). Since the forces affect the fluid motion, geometric similarity and dynamic similarity will automatically ensure kinematic similarity.

Notes:

(1) When modeling, we need to maintain similarity between all of the dimensionless parameters that are *important to the physics of the flow.* This means that we do not necessarily need to have similarity between all Π terms, just the ones that significantly affect the flow physics. Knowing a priori what dimensionless terms are important can be difficult, but with experience the task becomes easier.

(2) It is not uncommon to have the important physics of a system change at different scales. For example, surface tension forces become more pronounced at smaller geometric scales. If one was scaling up a small system in which surface tension was an important effect, but didn't consider the dynamic similarity of the surface tension force at the larger scale, then the scaling experiments would result in incorrect results.

7.5.1. Partial Similarity

True similarity may be difficult to achieve in practice. In such cases, one must either: (a) acknowledge that model testing may not be possible, or (b) relax one or more similarity requirements and use a combination of experimentation and analysis to scale the measurements.

For example, in modeling the flow around ships, both Reynolds number and Froude number similarity are important; however, both are difficult to achieve simultaneously. In such cases, one of the similarity requirements is relaxed (in boat modeling it's the Reynolds number similarity) and a combination of experiments and analysis is utilized to scale the measurements.

The two primary components of drag on a ship's hull are viscous drag, i.e., the friction of the water against the hull's surface, and wave drag, i.e., the force required to create the waves generated by the hull. The two significant dimensionless parameters corresponding to these phenomena are the,

$$\text{Reynolds number: ratio of inertial to viscous forces} \qquad \mathrm{Re} = \frac{VL}{\nu}, \tag{7.79}$$

$$\text{Froude number: ratio of inertial to gravitational forces} \qquad \mathrm{Fr} = \frac{V}{\sqrt{gL}}. \tag{7.80}$$

Maintaining both Reynolds number and Froude number similarity is difficult to achieve in practice,

$$\mathrm{Fr}_M = \mathrm{Fr}_P \implies \left(\frac{V}{\sqrt{gL}}\right)_M = \left(\frac{V}{\sqrt{gL}}\right)_P \implies V_M = V_P\sqrt{\frac{L_M}{L_P}}\sqrt{\frac{g_M}{g_P}} \implies V_M = V_P\sqrt{\frac{L_M}{L_P}}, \tag{7.81}$$

$$\mathrm{Re}_M = \mathrm{Re}_P \implies \left(\frac{VL}{\nu}\right)_M = \left(\frac{VL}{\nu}\right)_P \implies \nu_M = \nu_P\left(\frac{V_M}{V_P}\right)\left(\frac{L_M}{L_P}\right) \implies \nu_M = \nu_p\left(\frac{L_M}{L_P}\right)^{3/2}, \tag{7.82}$$

where the gravitational acceleration is assumed constant across scales ($g_M = g_P$). As an example, consider a scale model that has $L_P = 100L_M$, $\nu_P = \nu_{H2O} = 1\,\mathrm{cSt} \implies \nu_M = 0.001\,\mathrm{cSt}$. There is no such common model fluid available! Thus, we cannot easily maintain both Froude number and Reynolds number similarity.

How do we resolve this difficulty? In practice, Froude number similarity is maintained with water as both the prototype and model fluid (i.e., Eq. (7.81) holds). Reynolds number similarity is neglected in the experiment and instead analysis or computation is used to estimate the viscous drag contribution. The procedure is as follows:

(1) The total drag acting on the model is measured in the experiment. This drag force is usually expressed in terms of a dimensionless resistance coefficient.
(2) The viscous drag contribution to the total drag is calculated using analysis, e.g., boundary layer analysis, or computation, e.g., computational fluid dynamics.
(3) The difference between the total drag and the viscous drag is the wave drag.
(4) The viscous drag contribution to the total drag is calculated using analysis (e.g., boundary layer analysis) or computation (e.g., computational fluid dynamics).
(5) Estimate the viscous drag contribution for the prototype using analysis or computation.
(6) Sum the predicted viscous drag force (step 5) with the scaled wave drag force (step 4) to get the total prototype drag force.

As a demonstration of how well the procedure works, consider the resistance coefficient data from a 1:80 scale model test of the U.S. Navy guided missile frigate Oliver Hazard Perry (FFG-7) as shown in Figure 7.8. The error between the scaled and actual total drag force measurements is approximately ±5%.

Notes:

(1) Experimental observations have shown that in many (but not all!) cases, Reynolds number similarity may be neglected for sufficiently large Reynolds numbers. For example, consider the Moody plot in Figure 7.9, which plots the dimensionless wall friction coefficient (aka the friction factor) as a function of Reynolds number for varying dimensionless wall roughnesses. At sufficiently large Reynolds numbers, known as the "fully rough zone", the friction factor no longer is a function of the Reynolds number.

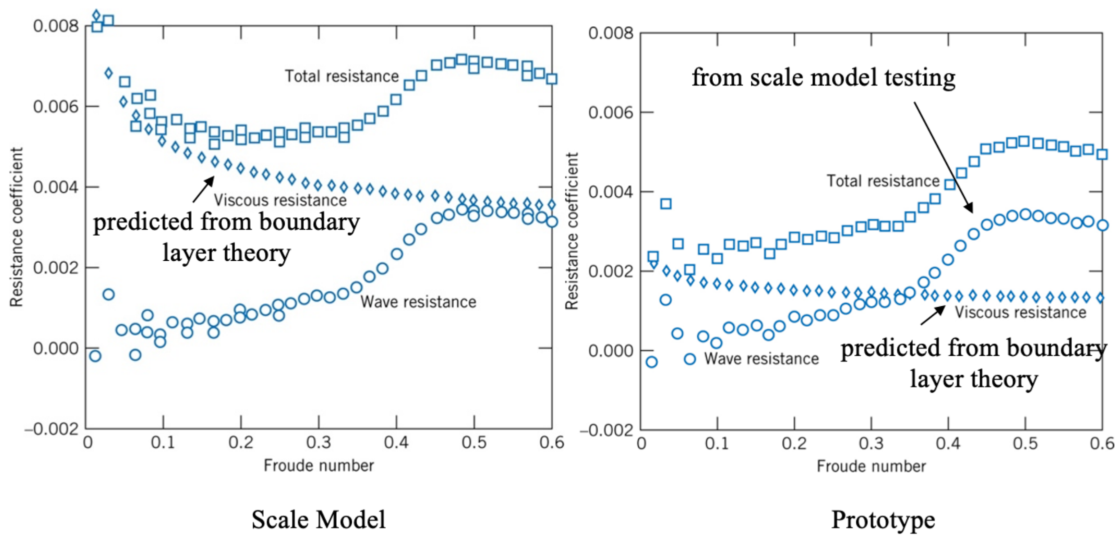


Figure 7.8. The resistance coefficient plotted as a function of Froude number for a scale model (left) and prototype (right). These plots are from Figs. 7.2 and 7.3 in Fox, R.W., Pritchard, P.J., and McDonald, A.T., 2008, *Introduction to Fluid Mechanics*, 7th ed., Wiley.

