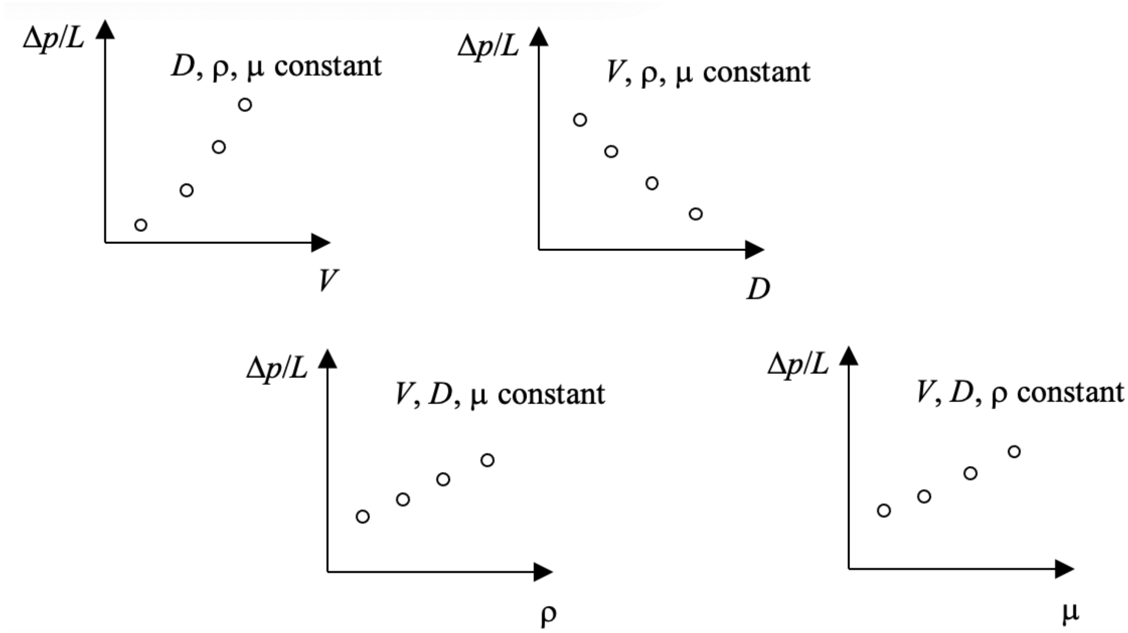


FIGURE 7.5. An illustration of the measured pressure gradient plotted as a function of the various independent variables.

7.2. Buckingham-Pi Theorem

The key component to dimensional analysis is the Buckingham-Pi Theorem: If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among $k-r$ independent dimensionless products (referred to as Π terms), where r is the minimum number of reference dimensions required to describe the variables, i.e.,

$$(\#\text{ of }\Pi\text{ terms}) = \underbrace{(\#\text{ of variables})}_{=k} - \underbrace{(\#\text{ of reference dimensions})}_{=r}. \tag{7.15}$$

The proof to this theorem will not be presented here.

Notes:

(1) Dimensionally homogeneous means that each term in the equation has the same units. For example, the following form of Bernoulli's equation,

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{constant}, \tag{7.16}$$

is dimensionally homogeneous since each term has units of length (L).

(2) A dimensionless product, also commonly referred to as a Pi (Π) term, is a term that has no dimensions. For example,

$$\frac{p}{\rho V^2}, \tag{7.17}$$

is a dimensionless product since both the numerator and denominator have the same dimensions.

(3) Reference dimensions are usually basic dimensions such as mass (M), length (L), and time (T) or force (F), length (L), and time (T). We'll discuss the "usually" modifier a little later when discussing the Method of Repeating Variables.

7.3. Method of Repeating Variables

The Buckingham-Pi Theorem merely states that a relationship among dimensional variables may be written, perhaps in a more compact form, in terms of dimensionless variables (Π terms). The Pi Theorem does not, however, tell us what these dimensionless variables are. The Method of Repeating Variables is an algorithm that can be used to determine these dimensionless variables.

