13.6. Adiabatic, Steady, 1D Compressible Flow of a Perfect Gas

Now let's consider the 1D, steady, adiabatic flow of a compressible gas. Recall that from the Energy Equation we have,

$$dh + V dV = 0. \tag{13.69}$$

For an ideal gas we can re-write the specific enthalpy in terms of the specific heat at constant pressure, c_p, which is a function of temperature, in general, and the absolute temperature, T,

$$dh = c_p(T) dT. \tag{13.70}$$

Substituting,

$$c_p(T) dT + V dV = 0. \tag{13.71}$$

Integrating along a stream tube,

$$\int_{T_{\text{in}}}^{T_{\text{out}}} c_p(T) dT + \frac{1}{2} V^2 = \text{constant}. \tag{13.72}$$

If the gas can be assumed perfect, i.e., c_p = constant, then the previous equation becomes,

$$c_p T + \frac{1}{2} V^2 = \text{constant}, \tag{13.73}$$

$$\boxed{T + \frac{V^2}{2c_p} = \text{constant}}. \tag{13.74}$$

We can re-write this equation in terms of the Mach number,

$$\text{Ma} = \frac{V}{c} = \frac{V}{\sqrt{kRT}} \implies V^2 = (kRT)\text{Ma}^2 \implies T + \frac{kRT\text{Ma}^2}{2c_p} = \text{constant}. \tag{13.75}$$

Substituting the following ideal gas relation,

$$\frac{R}{c_p} = \frac{k-1}{k}, \tag{13.76}$$

results in,

$$\boxed{T\left(1 + \frac{k-1}{2}\text{Ma}^2\right) = \text{constant}} \quad \underline{\text{adiabatic, 1D, steady flow of a perfect gas}}. \tag{13.77}$$

If the flow can also be considered internally reversible, making the flow isentropic, then we can use the isentropic relations for a perfect gas,

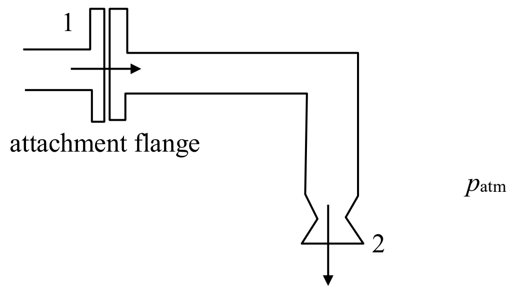
$$p = (\text{constant})T^{\frac{k}{k-1}} \quad \text{and} \quad \rho = (\text{constant})T^{\frac{1}{k-1}}, \tag{13.78}$$

to give,

$$\boxed{p\left(1 + \frac{k-1}{2}\text{Ma}^2\right)^{\frac{k}{k-1}} = \text{constant}} \quad \text{and} \quad \boxed{\rho\left(1 + \frac{k-1}{2}\text{Ma}^2\right)^{\frac{1}{k-1}} = \text{constant}}. \tag{13.79}$$

These relations are for <u>isentropic, 1D, steady flow of a perfect gas</u>.

A steady flow of air passes through the elbow-nozzle assembly shown. At the inlet (1), the pipe diameter is D_1 = 0.1524 m and the air properties are p_1 = 871.7 kPa (abs), T_1 = 529.0 K, and V_1 = 230.4 m/s. The air is expanded through a converging-diverging nozzle discharging into the atmosphere where p_{atm} = 101.3 kPa (abs). At the nozzle exit (2), the nozzle diameter is D_2 = 0.3221 m and the air properties are T_2 = 475.7 K and V_2 = 400.0 m/s.



a. Is the flow through the elbow-nozzle assembly adiabatic?
b. Determine the components of the force in the attachment flange required to hold the elbow-nozzle assembly in place. You may neglect the effects of gravity.