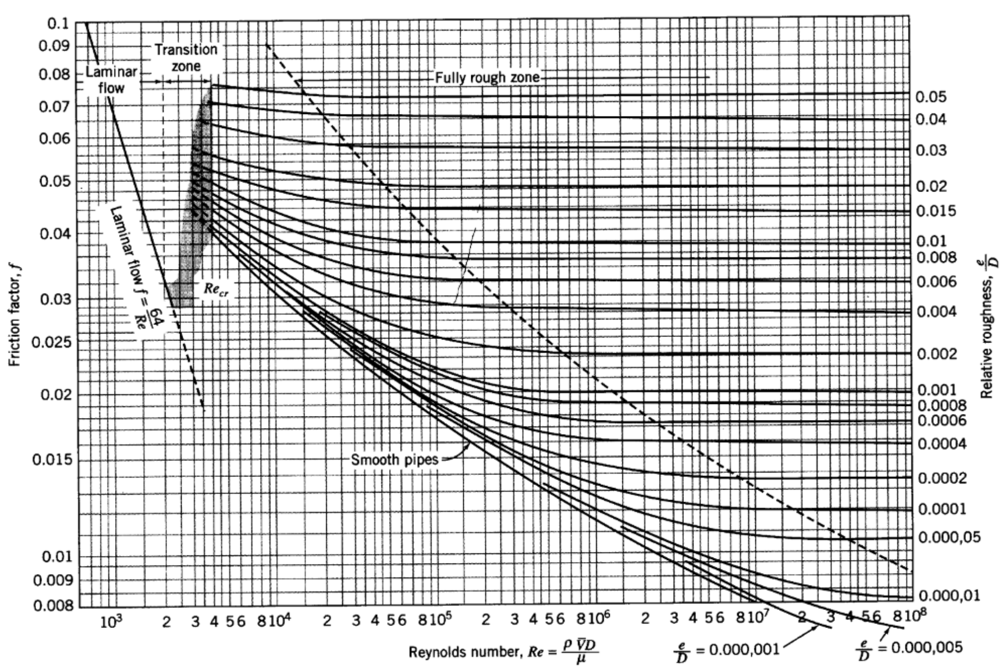


Figure 7.9. The Moody plot, which plots the friction coefficient as a function of Reynolds number for varying relative roughness. Note that in the fully rough zone, the friction factor is independent of the Reynolds number. This plot is from Fox, R.W., Pritchard, P.J., and McDonald, A.T., 2008, *Introduction to Fluid Mechanics*, 7th ed., Wiley.

(2) The drag coefficients for flow around sphere and circular disk are insensitive to the Reynolds number over a wide range of Reynolds numbers, as shown in Figure 7.10.

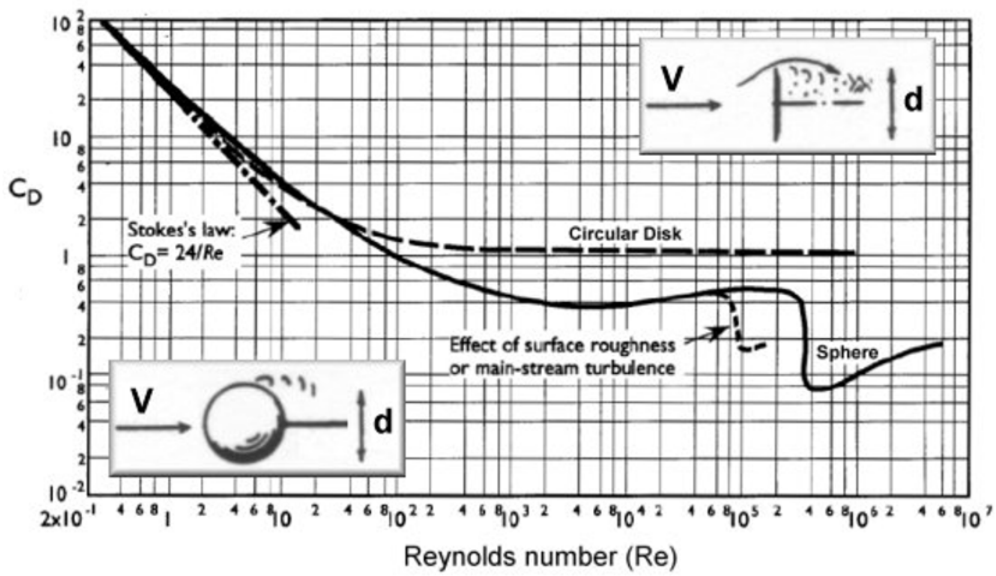


Figure 7.10. The drag coefficients for a sphere and circular disk plotted as a function of the Reynolds number. The drag coefficient is nearly independent of the Reynolds number over a wide range of Reynolds numbers. This figure is from Fox, R.W., Pritchard, P.J., and McDonald, A.T., 2008, *Introduction to Fluid Mechanics*, 7th ed., Wiley.

7.6. The Stokes Number (St) for Small Particles in a Flow

The Stokes number, St, is defined as the ratio of the particle response time, τ_p, to the fluid response time, τ_f,

$$\mathrm{St} := \frac{\tau_p}{\tau_f}. \tag{7.83}$$

A response time is a measure of how rapidly a quantity responds to rapid changes. The Stokes number for a particle is essentially a measure of how well the particle follows fluid streamlines. If $\mathrm{St} \ll 1$ then the particle will be able to follow the fluid streamlines whereas if $\mathrm{St} \gg 1$ then the particle will not be able to follow sudden changes in the fluid velocity. For example, consider driving down a country road late at night during the summer when a lot of bugs are out. If the Stokes number for a bug is small, then it will follow the fluid streamlines as you drive past it and it won't impact your car (Figure 7.11). However, if the Stokes number for the bug is large, it will end up hitting your windshield since it won't be able to follow the fluid streamlines that contour around your car.

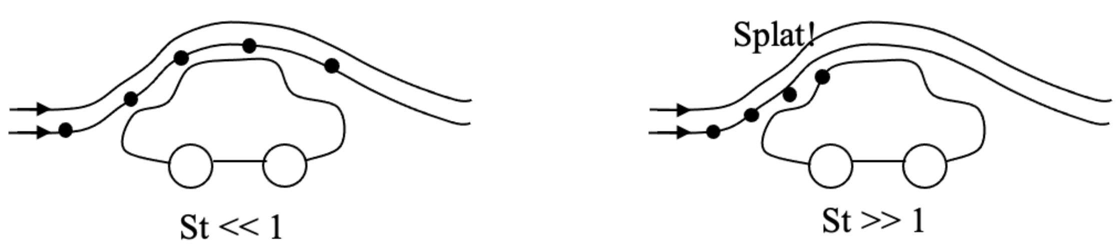


Figure 7.11. An illustration of a bug following the streamlines over a car when $\mathrm{St} \ll 1$ (yeah!) or not following the streamlines when $\mathrm{St} \gg 1$ (oh no!).

The particle response time can be found by considering the particle equation of motion (assuming spherical particles) for a particle with a speed slower than the surrounding fluid (so the particle accelerates),

$$\left(\rho \frac{\pi}{6} d_p^3\right) \frac{du_p}{dt} = C_D \frac{1}{2} \rho_f \left(u_f - u_p\right)^2 \left(\frac{\pi d_p^2}{4}\right), \tag{7.84}$$

where ρ_p an ρ_f are the particle and fluid densities, d_p is the particle diameter, u_p and u_f are the particle and fluid velocities, t is time, and C_D is the particle drag coefficient. Define the Reynolds number for the particle using the local relative velocity,

$$\mathrm{Re}_d = \frac{\rho(u_f - u_p)d_p}{\mu_f} \quad \text{(Note that } u_f > u_p \text{ is assumed.)} \tag{7.85}$$

Substitute Eq. (7.85) into Eq. (7.84) and simplify,

$$\frac{du_p}{dt} = C_D \mathrm{Re}_d \left[\frac{\mu_f}{\rho_f(u_f - u_p)d_p}\right] \rho_f (u_f - u_p)^2 \left(\frac{3}{4\rho_p d_p}\right), \tag{7.86}$$

$$= C_D \mathrm{Re}_d \left(\frac{3\mu_f}{4\rho_p d_p^2}\right)(u_f - u_p). \tag{7.87}$$

For small Reynolds numbers the drag coefficient approaches the Stokes drag,

$$C_D = \frac{24}{\mathrm{Re}_d} \quad \text{(We're now assuming that we're dealing with small particles.)}, \tag{7.88}$$

so that the particle equation of motion becomes,

$$\frac{du_p}{dt} = \frac{18\mu_f}{\rho_p d_p^2}(u_f - u_p). \tag{7.89}$$

The solution to this equation, assuming a constant fluid velocity and a particle released from rest ($u_p(t = 0) = 0$), is,

$$u_p = u_f \left[1 - \exp\left(-\frac{t}{\tau_p}\right)\right], \tag{7.90}$$

where τ_p is the particle response time,

$$\boxed{\tau_p = \frac{\rho_p d_p^2}{18\mu_f}}. \tag{7.91}$$

Let the fluid response time, τ_f, for the flow geometry be,

$$\boxed{\tau_f = \frac{L}{u_f}}, \tag{7.92}$$

where L is a typical flow dimension, e.g., the effective frontal diameter of the car in the car/bug example discussed previously. Therefore, the Stokes number for the particle is,

$$\mathrm{St} = \frac{\tau_p}{\tau_f} = \frac{\rho_p d_p^2 u_f}{18\mu_f L}. \tag{7.93}$$