The Method of Repeating Variables algorithm is as follows:

(1) *List all variables involved in the problem.*
   (a) This is the most difficult step since it requires experience and insight.
   (b) Variables are things like pressure, velocity, gravitational acceleration, viscosity, etc.
   (c) List only independent variables. For example, you can list ρ (density) and g (gravitational acceleration), or ρ and γ (specific weight), or g and γ, but you should not list ρ, g, and γ since one of the variables is dependent on the others.
   (d) If you include variables that are unimportant to the system, then you'll form Π terms that won't have an impact in practice. This situation is the same one you'd have if dimensional variables were used.
   (e) If you leave out an important variable, then you'll find that your relationship between dimensionless terms won't fully describe the system behavior. Again, this situation is the same one you'd have if you used dimensional terms.

(2) *Express each variable in terms of basic dimensions.*
   (a) For fluid mechanics problems we typically use mass (M), length (L), and time (T) or force (F), length (L), and time (T) as basic dimensions. We may occasionally need other basic dimensions such as temperature (θ).
   (b) For example, the dimensions of density can be written as,

$$[\rho] = \frac{M}{L^3} = \frac{FT^2}{L^4}. \tag{7.18}$$

   Note that the square brackets are used to indicate "dimensions of".

(3) *Determine the number of Π terms using the Buckingham-Pi Theorem.*
   (a) (# of Π terms) = (# of variables) – (# of reference dimensions)
   (b) Usually the number of reference dimensions will be the same as the number of basic dimensions found in the previous step. There are (rare) cases where some of the basic dimensions always appear in particular combinations so that the number of reference dimensions is less than the number of basic dimensions. For example, say that the variables in the problem are A, B, and C, and their corresponding basic dimensions, are,

$$[A] = \frac{M}{L^3} \quad [B] = \frac{M}{L^3T^2} \quad [C] = \frac{MT}{L^3}. \tag{7.19}$$

   The basic dimensions are M, L, and T (there are three basic dimensions). Notice, however, that the dimensions M and L always appear in the combination M/L^3. Thus, we really only need two reference dimensions, M/L^3 and T, to describe all of the variable dimensions.

(4) *Select repeating variables where the number of repeating variables is equal to the number of reference dimensions.*
   (a) The repeating variables should come from the list of independent variables. In our previous example, the list of independent variables is V, D, ρ, and μ.
   (b) Each repeating variable must have units independent of the other repeating variables.
   (c) Don't make the dependent variable one of the repeating variables. In our previous example, $\Delta p/L$ is the dependent variable. If we do that, then the resulting Π terms may have the dependent variable embedded in them.
   (d) All of the reference dimensions must be included in the group of repeating variables.

(5) *Form a Π term by multiplying one of the non-repeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.*
   (a) This step is most clearly illustrated in an example and will not be discussed here.

(b) Repeat this step for all non-repeating variables.

(6) *Check that all* Π *terms are dimensionless.*

(a) This is an important, but often overlooked, step to verify that your Π terms are, in fact, dimensionless.

(7) *Express the final form of the dimensional analysis as a relationship among the* Π *terms.*

(a) For example, $\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$.

Let's use our pipe flow experiment to demonstrate the procedure.

(1) *List all variables involved in the problem.*
The variables that are important in this problem are,

$$\Delta p/L \coloneqq \text{average pressure gradient over length } L \tag{7.20}$$

$$V \coloneqq \text{average flow velocity,} \tag{7.21}$$

$$D \coloneqq \text{pipe diameter,} \tag{7.22}$$

$$\rho \coloneqq \text{fluid density,} \tag{7.23}$$

$$\mu \coloneqq \text{fluid dynamic viscosity.} \tag{7.24}$$

Thus,

$$\Delta p/L = f_1(V, D, \rho, \mu). \tag{7.25}$$

(2) *Express each variable in terms of basic dimensions.*
The basic dimensions of each variable are,

$$\left[\frac{\Delta p}{L}\right] = \frac{F}{L^3} = \frac{M}{L^2T^2}, \tag{7.26}$$

$$[V] = \frac{L}{T}, \tag{7.27}$$

$$[D] = L, \tag{7.28}$$

$$[\rho] = \frac{M}{L^3}, \tag{7.29}$$

$$[\mu] = \frac{FT}{L^2} = \frac{M}{LT}. \tag{7.30}$$

(3) *Determine the number of* Π *terms using the Buckingham-Pi Theorem.*

- (# of variables) = 5 ($\Delta p/L$, V, D, ρ, μ)
- (# of reference dimensions) = 3 (F, L, T or M, L, T)
- (# of Π terms) = (# of variables) − (# of reference dimensions) = 2

Thus, instead of having a relation involving five terms, we actually have a relationship involving just two terms!