SOLUTION:

If the flow is adiabatic in going from 1 to 2, then the energy equation will give:

$$T_1 + \frac{V_1^2}{2c_P} = T_2 + \frac{V_2^2}{2c_P} \quad (1)$$

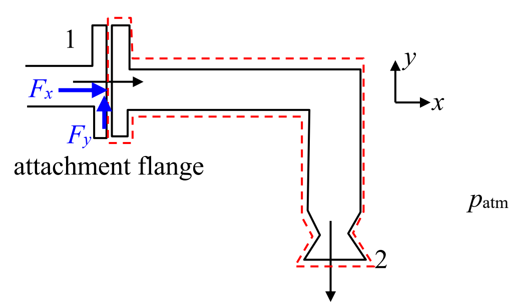
Using the given data:

T_1 = 529.0 K
V_1 = 230.4 m/s
c_P = 1004 J/(kg·K)
T_2 = 475.7 K
V_2 = 400.0 m/s

$$\Rightarrow \quad T_1 + \frac{V_1^2}{2c_P} = 555.4 \text{ K and } \quad T_2 + \frac{V_2^2}{2c_P} = 555.1 \text{ K}$$

Since the stagnation temperatures are approximately the same, the flow can be considered adiabatic in going from 1 to 2.

To determine the force components, apply the linear momentum equation to the control volume shown below using the indicated fixed frame of reference.



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = \left(V_1\right)\left(-\rho_1 V_1 \frac{\pi D_1^2}{4}\right) = -\rho_1 V_1^2 \frac{\pi D_1^2}{4}$$

$$F_{B,x} = 0$$

$$F_{S,x} = \left(p_1 - p_{atm}\right)\frac{\pi D_1^2}{4} + F_x$$

Substitute and simplify.

$$-\rho_1 V_1^2 \frac{\pi D_1^2}{4} = \left(p_1 - p_{atm}\right)\frac{\pi D_1^2}{4} + F_x$$

$$\boxed{F_x = -\rho_1 V_1^2 \frac{\pi D_1^2}{4} - \left(p_1 - p_{atm}\right)\frac{\pi D_1^2}{4}} \qquad (2)$$

Using the given numerical data:

ρ_1 = $p_1/(RT_1)$ = (871.7 kPa)/[287 J/(kg·K) ·529 K] = 5.738 kg/m^3
V_1 = 230.4 m/s
D_1 = 0.1524 m
p_1 = 871.7 kPa (abs)
p_{atm} = 101.3 kPa (abs)
$\Rightarrow$ $\boxed{F_x = -19.60 \text{ kN}}$

Also note that:

$$\dot{m}_1 = \rho_1 V_1 \frac{\pi D_1^2}{4} = 24.11 \text{ kg/s} \qquad (3)$$

Now consider the y-direction.

$$\frac{d}{dt}\int_{CV} u_y \rho dV + \int_{CS} u_y \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,y} + F_{S,y}$$

where

$$\frac{d}{dt}\int_{CV} u_y \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u_y \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = \left(-V_2\right)\left(\rho_2 V_2 \frac{\pi D_2^2}{4}\right) = -\rho_2 V_2^2 \frac{\pi D_2^2}{4}$$

$$F_{B,y} = 0$$

$$F_{S,y} = \left(p_2 - p_{atm}\right)\frac{\pi D_2^2}{4} + F_y$$

Substitute and simplify.

$$-\rho_2 V_2^2 \frac{\pi D_2^2}{4} = \left(p_2 - p_{atm}\right)\frac{\pi D_2^2}{4} + F_y$$

$$\boxed{F_y = -\rho_2 V_2^2 \frac{\pi D_2^2}{4} - \left(p_2 - p_{atm}\right)\frac{\pi D_2^2}{4}} \qquad (4)$$

Note that from conservation of mass on the same control volume:

$$\dot{m}_1 = \dot{m}_2 = \rho_2 V_2 \frac{\pi D_2^2}{4} \qquad (5)$$

The pressure at point 2 will depend on whether the flow at that point is subsonic or supersonic.

$$\mathrm{Ma}_2 = \frac{V_2}{c_2} = \frac{V_2}{\sqrt{\gamma R T_2}} \tag{6}$$

Using the given data:

V_2 = 400.0 m/s
γ = 1.4
R = 287 J/(kg·K)
T_2 = 475.7 K

$\Rightarrow$ $c_2 = 437.2$ m/s $\Rightarrow$ $\mathrm{Ma}_2 = 0.9148$ $\Rightarrow$ The flow at point 2 is subsonic. $\Rightarrow$ The pressure at point 2 is equal to atmospheric pressure, i.e., $p_2 = 101.3$ kPa (abs).

Substituting the given numerical data in Eq. (4) gives $\boxed{F_y = -9.644 \text{ kN}}$.