Re-writing in terms of the Reynolds number, Re_L, based on the typical flow dimension, L,

$$\mathrm{Re}_L = \frac{\rho_f u_f L}{\mu_f}, \tag{7.94}$$

the Stokes number is,

$$\mathrm{St} = \frac{\rho_p d_p^2 u_f}{18\mu_f L}\left(\mathrm{Re}_L \frac{\mu_f}{\rho_f u_f L}\right), \tag{7.95}$$

$$\boxed{\mathrm{St} = \frac{\mathrm{Re}_L}{18}\left(\frac{\rho_p}{\rho_f}\right)\left(\frac{d_p}{L}\right)^2}. \tag{7.96}$$

For very small particles compared to the flow dimension, i.e., ($d_p/L \ll 1$), and moderate flow Reynolds numbers and density ratios, we observe that $\mathrm{St} \ll 1$ and the particle should follow the fluid streamlines.

The power, P, to drive an axial flow pump depends on the following variables:

density of the fluid, ρ
angular speed of the rotor, Ω
diameter of the rotor, D
head rise across the pump, ΔH ($= \Delta p/\rho g$)
volumetric flow through the pump, Q

a. Rewrite the functional relationship in dimensionless form.
b. A model scaled to one-third the size of the prototype has the following characteristics:

$\Omega_m = 900$ rpm
$D_m = 5$ in
$\Delta H_m = 10$ ft
$Q_m = 3$ ft^3/s
$P_m = 2$ hp

If the full-size pump is to run at 300 rpm, what is the power required for this pump? What head will the pump maintain? What will the volumetric flow rate be in the prototype?

SOLUTION:

1. Write the dimensional functional relationship.

$$P = f_1(\rho, \Omega, D, \Delta H, Q)$$

2. Determine the basic dimensions of each parameter.

$$[P] = FL/T = ML^2/T^3$$
$$[\rho] = M/L^3$$
$$[\Omega] = 1/T$$
$$[D] = L$$
$$[\Delta H] = L$$
$$[Q] = L^3/T$$

3. Determine the number of Π terms required to describe the functional relationship.

\# of variables = 6 (P, ρ, Ω, D, ΔH, Q)
\# of reference dimensions = 3 (M, L, T)

(# Π terms) = (# of variables) – (# of reference dimensions) = 6 – 3 = 3

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

ρ, D, Ω (Note that these repeating variables have independent dimensions.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = P\rho^a D^b \Omega^c$$

$$\Rightarrow M^0 L^0 T^0 = \left(\frac{ML^2}{T^3}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{1}\right)^b \left(\frac{1}{T}\right)^c$$

$$\begin{array}{lll} M: & 0 = 1 + a & \Rightarrow a = -1 \\ L: & 0 = 2 - 3a + b & \Rightarrow b = -5 \\ T: & 0 = -3 - c & \Rightarrow c = -3 \end{array}$$

$$\therefore \Pi_1 = \frac{P}{\rho \Omega^3 D^5}$$

$$\Pi_2 = \Delta H \rho^a D^b \Omega^c$$

$$\Rightarrow M^0 L^0 T^0 = \left(\frac{L}{1}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{1}\right)^b \left(\frac{1}{T}\right)^c$$

$$\begin{array}{lll} M: & 0 = a & \Rightarrow a = 0 \\ L: & 0 = 1 - 3a + b & \Rightarrow b = -1 \\ T: & 0 = -c & \Rightarrow c = 0 \end{array}$$

$$\therefore \Pi_2 = \frac{\Delta H}{D}$$

$$\Pi_3 = Q\rho^a D^b \Omega^c$$

$$M^0 L^0 T^0 = \left(\frac{L^3}{T}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{1}\right)^b \left(\frac{1}{T}\right)^c$$

$$\begin{array}{lll} M: & 0 = a & \Rightarrow a = 0 \\ L: & 0 = 3 - 3a + b & \Rightarrow b = -3 \\ T: & 0 = -1 - c & \Rightarrow c = -1 \end{array}$$

$$\therefore \Pi_3 = \frac{Q}{\Omega D^3}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{P}{\rho \Omega^3 D^5}\right] = \frac{ML^2}{T^3}\frac{L^3}{M}\frac{T^3}{1}\frac{1}{L^5} = 1 \text{ OK!}$$

$$[\Pi_2] = \left[\frac{\Delta H}{D}\right] = \frac{L}{1}\frac{1}{L} = 1 \text{ OK!}$$

$$[\Pi_3] = \left[\frac{Q}{\Omega D^3}\right] = \frac{L^3}{T}\frac{T}{1}\frac{1}{L^3} = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{P}{\rho \Omega^3 D^5} = f_2\left(\frac{\Delta H}{D}, \frac{Q}{\Omega D^3}\right)}$$

Now perform a scaling analysis assuming that the same fluid is used in the model and prototype, i.e., $\rho_M = \rho_P$. Note that since a one-third scale model is being used, $D_P/D_M = 3/1$.

$$\left(\frac{P}{\rho\Omega^3 D^5}\right)_M = \left(\frac{P}{\rho\Omega^3 D^5}\right)_P$$

$$P_P = P_M\left(\frac{\Omega_P}{\Omega_M}\right)^3\left(\frac{D_P}{D_M}\right)^5$$

$$P_P = (2\text{ hp})\left(\frac{300\text{ rpm}}{900\text{ rpm}}\right)^3\left(\frac{3}{1}\right)^5$$

$$\boxed{P_P = 18\text{ hp}}$$

$$\left(\frac{\Delta H}{D}\right)_M = \left(\frac{\Delta H}{D}\right)_P$$

$$\Delta H_P = \Delta H_M\left(\frac{D_P}{D_M}\right)$$

$$\Delta H_P = (10\text{ ft})\left(\frac{3}{1}\right)$$

$$\boxed{\Delta H_P = 30\text{ ft}}$$

$$\left(\frac{Q}{\Omega D^3}\right)_M = \left(\frac{Q}{\Omega D^3}\right)_P$$

$$Q_P = Q_M\left(\frac{\Omega_P}{\Omega_M}\right)\left(\frac{D_P}{D_M}\right)^3$$

$$\boxed{Q_P = 27\text{ ft}^3/\text{s}}$$

The drag characteristics of a blimp 5 m in diameter and 60 m long are to be studied in a wind tunnel. If the speed of the blimp through still air is 10 m/s, and if a 1/10 scale model is to be tested, what airspeed in the wind tunnel is needed for dynamic similarity? Assume the same air temperature and pressure for both the prototype and model.

SOLUTION:

For dynamic similarity, equate the model and prototype Reynolds numbers.

$$\mathrm{Re}_P = \mathrm{Re}_M$$

$$\Rightarrow \left(\frac{VD}{\nu}\right)_P = \left(\frac{VD}{\nu}\right)_M$$

Since both the model and prototype use air at the same temperature and pressure as the working fluid, $\nu_P = \nu_M$.

$$\Rightarrow V_M = V_P\left(\frac{D_P}{D_M}\right) = \left(10\ \mathrm{m/s}\right)\left(\frac{10}{1}\right)$$

$$\boxed{\therefore V_M = 100\ \mathrm{m/s}}$$

Note that the model speed is still low enough that Mach number effects (i.e., compressibility effects) do not come into play.