(4) *Select repeating variables where the number of repeating variables is equal to the number of reference dimensions.*
Select the following three repeating variables (three since the number of reference dimensions is three): ρ, V, D.

(a) These three repeating variables have independent dimensions.

(b) The dependent variable ($\Delta p/L$) is not one of the repeating variables.

(c) We could have also selected (V, μ, D) or (V, ρ, μ) or (μ, D, ρ) as repeating variables. The choice of repeating variables is somewhat arbitrary as long as they have independent reference dimensions and do not include the dependent variable.

(5) *Form a* Π *term by multiplying one of the non-repeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.*

$$\Pi_1 = \left(\frac{\Delta p}{L}\right)\rho^a V^b D^c \tag{7.31}$$

$$(MLT)^0 = \left(\frac{M}{L^2T^2}\right)\left(\frac{M}{L^3}\right)^a\left(\frac{L}{T}\right)^b\left(\frac{L}{1}\right)^c \tag{7.32}$$

Examining the M, L, and T terms individually,

$$M^0 = M^1 M^a \implies 0 = 1 + a \tag{7.33}$$

$$L^0 = L^{-2}L^{-3a}L^bL^c \implies 0 = -2 - 3a + b + c \tag{7.34}$$

$$T^0 = T^{-2}T^{-b} \implies 0 = -2 - b \tag{7.35}$$

Solving this system of equations gives: $a = -1, b = -2, c = 1$,

$$\therefore \Pi_1 = \frac{(\Delta p/L)D}{\rho V^2}. \tag{7.36}$$

Now consider the second Π term.

$$\Pi_2 = \mu\rho^a V^b D^c \tag{7.37}$$

$$(MLT)^0 = \left(\frac{M}{LT}\right)\left(\frac{M}{L^3}\right)^a\left(\frac{L}{T}\right)^b\left(\frac{L}{1}\right)^c \tag{7.38}$$

Examining the M, L, and T terms individually,

$$M^0 = M^1 M^a \implies 0 = 1 + a \tag{7.39}$$

$$L^0 = L^{-1}L^{-3a}L^bL^c \implies 0 = -1 - 3a + b + c \tag{7.40}$$

$$T^0 = T^{-1}T^{-b} \implies 0 = -1 - b \tag{7.41}$$

Solving this system of equations gives: $a = -1, b = -1, c = -1$,

$$\therefore \Pi_2 = \frac{\mu}{\rho V D}. \tag{7.42}$$

(6) *Check that all* Π *terms are dimensionless.*

$$[\Pi_1] = \left[\frac{(\Delta p/L)D}{\rho V^2}\right] = \frac{M}{L^2T^2}\frac{L}{1}\frac{L^3}{M}\frac{T^2}{L^2} = M^0L^0T^0 \quad \text{OK!} \tag{7.43}$$

$$[\Pi_2] = \left[\frac{\mu}{\rho V D}\right] = \frac{M}{LT}\frac{L^3}{M}\frac{T}{L}\frac{1}{L} = M^0L^0T^0 \quad \text{OK!} \tag{7.44}$$

(7) *Express the final form of the dimensional analysis as a relationship among the* Π *terms.* Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{(\Delta p/L)D}{\rho V^2} = f_2\left(\frac{\mu}{\rho V D}\right)}. \tag{7.45}$$

Notes:

(1) Instead of having to run four different sets of experiments as was discussed at the beginning of this chapter, we only really need to run one set of experiments where we vary,

$$\Pi_2 = \frac{\mu}{\rho V D}, \tag{7.46}$$

and measure,

$$\Pi_1 = \frac{(\Delta p/L)D}{\rho V^2}. \tag{7.47}$$

All of the information contained in Figure 7.5 is contained within Figure 7.6! This reduces the complexity, cost, and time required to determine the relationship between the average pressure gradient and the other variables.