13.7. Stagnation and Sonic Conditions

It's convenient to choose some useful reference points in the flow where we can evaluate the constants in Eqs. (13.77) and (13.79). Two such reference points are commonly used in compressible flows: stagnation conditions and sonic conditions.

Stagnation Conditions: Stagnation conditions are those conditions that would occur if the fluid is brought to rest ($V = 0 \implies \mathrm{Ma} = 0$). These conditions are typically indicated by the subscript "$_0$". Equations (13.77) and (13.79) can be written in terms of stagnation conditions as,

$$\boxed{\frac{T}{T_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{-1}} \quad \underline{\text{adiabatic, steady, 1D flow of a perfect gas}}, \tag{13.80}$$

$$\boxed{\frac{p}{p_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{k}{1-k}}} \quad \underline{\text{isentropic, steady, 1D flow of a perfect gas}}, \tag{13.81}$$

$$\boxed{\frac{\rho}{\rho_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{1}{1-k}}} \quad \underline{\text{isentropic, steady, 1D flow of a perfect gas}}. \tag{13.82}$$

We can also determine the speed of sound at stagnation conditions using the fact that $c = \sqrt{kRT}$,

$$\boxed{\frac{c}{c_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{-\frac{1}{2}}} \quad \underline{\text{adiabatic, steady, 1D flow of a perfect gas}}. \tag{13.83}$$

These various stagnation ratios are plotted in Figure 13.15 as a function of Mach number for air.

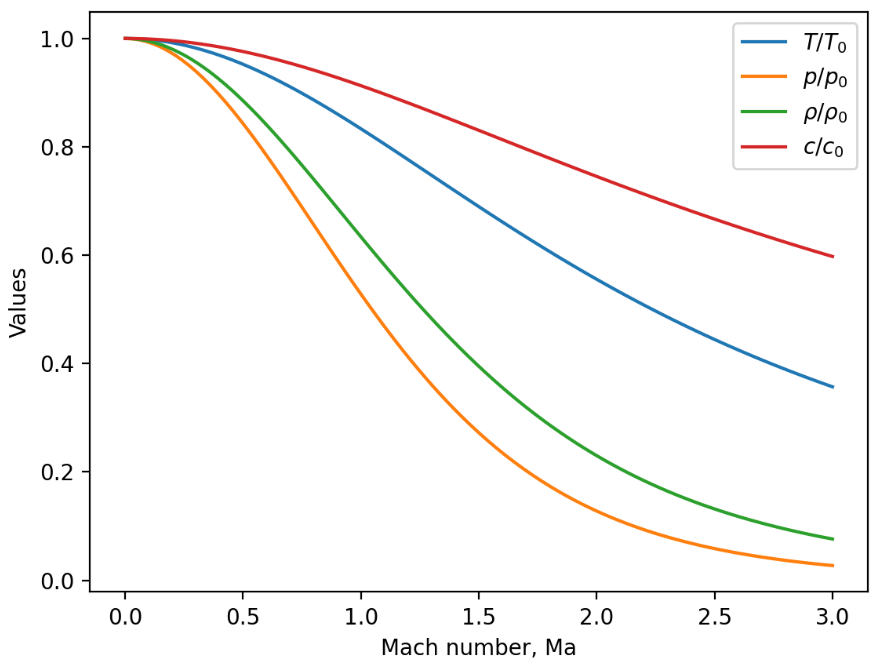


Figure 13.15. Various stagnation ratios, referring to Eqs. (13.80) - (13.83), plotted as a function of the Mach number. These ratios are for a specific heat ratio of $k = 1.4$, which corresponds to air.

Notes:

(1) Stagnation conditions are also commonly referred to as total conditions, given by the subscript "T".
(2) Stagnation conditions can be determined even for a moving fluid. The fluid doesn't necessarily have to be at rest to state its stagnation conditions. To determine stagnation conditions we only need to imagine the conditions *if* the flow is brought to rest.
(3) Equations (13.81) and (13.82) are for a flow brought to rest isentropically. Equations (13.80) and (13.83) are for a flow brought to rest adiabatically.
(4) Tables listing the values of Eqs. (13.80) - (13.83) for various Mach numbers are typically given in the back of most textbooks concerning compressible flows.
(5) Note that the stagnation temperature is greater than the flow temperature since when the flow is decelerated to zero velocity, the macroscopic kinetic energy is converted into internal energy (microscopic kinetic energy) and, thus, the temperature increases.
(6) The stagnation pressure is a significant property for a flow because it's directly related to the amount of work that can be extracted from the flow. For example, imagine bringing a flow to rest so we have stagnation conditions within the tank shown in Figure 13.16. The larger the stagnation pressure in the tank, the greater the force we can exert on the piston that can be used to perform useful work.