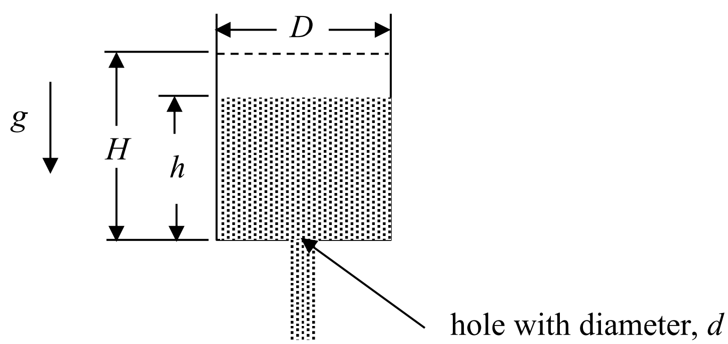
The height of the free surface, h, in a tank of diameter, D, that is draining fluid through a small hole at the bottom with diameter, d, decreases with time, t. This change in free surface height is studied experimentally with a half-scale model. For the prototype tank:

H = 16 in. (the initial height of the free surface)
D = 4.0 in.
d = 0.25 in.

Experimental data is obtained from the prototype and half-scale model and is given below:

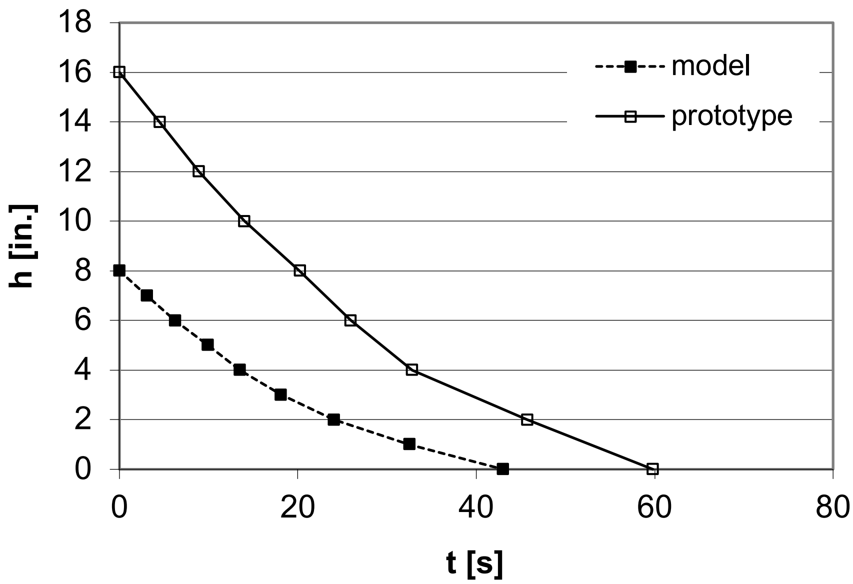
Model Data		Prototype Data	
h [in.]	t [s]	h [in.]	t [s]
8.0	0.0	16.0	0.0
7.0	3.1	14.0	4.5
6.0	6.2	12.0	8.9
5.0	9.9	10.0	14.0
4.0	13.5	8.0	20.2
3.0	18.1	6.0	25.9
2.0	24.0	4.0	32.8
1.0	32.5	2.0	45.7
0.0	43.0	0.0	59.8

1. Plot, on the same graph, the height data as a function of time for both the model and the prototype.
2. Develop a set of dimensionless parameters for this problem assuming that: $h = f(H, D, d, g, t)$
3. Re-plot, on the same graph, the height data as a function of time in non-dimensional form for both the model and prototype.



SOLUTION:

First plot the model and prototype dimensional data.



Now perform a dimensional analysis to determine the dimensionless terms describing the relationship.

1. Write the dimensional functional relationship.
   $$h = f_1(H, D, d, g, t)$$

2. Determine the basic dimensions of each parameter.
   $$[h] = L$$
   $$[H] = L$$
   $$[D] = L$$
   $$[d] = L$$
   $$[g] = L/T^2$$
   $$[t] = T$$

3. Determine the number of Π terms required to describe the functional relationship.
   \# of variables = 6 (h, H, D, d, g, t)
   \# of reference dimensions = 2 (L, T)
   (Note that the number of reference dimensions and the number of basic dimensions are equal for this problem.)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 6 – 2 = 4

4. Choose two repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
   H, g (Note that the dimensions for these variables are independent.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = \frac{h}{H} \quad \text{(Found via inspection.)}$$

$$\Pi_2 = \frac{D}{H} \quad \text{(Found via inspection.)}$$

$$\Pi_3 = \frac{d}{H} \quad \text{(Found via inspection.)}$$

$$\Pi_4 = tH^a g^b$$

$$\Rightarrow \; L^0 T^0 = \left(\frac{T}{1}\right)\left(\frac{L}{1}\right)^a \left(\frac{L}{T^2}\right)^b$$

$$\begin{aligned} L&: \quad 0 = a + b \\ T&: \quad 0 = 1 - 2b \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= -\tfrac{1}{2} \\ b &= \tfrac{1}{2} \end{aligned}$$

$$\therefore \Pi_4 = t\sqrt{\frac{g}{H}}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{h}{H}\right] = \frac{L}{1}\frac{1}{L} = 1 \quad \text{OK!}$$

$$[\Pi_2] = \left[\frac{D}{H}\right] = \frac{L}{1}\frac{1}{L} = 1 \quad \text{OK!}$$

$$[\Pi_3] = \left[\frac{d}{H}\right] = \frac{L}{1}\frac{1}{L} = 1 \quad \text{OK!}$$

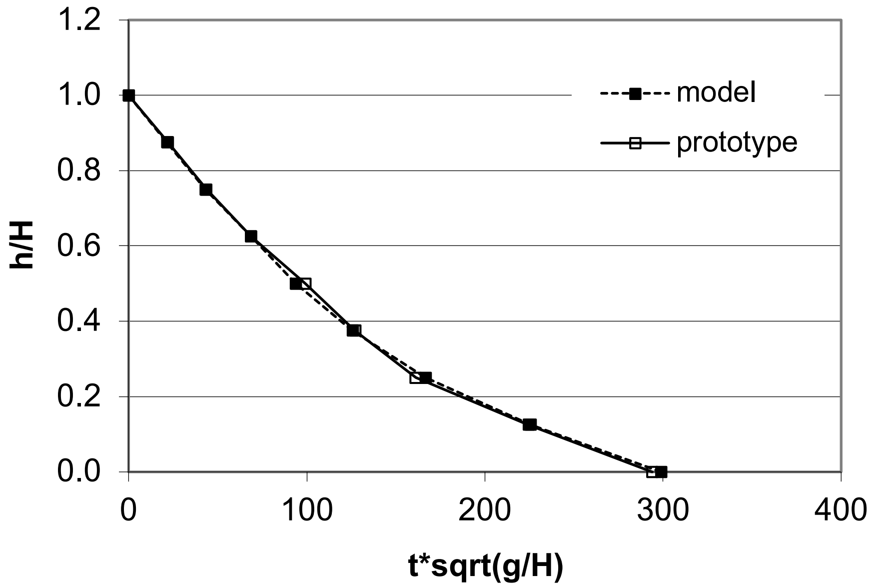
$$[\Pi_4] = \left[t\sqrt{\frac{g}{H}}\right] = \frac{T}{1}\frac{L^{1/2}}{T}\frac{1}{L^{1/2}} = 1 \quad \text{OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{h}{H} = f_2\left(\frac{D}{H}, \frac{d}{H}, t\sqrt{\frac{g}{H}}\right)}$$

Now plot the model and prototype data in dimensionless form. Note that since there is geometric similarity (the model is one-half the size of the prototype):

$$\left(\frac{d}{H}\right)_M = \left(\frac{d}{H}\right)_P \quad \text{and} \quad \left(\frac{D}{H}\right)_M = \left(\frac{D}{H}\right)_P$$



Notice that the data collapse to a single curve when plotted in dimensionless terms.

A model test of a tractor-trailer rig is performed in a wind tunnel. The drag force, F_D, is found to depend on the frontal area, A, wind speed, V, air density, ρ, and air viscosity, μ. The model scale is 1:4 (e.g., 1 m in the model is equivalent to 4 m in the prototype), frontal area of the model is $A = 0.625\text{ m}^2$.

a. Obtain a set of dimensionless parameters suitable to characterize the model test results.
b. If the drag force on the full-scale vehicle traveling at 22.4 m/s is to be predicted from model testing, what should be the wind tunnel air speed? Assume that the air conditions are the same for the model and prototype.
c. When tested at the wind speed found in part (b), the measured drag force on the model was $F_D = 2.46$ kN. Estimate the aerodynamic drag force on the full-scale vehicle.
d. Calculate the power needed to overcome the full-scale drag force.