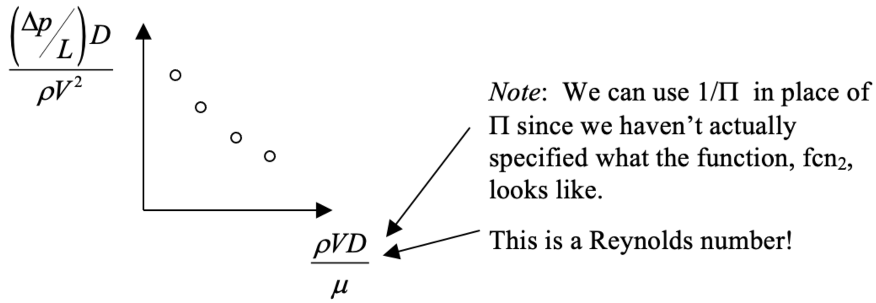


Figure 7.6. An illustration of the dimensionless pressure gradient plotted as a function of the Reynolds number. Compare this plot to the ones in Figure 7.5.

(2) Dimensional analysis is a very powerful tool because it tells us what terms really are important in an equation. For example, we started with the relation,

$$\frac{\Delta p}{L} = f_1(V, D, \rho, \mu), \tag{7.48}$$

leading us to believe that V, D, ρ, and μ are all important terms by themselves. However, dimensional analysis shows us that instead of the terms by themselves, it is the following grouping of terms,

$$\frac{(\Delta p/L)D}{\rho V^2} = f_2\left(\frac{\mu}{\rho V D}\right), \tag{7.49}$$

that is important in the relationship. This is a subtle but very important point.

(3) Dimensional analysis tells us how many dimensionless terms are important in a relation. It does not tell us what the functional relationship is. We need to rely on other analyses or experiments to determine the functional relationship.

(4) The dimensionless Π terms found via dimensionless analysis are not necessarily unique. Had we chosen different repeating variables in the previous example, we would have ended up with different Π terms. One can multiply, divide, or raise their set of Π terms to form the Π terms found by another. The number of Π terms, however, is unique.

(5) After a bit of practice, one can quickly form Π terms by inspection rather than having to go through the method of repeating variables.

An open cylindrical tank having a diameter D is supported around its bottom circumference and is filled to a depth h with a liquid having a specific weight γ. The vertical deflection, δ, of the center of the bottom is a function of D, h, d, γ, and E where d is the thickness of the bottom and E is the modulus of elasticity of the bottom material. Form the dimensionless groups describing this relationship.

SOLUTION:

1. Write the dimensional functional relationship.

$$\delta = f_1\left(D, h, d, \gamma, E\right)$$

2. Determine the basic dimensions of each parameter.

$$[\delta] = L$$
$$[h] = L$$
$$[D] = L$$
$$[d] = L$$
$$[\gamma] = \frac{M}{L^2T^2} = \frac{F}{L^3}$$
$$[E] = \frac{M}{LT^2} = \frac{F}{L^2}$$

3. Determine the number of Π terms required to describe the functional relationship.

\# of variables = 6 (δ, D, h, d, γ, E)
\# of reference dimensions = 2 (L, F/L^2 or L, M/T^2)
(Note that the number of reference dimensions and the number of basic dimensions are not the same for this problem!)

(# Π terms) = (# of variables) – (# of reference dimensions) = 6 – 2 = 4

4. Choose two repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

D, γ (Note that the dimensions for D and γ are independent.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = \delta D^a \gamma^b$$
$$\Rightarrow F^0 L^0 = \left(L/1\right)\left(L/1\right)^a \left(F/L^3\right)^b$$
$$F: \ 0 = b$$
$$L: \ 0 = 1 + a - 3b \ \Rightarrow \ a = -1$$
$$\therefore \Pi_1 = \delta/D$$

$$\Pi_2 = hD^a \gamma^b$$
$$\Rightarrow F^0 L^0 = \left(L/1\right)\left(L/1\right)^a \left(F/L^3\right)^b$$
$$F: \ 0 = b$$
$$L: \ 0 = 1 + a - 3b \ \Rightarrow \ a = -1$$
$$\therefore \Pi_2 = h/D$$