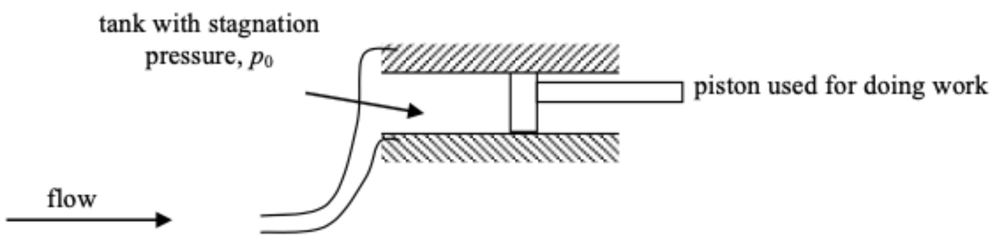


Figure 13.16. An illustration showing how stagnation pressure is related to the ability to do work.

Sonic Conditions: Another convenient reference point is where the flow has a Mach number of one (Ma = 1). Conditions where the Mach number is one are known as sonic conditions and are typically specified using the superscript "*".

Equations (13.80) - (13.83) evaluated at sonic conditions are:

$$\boxed{\frac{T^*}{T_0} = \left(1 + \frac{k-1}{2}\right)^{-1}} \quad \text{adiabatic, steady, 1D flow of a perfect gas,} \tag{13.84}$$

$$\boxed{\frac{p^*}{p_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{1-k}}} \quad \text{isentropic, steady, 1D flow of a perfect gas,} \tag{13.85}$$

$$\boxed{\frac{\rho^*}{\rho_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{1}{1-k}}} \quad \text{isentropic, steady, 1D flow of a perfect gas,} \tag{13.86}$$

$$\boxed{\frac{c^*}{c_0} = \left(1 + \frac{k-1}{2}\right)^{-\frac{1}{2}}} \quad \text{adiabatic, steady, 1D flow of a perfect gas.} \tag{13.87}$$

Notes:

(1) For air ($k = 1.4$), Eqs. (13.84) - (13.87) become,

$$\boxed{\frac{T^*}{T_0} = 0.8333} \quad \underline{\text{adiabatic, steady, 1D flow of air as a perfect gas}}, \tag{13.88}$$

$$\boxed{\frac{p^*}{p_0} = 0.5283} \quad \underline{\text{isentropic, steady, 1D flow of air as a perfect gas}}, \tag{13.89}$$

$$\boxed{\frac{\rho^*}{\rho_0} = 0.6339} \quad \underline{\text{isentropic, steady, 1D flow of air as a perfect gas}}, \tag{13.90}$$

$$\boxed{\frac{c^*}{c_0} = 0.9129} \quad \underline{\text{adiabatic, steady, 1D flow of air as a perfect gas}}. \tag{13.91}$$

The p^*/p_0 ratio value is a useful one to memorize since differences in pressure are what drive most compressible flows.

On the assumption of isentropic flow: In many engineering gas dynamics flows, the assumption that the entropy of the fluid remains constant (an isentropic process) is a good one. If viscous and heat transfer effects can be neglected, then we can reasonably assume that the flow is isentropic (isentropic = internally reversible + adiabatic). This situation is often the case for flows through short, insulated ducts or through stream tubes not passing through a boundary layer or a shock wave (strong viscous effects occur in both cases). Experiments have verified that the isentropic assumption under these conditions is reasonable.

13.8. Mollier (aka h-s) Diagrams

Mollier diagrams are diagrams that plot the enthalpy (h) as a function of entropy (s) for a process. They are often useful in visualizing trends.

Notes:

(1) Sketches of constant pressure and constant volume (or density) curves are shown in Figure 13.17.