SOLUTION:

1. Write the dimensional functional relationship.

$$F_D = f_1\left(A, V, \rho, \mu\right)$$

2. Determine the basic dimensions of each parameter.

$$[F_D] = \frac{ML}{T^2}$$
$$[A] = L^2$$
$$[V] = \frac{L}{T}$$
$$[\rho] = \frac{M}{L^3}$$
$$[\mu] = \frac{M}{LT}$$

3. Determine the number of Π terms required to describe the functional relationship.

   # of variables = 5 (F_D, A, V, ρ, μ)
   # of reference dimensions = 3 (L, T, M)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 5 – 3 = 2

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

   A, V, ρ

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = F_D A^a V^b \rho^c$$

$$\Rightarrow\ M^0L^0T^0 = \left(\frac{ML}{T^2}\right)\left(\frac{L^2}{1}\right)^a\left(\frac{L}{T}\right)^b\left(\frac{M}{L^3}\right)^c$$

$$\begin{array}{llll} M: & 0 = 1 + c & & a = -1 \\ L: & 0 = 1 + 2a + b - 3c & \Rightarrow & b = -2 \\ T: & 0 = -2 - b & & c = -1 \end{array}$$

$$\therefore \Pi_1 = \frac{F_D}{\rho V^2 A} \quad \text{(This is a drag coefficient!)}$$

$$\Pi_2 = \mu A^a V^b \rho^c$$

$$\Rightarrow\ M^0L^0T^0 = \left(\frac{M}{LT}\right)\left(\frac{L^2}{1}\right)^a\left(\frac{L}{T}\right)^b\left(\frac{M}{L^3}\right)^c$$

$$\begin{array}{llll} M: & 0 = 1 + c & & a = -\tfrac{1}{2} \\ L: & 0 = -1 + 2a + b - 3c & \Rightarrow & b = -1 \\ T: & 0 = -1 - b & & c = -1 \end{array}$$

$$\therefore \Pi_2 = \frac{\mu}{\rho V\sqrt{A}} \quad \text{or} \quad \Pi_2 = \frac{\rho V\sqrt{A}}{\mu} \quad \text{(This is a Reynolds number!)}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{F_D}{\rho V^2 A}\right] = \frac{ML}{T^2}\frac{L^3}{M}\frac{T^2}{L^2}\frac{1}{L^2} = 1 \quad \text{OK!}$$

$$[\Pi_2] = \left[\frac{\rho V\sqrt{A}}{\mu}\right] = \frac{M}{L^3}\frac{L}{T}\frac{L}{1}\frac{LT}{M} = 1 \quad \text{OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{F_D}{\rho V^2 A} = f_2\left(\frac{\rho V\sqrt{A}}{\mu}\right)} \tag{1}$$

To maintain similarity, the dimensionless terms must be the same between the model and prototype,

$$\boxed{\left(\frac{\rho V\sqrt{A}}{\mu}\right)_M = \left(\frac{\rho V\sqrt{A}}{\mu}\right)_P} \Rightarrow \boxed{\left(\frac{F_D}{\rho V^2 A}\right)_M = \left(\frac{F_D}{\rho V^2 A}\right)_P} \tag{2}$$

To determine the model testing wind speed, keep the Reynolds numbers the same between scales (the Pi term on the right hand side of Eq. (1)),

$$\left(\frac{\rho V\sqrt{A}}{\mu}\right)_M = \left(\frac{\rho V\sqrt{A}}{\mu}\right)_P, \tag{3}$$

$$V_M = V_P\underbrace{\left(\frac{\rho_P}{\rho_M}\right)}_{=1}\underbrace{\left(\frac{\mu_M}{\mu_P}\right)}_{=1}\underbrace{\left(\frac{A_P}{A_M}\right)^{1/2}}_{=(16/1)^{1/2}} \quad \text{(same air properties; } L_P/L_M = 4/1 \Rightarrow A_P/A_M = (4/1)^2 = 16/1\text{)} \tag{4}$$

$$\Rightarrow \boxed{V_M = 4V_P = 89.6\ \text{m/s}}. \tag{5}$$

The force on the prototype is found using the other Pi term,

$$\left(\frac{F_D}{\rho V^2 A}\right)_M = \left(\frac{F_D}{\rho V^2 A}\right)_P$$

$$F_D\big|_P = \underbrace{F_D\big|_M}_{=2.46\text{ kN}} \underbrace{\left(\frac{\rho_P}{\rho_M}\right)}_{=1} \underbrace{\left(\frac{V_P}{V_M}\right)^2}_{=\left(\frac{22.4\text{ m/s}}{89.6\text{ m/s}}\right)^2} \underbrace{\left(\frac{A_P}{A_M}\right)}_{=\left(\frac{16}{1}\right)} \quad \text{(Note the same air is used in both the model and prototype.)} \tag{6}$$

$$\boxed{\therefore F_D\big|_P = 2.46\text{ kN}}$$

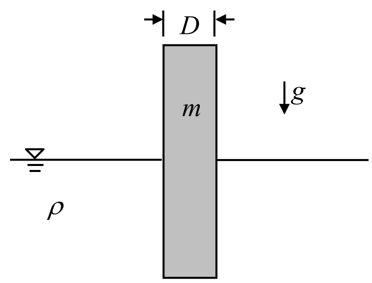
The power required to overcome the prototype drag force is:

$$P_P = \underbrace{F_{D_P}}_{=2.46\text{ kN}} \cdot \underbrace{V_P}_{=22.4\text{ m/s}} \tag{7}$$

$$\boxed{\therefore P_P = 55.1\text{ kW}}$$

A cylinder with a diameter, D, floats upright in a liquid as shown in the figure. When the cylinder is displaced slightly along its vertical axis it will oscillate about its equilibrium position with a frequency, ω. Assume that this frequency is a function of the diameter, D, the mass of the cylinder, m, the liquid density, ρ, and the acceleration due to gravity, g.

If the mass of the cylinder were doubled (assuming the same cylinder material density), by how much would the oscillation frequency change?



SOLUTION:

1. Write the dimensional functional relationship.
$$\omega = f_1\left(D, m, \rho, g\right) \tag{1}$$

2. Determine the basic dimensions of each parameter.
$$[\omega] = 1/T$$
$$[D] = L$$
$$[m] = M$$
$$[\rho] = M/L^3$$
$$[g] = L/T^2$$

3. Determine the number of Π terms required to describe the functional relationship.
\# of variables = 5 (ω, D, m, ρ, g)
\# of reference dimensions = 3 (M, L, T)

(# Π terms) = (# of variables) – (# of reference dimensions) = 5 – 3 = 2

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
ρ, D, g (Note that these repeating variables have independent dimensions.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.
$$\Pi_1 = \omega\rho^a D^b g^c$$
$$\Rightarrow M^0L^0T^0 = \left(1/T\right)\left(M/L^3\right)^a\left(L/1\right)^b\left(L/T^2\right)^c$$
M: $\quad 0 = a \quad \Rightarrow a = 0$
T: $\quad 0 = -1 - 2c \quad \Rightarrow c = -\frac{1}{2}$
L: $\quad 0 = -3a + b + c \quad \Rightarrow b = \frac{1}{2}$
$$\therefore \Pi_1 = \omega\sqrt{\frac{D}{g}}$$

$$\Pi_2 = m\rho^a D^b g^c$$
$$\Rightarrow \; M^0 L^0 T^0 = \left(\tfrac{M}{1}\right)\left(\tfrac{M}{L^3}\right)^a \left(\tfrac{L}{1}\right)^b \left(\tfrac{L}{T^2}\right)^c$$
$$\begin{array}{lll} M: & 0 = 1 + a & \Rightarrow \; a = -1 \\ T: & 0 = -2c & \Rightarrow \; c = 0 \\ L: & 0 = -3a + b + c & \Rightarrow \; b = -3 \end{array}$$
$$\therefore \Pi_2 = \frac{m}{\rho D^3}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\omega\sqrt{\frac{D}{g}}\right] = \tfrac{1}{T}\tfrac{L^{1/2}}{1}\tfrac{T}{L^{1/2}} = 1 \;\text{ OK!}$$
$$[\Pi_2] = \left[\frac{m}{\rho D^3}\right] = \tfrac{M}{1}\tfrac{L^3}{M}\tfrac{1}{L^3} = 1 \;\text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\omega\sqrt{\frac{D}{g}} = f_2\left(\frac{m}{\rho D^3}\right) \tag{2}$$