$$\Pi_3 = dD^a\gamma^b$$
$$\Rightarrow\ F^0L^0 = \left(L/1\right)\left(L/1\right)^a\left(F/L^3\right)^b$$
$$F:\ 0 = b$$
$$L:\ 0 = 1 + a - 3b\ \Rightarrow\ a = -1$$
$$\therefore \Pi_3 = d/D$$

$$\Pi_4 = ED^a\gamma^b$$
$$\Rightarrow\ F^0L^0 = \left(F/L^2\right)\left(L/1\right)^a\left(F/L^3\right)^b$$
$$F:\ 0 = 1 + b\ \Rightarrow\ b = -1$$
$$L:\ 0 = -2 + a - 3b\ \Rightarrow\ a = -1$$
$$\therefore \Pi_4 = E/(D\gamma)$$

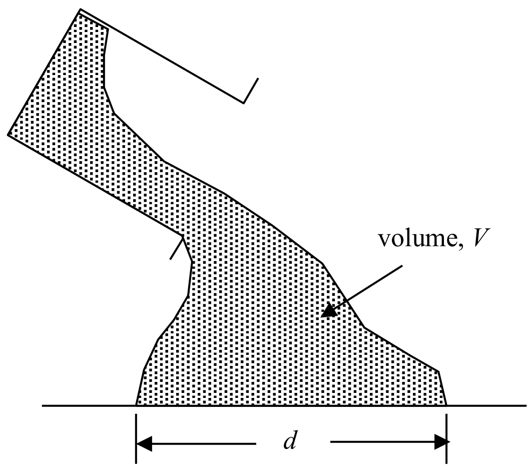
6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\delta/D\right] = L/L = 1\ \text{ OK!}$$
$$[\Pi_2] = \left[h/D\right] = L/L = 1\ \text{ OK!}$$
$$[\Pi_3] = \left[d/D\right] = L/L = 1\ \text{ OK!}$$
$$[\Pi_4] = \left[E/(D\gamma)\right] = \left(F/L^2\right)\left(1/L\right)\left(L^3/F\right) = 1\ \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{\delta}{D} = f_2\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{D\gamma}\right)}$$

A viscous fluid is poured onto a horizontal plate as shown in the figure. Assume that the time, *t*, required for the fluid to flow a certain distance, *d*, along the plate is a function of the volume of fluid poured, *V*, acceleration due to gravity, *g*, fluid density, ρ, and fluid dynamic viscosity, μ. Determine an appropriate set of dimensionless terms to describe this process.



SOLUTION:

1. Write the dimensional functional relationship.

$$t = f_1(d, V, g, \rho, \mu)$$

2. Determine the basic dimensions of each parameter.

$$[t] = T$$
$$[d] = L$$
$$[V] = L^3$$
$$[g] = L/T^2$$
$$[\rho] = M/L^3$$
$$[\mu] = M/(LT)$$

3. Determine the number of Π terms required to describe the functional relationship.

   \# of variables = 6 (t, d, V, g, ρ, μ)
   \# of reference dimensions = 3 (T, L, M)
   (Note that the number of reference dimensions and the number of basic dimensions are equal for this problem.)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 6 – 3 = 3

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

   d, g, ρ (Note that the dimensions for these variables are independent.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = td^a g^b \rho^c$$
$$\Rightarrow\ M^0L^0T^0 = \left(\frac{T}{1}\right)\left(\frac{L}{1}\right)^a\left(\frac{L}{T^2}\right)^b\left(\frac{M}{L^3}\right)^c$$
$$\begin{array}{llll} M: & 0 = c & & a = -\tfrac{1}{2} \\ L: & 0 = a + b - 3c & \Rightarrow & b = \tfrac{1}{2} \\ T: & 0 = 1 - 2b & & c = 0 \end{array}$$
$$\therefore \Pi_1 = t\sqrt{\frac{g}{d}}$$

$$\Pi_2 = Vd^a g^b \rho^c$$
$$\Rightarrow\ M^0L^0T^0 = \left(\frac{L^3}{1}\right)\left(\frac{L}{1}\right)^a\left(\frac{L}{T^2}\right)^b\left(\frac{M}{L^3}\right)^c$$
$$\begin{array}{llll} M: & 0 = c & & a = -3 \\ L: & 0 = 3 + a + b - 3c & \Rightarrow & b = 0 \\ T: & 0 = -2b & & c = 0 \end{array}$$
$$\therefore \Pi_2 = \frac{V}{d^3}$$