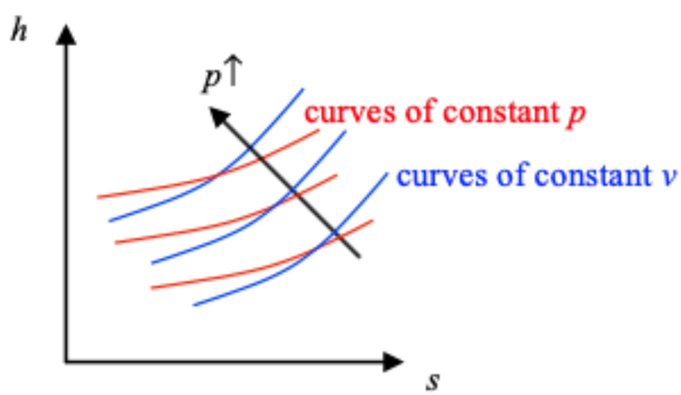


Figure 13.17. An example Mollier plot showing isobars (curves of constant pressure) and isochores (curves of constant specific volume). The curves become steeper as the specific enthalpy increases. The pressure and specific volume increase as one moves to curves approaching the upper left of the plot.

(2) For a perfect gas, curves of constant volume (or density) and constant pressure have slopes given, respectively, by,

$$Tds = du + pdv, \tag{13.92}$$

$$Tds = \frac{c_v}{c_p}\underbrace{(c_p dT)}_{=dh} + pdv, \tag{13.93}$$

$$\therefore \left.\frac{dh}{ds}\right|_v = kT, \tag{13.94}$$

and,

$$Tds = dh - vdp, \tag{13.95}$$

$$\therefore \left.\frac{dh}{ds}\right|_p = T. \tag{13.96}$$

Thus, larger temperatures (and, thus, larger specific enthalpies) will result in steeper slopes for curves of constant pressure and constant volume.

(3) From the Energy Equation (refer to Eq. (13.69)), the difference between the flow specific enthalpy and the stagnation specific enthalpy for an isentropic process is equal to the specific kinetic energy,

$$h_0 = h + \frac{1}{2}V^2. \tag{13.97}$$

Figure 13.18 shows this relationship specifically for an isentropic process, but the relationship holds for non-isentropic processes too.

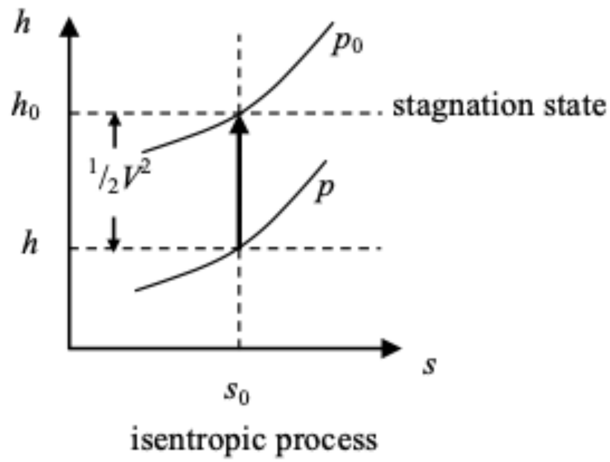
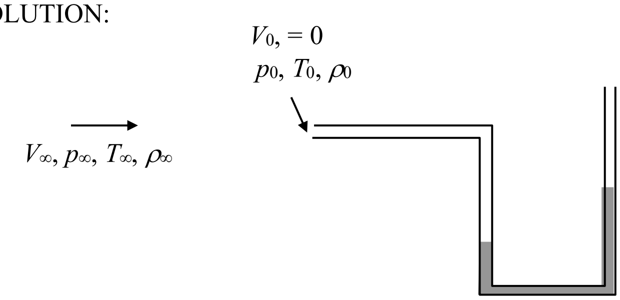


Figure 13.18. A Mollier plot showing the difference between the flow specific enthalpy and specific stagnation enthalpy, in this case for an isentropic process. The difference between the different specific enthalpies is equal to the specific kinetic energy in the flow. This relationship is true even for non-isentropic processes.

(4) For perfect gases, the h-s plots are usually shown as T-s plots since $\Delta h = c_p \Delta T$.

A pitot tube is used to measure the velocity of air. At low speeds, we can reasonably treat the air as an incompressible fluid; however, at high speeds this assumption is not very good due to compressibility effects. At what Mach number does the incompressibility assumption become inaccurate for engineering calculations? Justify your answer with appropriate calculations.