For similarity:

$$\left(\omega\sqrt{\frac{D}{g}}\right)_1 = \left(\omega\sqrt{\frac{D}{g}}\right)_2 \tag{3}$$
$$\left(\frac{m}{\rho D^3}\right)_1 = \left(\frac{m}{\rho D^3}\right)_2 \tag{4}$$

Assuming the same liquid (*i.e.* $\rho_1 = \rho_2$), Eq. (4) indicates:

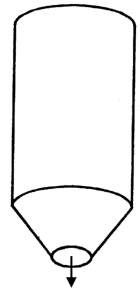
$$\frac{D_2}{D_1} = \left(\frac{m_2}{m_1}\right)^{1/3} \tag{5}$$

Using Eq. (5) with Eq. (3), assuming the same gravitational acceleration (i.e., $g_1 = g_2$), gives:

$$\frac{\omega_2}{\omega_1} = \left(\frac{D_1}{D_2}\right)^{1/2} = \left(\frac{m_1}{m_2}\right)^{1/6} \tag{6}$$

Hence, doubling the mass (i.e., $m_2 = 2m_1$) will result in a smaller frequency with $\omega_2 = 2^{-1/6}\,\omega_1$.

Hoppers are a commonly used device in the handling and storage of particulate materials. A hopper design typically consists of a bin section located above a converging section with a hole located in the bottom through which the particulate material flows (refer to the figures below).



One interesting observation with hopper flows is that the mass flow rate from the hopper exit is independent of the height of the material above the exit and the bin diameter (except when the hopper is nearly empty). The parameters that do affect the discharge rate (assuming cohesionless particles) include the hopper exit diameter, the acceleration due to gravity, the angle of the hopper walls, the friction coefficient between the particulate material and the walls and between the particles themselves, and the bulk density of the material at the discharge plane.

a. Perform a dimensional analysis to determine the dimensionless quantities that govern flow from a hopper.
b. If the same hopper and particulate material are used (i.e., the wall angle and friction properties remain the same), how will the mass flow rate from the hopper change if the hopper exit diameter is doubled?
c. Compare the discharge rate found in part (a) with the mass discharge rate expected for a liquid.

SOLUTION:

1. Write the dimensional functional relationship.

$$\dot{m} = f_1\left(D_E, g, \theta, \mu_{pp}, \mu_{pw}, \rho_b\right) \tag{1}$$

where $\dot{m}$ is the mass discharge rate from the hopper, D_E is the hopper exit diameter, g is the acceleration due to gravity, θ is the hopper wall angle, μ_{pp} and μ_{pw} are the friction coefficients between particles and between particles and the hopper walls, respectively, and ρ_b is the bulk density of the material at the hopper exit (the bulk density is the density of the particulate material including the void space between particles).

2. Determine the basic dimensions of each parameter.

$$[\dot{m}] = M/T$$
$$[D_E] = L$$
$$[g] = L/T^2$$
$$[\theta] = -$$
$$[\mu_{pp}] = [\mu_{pw}] = -$$
$$[\rho_b] = M/L^3$$

3. Determine the number of Π terms required to describe the functional relationship.

# of variables = 7 ($\dot{m}$, D_E, g, θ, μ_{pp}, μ_{pw}, ρ_b)

# of reference dimensions = 3 (L, T, M)

$$(\#\,\Pi\text{ terms}) = (\#\text{ of variables}) - (\#\text{ of reference dimensions}) = 7 - 3 = 4 \tag{2}$$

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
   D_E, g, ρ_b

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = \dot{m} D_E^a g^b \rho_b^c \tag{3}$$

$$\Rightarrow\ M^0 L^0 T^0 = \left(\frac{M}{T}\right)\left(\frac{L}{1}\right)^a\left(\frac{L}{T^2}\right)^b\left(\frac{M}{L^3}\right)^c \tag{4}$$

$$\begin{aligned} M:&\quad 0 = 1 + c & & a = -\tfrac{5}{2} \\ L:&\quad 0 = a + b - 3c \quad\Rightarrow & & b = \tfrac{1}{2} \\ T:&\quad 0 = -1 - 2b & & c = -1 \end{aligned} \tag{5}$$

$$\therefore \Pi_1 = \frac{\dot{m}}{\rho_b g^{1/2} D_E^{5/2}} \tag{6}$$

$\Pi_2 = \theta$ (angles are dimensionless)

$\Pi_3 = \mu_{pp}$ (friction coefficients are dimensionless)

$\Pi_4 = \mu_{pw}$ (friction coefficients are dimensionless)

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{\dot{m}}{\rho_b g^{1/2} D_E^{5/2}}\right] = \frac{M/T}{\left(M/L^3\right)\left(L^{1/2}/T\right)\left(L^{5/2}\right)} = 1\ \text{ OK!}$$

$$[\Pi_2] = [\theta] = 1\ \text{ OK!}$$

$$[\Pi_3] = [\mu_{pp}] = 1\ \text{ OK!}$$

$$[\Pi_4] = [\mu_{pw}] = 1\ \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{\dot{m}}{\rho_b g^{1/2} D_E^{5/2}} = f_2\left(\theta, \mu_{pp}, \mu_{pw}\right)} \tag{7}$$

If the wall angle and frictional properties remain constant, then doubling the exit diameter increases the mass flow rate by a factor of $2^{5/2} \approx 5.66$.

The mass discharge rate for a liquid discharging from the hopper is given by:

$$\dot{m} = \rho V_E \frac{\pi D_E^2}{4} \tag{8}$$

where ρ is the liquid density and V_E is the liquid speed at the hopper exit. The liquid speed may be found using Bernoulli's equation applied along a streamline from the hopper free surface, located a height, H, above the hopper exit, to the hopper exit. On both surfaces the fluid pressure is atmospheric and, hence:

$$V_E = \sqrt{2gH} \tag{9}$$

assuming that the kinetic energy of the upper free surface is negligible (i.e., it moves at a small velocity). Comparing Eqs. (7) and (8) shows that the mass discharge rate for a liquid depends upon the height of liquid above it while for a particulate material the discharge rate is independent of material height. In addition, the discharge rate for a particulate material is more sensitive to the hopper exit diameter (varying with $D_E^{5/2}$) than it is for a liquid (varying with D_E^2).

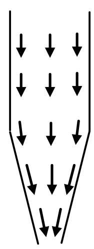
Notes:

1. Beverloo et al. (1961) observed that the experimental data for mass discharge rate from a flat-bottomed hopper is better fit using the following relation:

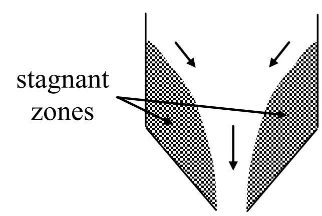
$$W = c\rho_b g^{1/2}\left(D_E - kd\right)^{5/2} \qquad \textbf{Beverloo Mass Flow Rate Correlation} \qquad (10)$$

where c is a constant incorporating the hopper wall angle and frictional properties (function f_2 in Eq. (7)), k is a constant that depends on the geometry of the exit and particle shape, and d is the effective diameter of the particles. The factor kd accounts for the fact that there is an annular zone at the periphery of the exit within which there are few particles. Hence, the effective exit diameter is reduced. The parameter k typically varies between 1.3 – 2.9 with a value of $k \approx 1.5$ for spherical particles. Angular particles have somewhat larger values for k. A value of $k \approx 1.4$ is a good general estimate if no discharge rate test data is available.

Beverloo et al. also observed that for funnel flow hoppers the parameter c is nearly independent of the friction coefficients, μ_{pp} and μ_{pw}, and the hopper wall angle, θ, and remains at a constant value of $c \approx 0.58$. A funnel flow hopper is one in which material remains stagnant adjacent to the hopper walls. A mass flow hopper is one in which all of the material flows simultaneously within the hopper.