$$\Pi_3 = \mu d^a g^b \rho^c$$
$$\Rightarrow\ M^0L^0T^0 = \left(\frac{M}{LT}\right)\left(\frac{L}{1}\right)^a\left(\frac{L}{T^2}\right)^b\left(\frac{M}{L^3}\right)^c$$
$$\begin{array}{llll} M: & 0 = 1 + c & & a = -\tfrac{3}{2} \\ L: & 0 = -1 + a + b - 3c & \Rightarrow & b = -\tfrac{1}{2} \\ T: & 0 = -1 - 2b & & c = -1 \end{array}$$
$$\therefore \Pi_3 = \frac{\mu}{\rho d\sqrt{gd}}$$

6. Verify that each Π term is, in fact, dimensionless.

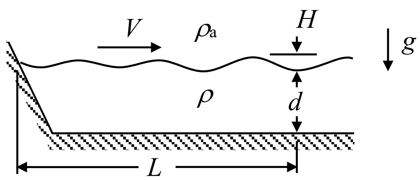
$$[\Pi_1] = \left[t\sqrt{\frac{g}{d}}\right] = \frac{T}{1}\frac{L^{1/2}}{T}\frac{1}{L^{1/2}} = 1 \text{ OK!}$$
$$[\Pi_2] = \left[\frac{V}{d^3}\right] = \frac{L^3}{1}\frac{1}{L^3} = 1 \text{ OK!}$$
$$[\Pi_3] = \left[\frac{\mu}{\rho d\sqrt{gd}}\right] = \frac{M}{LT}\frac{L^3}{M}\frac{1}{L}\frac{T}{L^{1/2}}\frac{1}{L^{1/2}} = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{t\sqrt{\frac{g}{d}} = f_2\left(\frac{V}{d^3}, \frac{\mu}{\rho d\sqrt{gd}}\right)}$$

It is desired to determine the wave height when wind blows across a lake. The wave height, H, is assumed to be a function of the wind speed, V, the water density, ρ, the air density, ρ_a, the water depth, d, the distance from the shore, L, and the acceleration of gravity, g. Use d, V, and ρ as repeating variables to determine a suitable set of pi terms that could be used to describe this problem.

SOLUTION:



1. Write the dimensional functional relationship.
   $H = f_1(V, \rho, \rho_a, d, L, g)$

2. Determine the basic dimensions of each parameter.
   $[H] = L$
   $[V] = L/T$
   $[\rho] = M/L^3$
   $[\rho_a] = M/L^3$
   $[d] = L$
   $[L] = L$
   $[g] = L/T^2$

3. Determine the number of Π terms required to describe the functional relationship.
   # of variables = 7 (H, V, ρ, ρ_a, d, L, g)
   # of reference dimensions = 3 (L, T, M)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 7 – 3 = 4

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
   d, V, ρ

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

   $\Pi_1 = \dfrac{H}{d}$ (by inspection)

   $\Pi_2 = \dfrac{\rho_a}{\rho}$ (by inspection)

   $\Pi_3 = \dfrac{L}{d}$ (by inspection)

   $\Pi_4 = \dfrac{V}{\sqrt{gd}}$ (by inspection, This is a Froude number!)

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{H}{d}\right] = \frac{L}{1}\frac{1}{L} = 1 \text{ OK!}$$

$$[\Pi_2] = \left[\frac{\rho_a}{\rho}\right] = \frac{M}{L^3}\frac{L^3}{M} = 1 \text{ OK!}$$

$$[\Pi_3] = \left[\frac{L}{d}\right] = \frac{L}{1}\frac{1}{L} = 1 \text{ OK!}$$

$$[\Pi_4] = \left[\frac{V}{\sqrt{gd}}\right] = \frac{L}{T}\frac{T}{L^{1/2}}\frac{1}{L^{1/2}} = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{H}{d} = f_2\left(\frac{\rho_a}{\rho}, \frac{L}{d}, \frac{V}{\sqrt{gd}}\right)}$$

Small droplets of liquid are formed when a liquid jet breaks up in spray and fuel injection processes. The resulting droplet diameter, d, is thought to depend on liquid density, ρ, viscosity, μ, and surface tension, σ, as well as jet speed, V, and diameter, D. How many dimensionless ratios are required to characterize this process? Determine these ratios.

SOLUTION:

1. Write the dimensional functional relationship.

$$d = f_1\left(\rho, \mu, \sigma, V, D\right)$$

2. Determine the basic dimensions of each parameter.

$$[d] = L$$
$$[\rho] = M/L^3$$
$$[\mu] = M/LT$$
$$[\sigma] = F/L = M/T^2$$
$$[V] = L/T$$
$$[D] = L$$

3. Determine the number of Π terms required to describe the functional relationship.

# of variables = 6 (d, ρ, μ, σ, V, D)
# of reference dimensions = 3 (M, L, T)

$$\boxed{(\#\ \Pi \text{ terms}) = (\#\text{ of variables}) - (\#\text{ of reference dimensions}) = 6 - 3 = 3}$$

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

ρ, V, D (Note that these repeating variables have independent dimensions.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = d\rho^a V^b D^c$$
$$\Rightarrow\ M^0L^0T^0 = \left(L/1\right)\left(M/L^3\right)^a\left(L/T\right)^b\left(L/1\right)^c$$

M: $0 = a \Rightarrow a = 0$
T: $0 = -b \Rightarrow b = 0$
L: $0 = 1 - 3a + b + c \Rightarrow c = -1$

$$\therefore \Pi_1 = \frac{d}{D}$$

$$\Pi_2 = \mu\rho^a V^b D^c$$

$$\Rightarrow \ M^0 L^0 T^0 = \left(\frac{M}{LT}\right)\left(M/L^3\right)^a \left(L/T\right)^b \left(L/1\right)^c$$

$$\begin{array}{lll} M: & 0 = 1 + a & \Rightarrow \ a = -1 \\ T: & 0 = -1 - b & \Rightarrow \ b = -1 \\ L: & 0 = -1 - 3a + b + c & \Rightarrow \ c = -1 \end{array}$$

$$\therefore \Pi_2 = \frac{\mu}{\rho V D} \text{ or } \Pi_2 = \frac{\rho V D}{\mu} \quad \text{(a Reynolds number!)}$$

$$\Pi_3 = \sigma\rho^a V^b D^c$$

$$\Rightarrow \ M^0 L^0 T^0 = \left(\frac{M}{T^2}\right)\left(M/L^3\right)^a \left(L/T\right)^b \left(L/1\right)^c$$

$$\begin{array}{lll} M: & 0 = 1 + a & \Rightarrow \ a = -1 \\ T: & 0 = -2 - b & \Rightarrow \ b = -2 \\ L: & 0 = -3a + b + c & \Rightarrow \ c = -1 \end{array}$$

$$\therefore \Pi_3 = \frac{\sigma}{\rho V^2 D} \text{ or } \Pi_3 = \frac{\rho V^2 D}{\sigma} \quad \text{(a Weber number!)}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{d}{D}\right] = L/1 \ 1/L = 1 \text{ OK!}$$

$$[\Pi_2] = \left[\frac{\rho V D}{\mu}\right] = M/L^3 \ L/T \ L/1 \ LT/M = 1 \text{ OK!}$$

$$[\Pi_3] = \left[\frac{\rho V^2 D}{\sigma}\right] = M/L^3 \ L^2/T^2 \ L/1 \ T^2/M = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{d}{D} = f_2\left(\underbrace{\frac{\rho V D}{\mu}}_{\text{Reynolds \#}}, \underbrace{\frac{\rho V^2 D}{\sigma}}_{\text{Weber \#}}\right)} \quad (1)$$

Spin plays an important role in the flight trajectory of golf, Ping-Pong, and tennis balls. Therefore, it is important to know the rate at which spin decreases for a ball in flight. The aerodynamic torque, T, acting on a ball in flight, is thought to depend on flight speed, V, air density, ρ, air viscosity, μ, ball diameter, D, spin rate (angular speed), ω, and diameter of the dimples on the ball, d. Determine the dimensionless parameters that result.