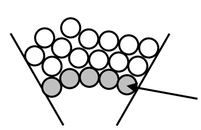
A mass-flow hopper.



A funnel-flow hopper.

3. The bulk density, ρ_b, in Eqs. (7) and (10) is <u>not</u> the bulk density of the material within the hopper. Studies have shown that the discharge rate from a hopper is independent of how the material is originally filled into the hopper. Instead, ρ_b is the bulk density of the *flowing* material. Since we often don't know the flowing bulk density of the material a priori, one can use the bulk density measured by loosely filling a container. The resulting predicted mass flow rate is typically within 5% of the measured value.

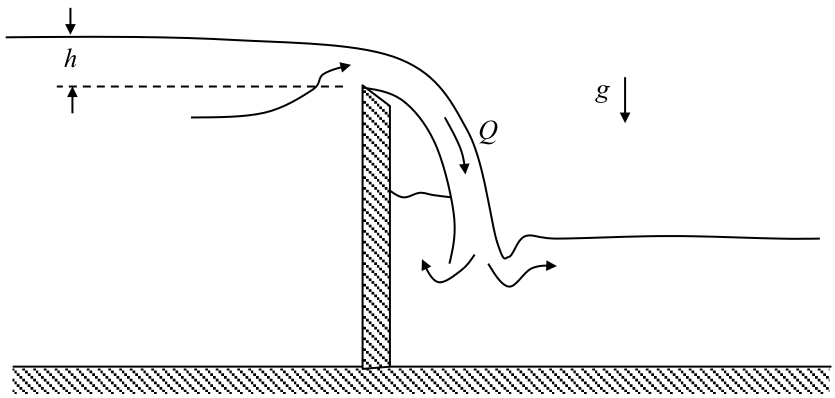
4. Blocking of the hopper exit can occur when the exit diameter is less than about six times the particle diameter. When the exit is smaller than this value, particles can form a mechanical arch that can support the force exerted by the material above it.



References:

Beverloo, W.A., Leniger, H.A., and Van de Velde, J., 1961, "The flow of granular solids through orifices," *Chemical Engineering Science*, Vol. 15, p. 260.

A 1/16th-scale model of a weir has a measured flow rate of Q = 2.1 ft^3/s when the upstream water height is h = 6.3 in. The flow rate is known to be a function of the acceleration due to gravity, g, the weir width (into the page), b, and the upstream water height, h. Furthermore, the flow rate is found to be directly proportional to the weir width, b. What is the flow rate over the prototype weir when the upstream water height is h = 3.2 ft.



SOLUTION:

1. Write the dimensional functional relationship.
$$Q = f_1(g, h, b)$$

2. Determine the basic dimensions of each parameter.
$$[Q] = L^3/T$$
$$[g] = L/T^2$$
$$[h] = L$$
$$[b] = L$$

3. Determine the number of Π terms required to describe the functional relationship.
# of variables = 4 (Q, g, h, b)
# of reference dimensions = 2 (L, T)

(# Π terms) = (# of variables) – (# of reference dimensions) = 4 – 2 = 2

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
g, h

5. Make the remaining non-repeating variables dimensionless using the repeating variables.
$$\Pi_1 = \frac{Q}{\sqrt{gh^5}} \quad \text{(by inspection)}$$
$$\Pi_2 = \frac{b}{h} \quad \text{(by inspection)}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{Q}{\sqrt{gh^5}}\right] = \frac{L^3/T}{\sqrt{L/T^2 \cdot L^5}} = 1 \quad \text{OK!}$$

$$[\Pi_2] = \left[\frac{b}{h}\right] = \frac{L}{L} = 1$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{Q}{\sqrt{gh^5}} = f_2\left(\frac{b}{h}\right)} \tag{1}$$

We are also told that $Q \propto b$ so that Eqn. (1) becomes:

$$\frac{Q}{\sqrt{gh^5}} = c\left(\frac{b}{h}\right) \tag{2}$$

$$\therefore \frac{Q}{b\sqrt{gh^3}} = c \tag{3}$$

where c is a constant of proportionality.

Since the right-hand side of Eq. (1) is a constant, then:

$$\left(\frac{Q}{b\sqrt{gh^3}}\right)_{\text{prototype}} = \left(\frac{Q}{b\sqrt{gh^3}}\right)_{\text{model}}$$

$$Q_{\text{prototype}} = Q_{\text{model}}\frac{\left(b\sqrt{gh^3}\right)_{\text{prototype}}}{\left(b\sqrt{gh^3}\right)_{\text{model}}}$$

The gravitational acceleration is the same for the model and prototype (i.e., $g_1 = g_2$):

$$\boxed{\therefore Q_{\text{prototype}} = Q_{\text{model}}\left(\frac{b_{\text{prototype}}}{b_{\text{model}}}\right)\left(\frac{h_{\text{prototype}}}{h_{\text{model}}}\right)^{3/2}} \tag{4}$$

Use the given data to determine Q_2.

Q_{model} = 2.1 ft³/s
h_{model} = 6.3 in. = 0.525 ft.
$b_{\text{model}}/b_{\text{prototype}}$ = 1/16
$h_{\text{prototype}}$ = 3.2 ft
$\Rightarrow$ $\boxed{Q_{\text{prototype}} = 506 \text{ ft}^3/\text{s}}$

In the late 1940s, much of the science concerning nuclear bombs was highly classified. In particular, information regarding the energy released in a nuclear explosion, *e.g.* the number of equivalent kilotons of TNT (nowadays the energy is measured in megatons), was top secret. G.I. Taylor, a famous fluid mechanics professor, was asked in 1941 by the British Civil Defence Research Committee of the Ministry of Home Security to predict the dynamics of a blast caused by a nuclear explosion. In his analysis, Taylor assumed that a finite amount of energy, E, is suddenly released in an infinitely concentrated form. The resulting blast wave, with a radius R, then propagates into the surrounding atmosphere, with density ρ_0 and specific heat ratio $\gamma = c_p/c_v$, as a function of time, t. Taylor's analysis resulted in a simple relationship between the blast radius as a function of the time, air density, blast energy, and specific heat ratio. Using declassified photographs of the first nuclear explosion, which occurred at the Trinity test site in New Mexico in 1945, Taylor was able to estimate the energy release to within remarkable accuracy.

Perform a dimensional analysis to determine an expression involving the blast radius as a function of the other significant parameters in the problem.

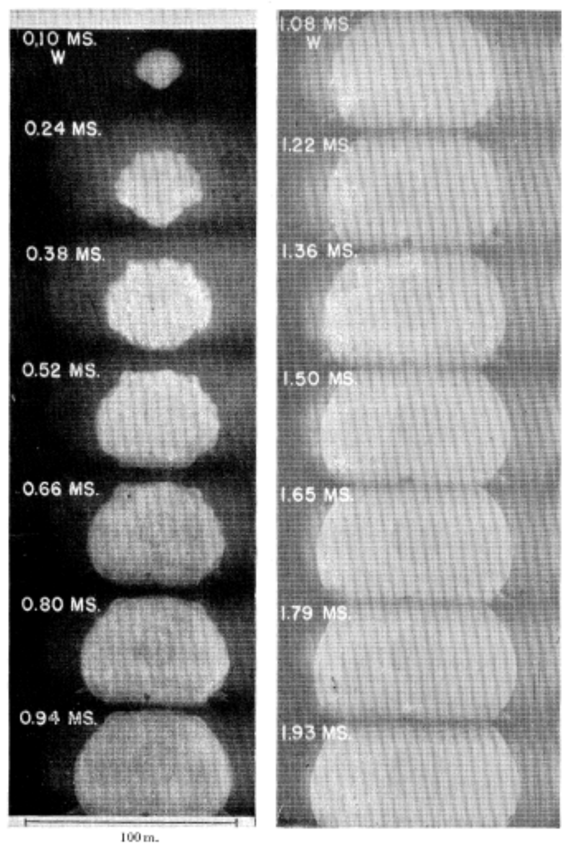


FIGURE 6. Succession of photographs of the 'ball of fire' from $t = 0{\cdot}10$ msec. to $1{\cdot}93$ msec.