SOLUTION:

1. Write the dimensional functional relationship.
$$T = f_1\left(V, \rho, \mu, D, \omega, d\right)$$

2. Determine the basic dimensions of each parameter.
$$[T] = F \cdot L = {ML^2}/{T^2}$$
$$[V] = {L}/{T}$$
$$[\rho] = {M}/{L^3}$$
$$[\mu] = {M}/{LT}$$
$$[D] = L$$
$$[\omega] = {1}/{T}$$
$$[d] = L$$

3. Determine the number of Π terms required to describe the functional relationship.
# of variables = 7 (T, V, ρ, μ, D, ω, d)
# of reference dimensions = 3 (M, L, T)

(# Π terms) = (# of variables) – (# of reference dimensions) = 7 – 3 = 4

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
ρ, V, D (Note that these repeating variables have independent dimensions.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.
$$\Pi_1 = T\rho^a V^b D^c$$
$$\Rightarrow\ M^0 L^0 T^0 = \left({ML^2}/{T^2}\right)\left({M}/{L^3}\right)^a \left({L}/{T}\right)^b \left({L}/{1}\right)^c$$
$$\begin{array}{lll} M: & 0 = 1 + a & \Rightarrow\ a = -1 \\ T: & 0 = -2 - b & \Rightarrow\ b = -2 \\ L: & 0 = 2 - 3a + b + c & \Rightarrow\ c = -3 \end{array}$$
$$\therefore \Pi_1 = \frac{T}{\rho V^2 D^3}$$

$$\Pi_2 = \mu\rho^a V^b D^c$$
$$\Rightarrow M^0 L^0 T^0 = \left(\frac{M}{LT}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b \left(\frac{L}{1}\right)^c$$
$$\begin{array}{lll} M: & 0 = 1 + a & \Rightarrow a = -1 \\ T: & 0 = -1 - b & \Rightarrow b = -1 \\ L: & 0 = -1 - 3a + b + c & \Rightarrow c = -1 \end{array}$$
$$\therefore \Pi_2 = \frac{\mu}{\rho V D} \text{ or } \Pi_2 = \frac{\rho V D}{\mu} \quad \text{(a Reynolds number!)}$$

$$\Pi_3 = \omega\rho^a V^b D^c$$
$$\Rightarrow M^0 L^0 T^0 = \left(\frac{1}{T}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b \left(\frac{L}{1}\right)^c$$
$$\begin{array}{lll} M: & 0 = a & \Rightarrow a = 0 \\ T: & 0 = -1 - b & \Rightarrow b = -1 \\ L: & 0 = -3a + b + c & \Rightarrow c = 1 \end{array}$$
$$\therefore \Pi_3 = \frac{\omega D}{V}$$

$$\Pi_4 = d\rho^a V^b D^c$$
$$\Rightarrow M^0 L^0 T^0 = \left(\frac{L}{1}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b \left(\frac{L}{1}\right)^c$$
$$\begin{array}{lll} M: & 0 = a & \Rightarrow a = 0 \\ T: & 0 = -b & \Rightarrow b = 0 \\ L: & 0 = 1 + c & \Rightarrow c = -1 \end{array}$$
$$\therefore \Pi_4 = \frac{d}{D}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{T}{\rho V^2 D^3}\right] = \frac{ML^2}{T^2}\frac{L^3}{M}\frac{T^2}{L^2}\frac{1}{L^3} = 1 \text{ OK!}$$
$$[\Pi_2] = \left[\frac{\rho V D}{\mu}\right] = \frac{M}{L^3}\frac{L}{T}\frac{L}{1}\frac{LT}{M} = 1 \text{ OK!}$$
$$[\Pi_3] = \left[\frac{\omega D}{V}\right] = \frac{1}{T}\frac{L}{1}\frac{T}{L} = 1 \text{ OK!}$$
$$[\Pi_4] = \left[\frac{d}{D}\right] = \frac{L}{1}\frac{1}{L} = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{T}{\rho V^2 D^3} = f_2\left(\underbrace{\frac{\rho V D}{\mu}}_{\text{Reynolds \#}}, \frac{\omega D}{V}, \frac{d}{D}\right)} \qquad (1)$$