References: Taylor, G., 1950, "The formation of a blast wave by a very intense explosion. I. Theoretical analyses," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 201, No. 1065, pp. 159 – 174. Taylor, G., 1950, "The formation of a blast wave by a very intense explosion. II. The atomic explosion of 1945," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 201, No. 1065, pp. 175 – 186.

SOLUTION:

1. Write the dimensional functional relationship.

$$R = f_1\left(t, E, \rho_0, \gamma\right) \tag{1}$$

2. Determine the basic dimensions of each parameter.

$[R] = L$
$[t] = T$
$[E] = FL = ML^2/T^2$
$[\rho_0] = M/L^3$
$[\gamma] = -$

3. Determine the number of Π terms required to describe the functional relationship.

# of variables = 5
# of reference dimensions = 3 (L, T, M)

(# Π terms) = (# of variables) – (# of reference dimensions) = 5 – 3 = 2

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions): t, E, ρ_0.

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = Rt^a E^b \rho_0^c \tag{2}$$

$$\Rightarrow \quad M^0 L^0 T^0 = \left(\frac{L}{1}\right)\left(\frac{T}{1}\right)^a \left(\frac{ML^2}{T^2}\right)^b \left(\frac{M}{L^3}\right)^c$$

$$\begin{array}{llll} M: & 0 = b + c & & a = -\tfrac{2}{5} \\ L: & 0 = 1 + 2b - 3c & \Rightarrow & b = -\tfrac{1}{5} \\ T: & 0 = a - 2b & & c = \tfrac{1}{5} \end{array}$$

$$\therefore \Pi_1 = \frac{R\rho_0^{1/5}}{t^{2/5} E^{1/5}} \tag{3}$$

$$\Pi_2 = \gamma \quad \text{(the specific heat ratio is already a dimensionless quantity)} \tag{4}$$

6. Verify that each Π term is, in fact, dimensionless.

$$\left[\Pi_1\right] = \left[\frac{R\rho_0^{1/5}}{t^{2/5} E^{1/5}}\right] = \frac{L}{1}\frac{M^{1/5}}{L^{3/5}}\frac{1}{T^{2/5}}\frac{T^{2/5}}{M^{1/5}L^{2/5}} = 1 \quad \text{OK!} \tag{5}$$

$$\left[\Pi_2\right] = \left[\gamma\right] = - \quad \text{OK!} \tag{6}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{R\rho_0^{1/5}}{t^{2/5} E^{1/5}} = f_2\left(\gamma\right)} \tag{7}$$

Note that the specific heat ratio and density of the atmosphere are well known (and assumed constant) so we could write Eq. (7) as:

$$R = c_1 t^{2/5} E^{1/5} \tag{8}$$

where c_1 is a constant (involving ρ_0 and γ). Taking the base 10 logarithm of both sides and re-arranging:

$$\tfrac{5}{2}\log_{10} R = \log_{10} t + \tfrac{5}{2}\left(\log_{10} c_1 + \tfrac{1}{5}\log_{10} E\right) \tag{9}$$

Thus, for a given explosion, the blast radius should follow a straight line when $(5/2)\log_{10}R$ is plotted as a function of $\log_{10}t$. The intercept of the line will be related to the atmospheric conditions (recall that c_1 is a function of ρ_0 and γ) and the blast energy, E. The following plots shows the measurements made from the video of the 1945 nuclear test. It's remarkable that the predictions performed by Taylor four years before the actual test are so accurate. In addition, simple radius vs. time measurements from a movie of the explosion could also easily give an estimate of the energy released. Taylor estimated the energy release to be between 23.7 kilotons of TNT. The actual energy release was estimated to be 20 kilotons.

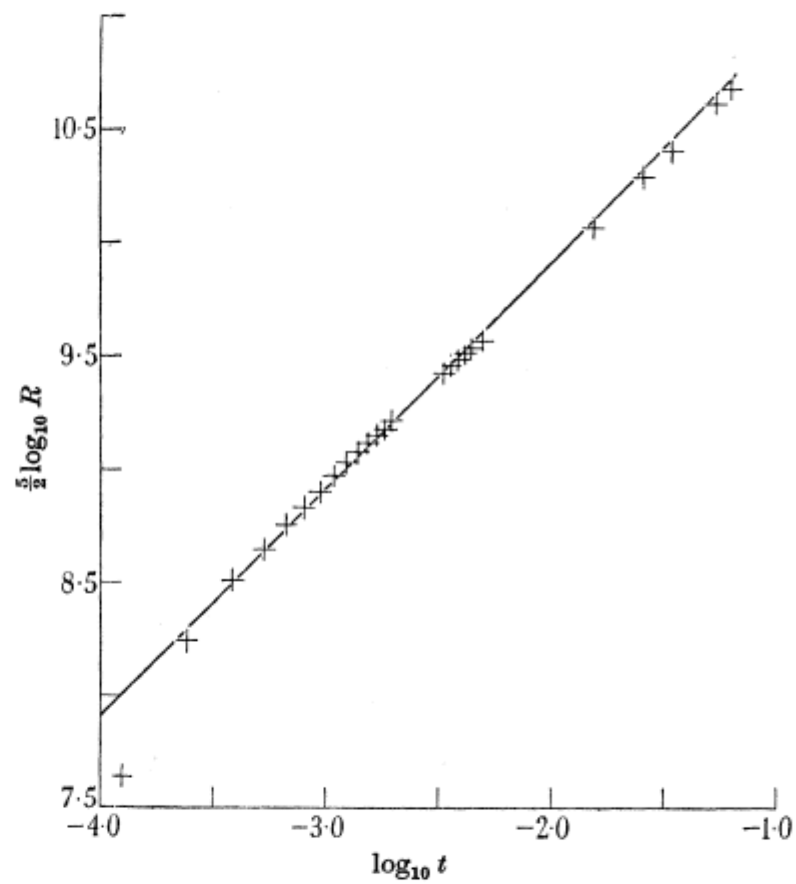


Figure 1. Logarithmic plot showing that $R^{5/2}$ is proportional to t.

The aerodynamic behavior of a flying insect is to be investigated in a wind tunnel using a ten-times scale model. If the insect flaps its wings 50 times a second when flying at 4 ft/s, determine the wind tunnel air speed and wing oscillation frequency required for dynamic similarity.

SOLUTION:

Maintain Reynolds number similarity,

$$\mathrm{Re}_M = \mathrm{Re}_P \;\Rightarrow\; \left.\frac{VL}{\nu}\right|_M = \left.\frac{VL}{\nu}\right|_P \;\Rightarrow\; V_M = V_P\left(\frac{L_P}{L_M}\right)\left(\frac{\nu_M}{\nu_P}\right) \tag{1}$$

where $L_P/L_M = 1/10$ and $\nu_M/\nu_P = 1$ (air used in both cases). Hence,

$$\boxed{V_M = (1/10)V_P = 0.4 \text{ ft/s}} \tag{2}$$

Also maintain Strouhal number similarity,

$$\mathrm{St}_M = \mathrm{St}_P \;\Rightarrow\; \left.\frac{\omega L}{V}\right|_M = \left.\frac{\omega L}{V}\right|_P \;\Rightarrow\; \omega_M = \omega_P\left(\frac{L_P}{L_M}\right)\left(\frac{V_M}{V_P}\right) \tag{3}$$

where $L_P/L_M = 1/10$ and $V_M/V_P = 1/10$. Hence,

$$\boxed{\omega_M = (1/100)\omega_P = 0.5 \text{ Hz}} \tag{4}$$

7.7. Review Questions

(1) Describe some of the benefits to performing a dimensional analysis of a problem.
(2) What does the Buckingham-Pi theorem state? Are the dimensionless terms resulting from the theorem unique?
(3) Describe the Method of Repeating Variables. Must this method always be followed to determine dimensionless terms?
(4) What is the difference between "basic dimensions" and "reference dimensions"?
(5) Describe the three types of similarity.
(6) Must there be exact similarity between a model and prototype in order to perform engineering modeling?
(7) In words, define the Reynolds, Froude, Strouhal, and Euler numbers.