SOLUTION:



First use the incompressible form of Bernoulli's equation to determine the incoming velocity.

$$p_\infty + \tfrac{1}{2}\rho_\infty V_\infty^2 = p_0 \tag{1}$$

$$\therefore \left(V_\infty\right)_{\text{incompressible}} = \sqrt{\frac{2\left(p_0 - p_\infty\right)}{\rho_\infty}} \tag{2}$$

Now consider the pressure difference for a perfect gas brought to rest isentropically (a reasonable model as long as a shock wave does not form in front of the tube).

$$p_0 - p_\infty = p_\infty\left(\frac{p_0}{p_\infty} - 1\right) \tag{3}$$

where

$$\frac{p_0}{p_\infty} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} \tag{4}$$

Substitute and simplify.

$$p_0 - p_\infty = p_\infty\left[\left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

$$p_0 - p_\infty = \left(\tfrac{1}{2}\rho_\infty V_\infty^2\right)\frac{p_\infty}{\left(\tfrac{1}{2}\rho_\infty V_\infty^2\right)}\left[\left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \tag{5}$$

Note that for an ideal gas:

$$\frac{p_\infty}{\tfrac{1}{2}\rho_\infty V_\infty^2} = \frac{RT_\infty}{\tfrac{1}{2}V_\infty^2} = \frac{2\gamma RT_\infty}{\gamma V_\infty^2} = \frac{2}{\gamma \text{Ma}_\infty^2} \tag{6}$$

Substitute and simplify.

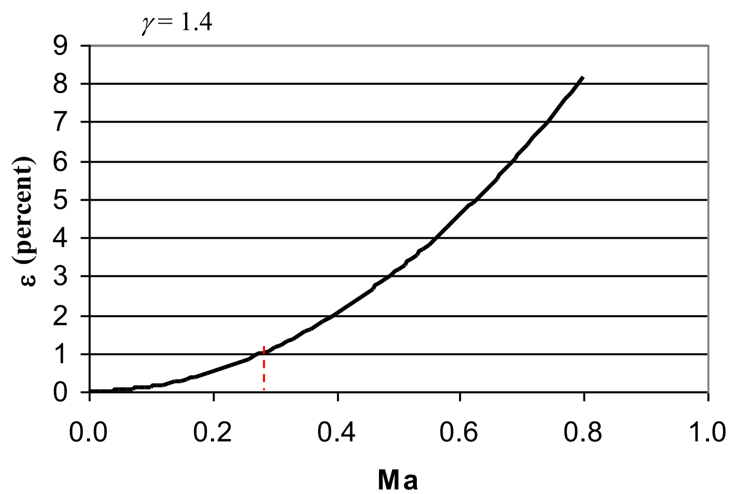
$$p_0 - p_\infty = \left(\tfrac{1}{2}\rho_\infty V_\infty^2\right)\frac{2}{\gamma \text{Ma}_\infty^2}\left[\left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

$$\therefore \left(V_\infty\right)_{\substack{\text{isentropic,}\\ \text{ideal gas}}} = \sqrt{\frac{2\left(p_0 - p_\infty\right)}{\rho_\infty}\left\{\frac{2}{\gamma \text{Ma}_\infty^2}\left[\left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]\right\}^{-1}} \tag{7}$$

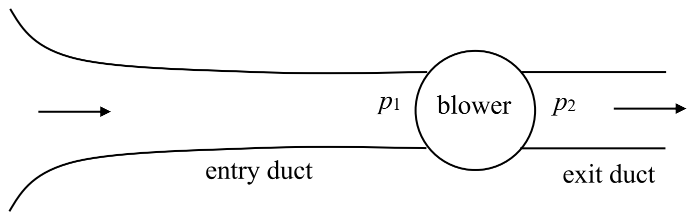
Define the relative error as:

$$\varepsilon \equiv \frac{\left(V_\infty\right)_{\substack{\text{isentropic}\\ \text{ideal gas}}} - \left(V_\infty\right)_{\text{incompressible}}}{\left(V_\infty\right)_{\substack{\text{isentropic}\\ \text{ideal gas}}}} = 1 - \sqrt{\frac{2}{\gamma \text{Ma}_\infty^2}\left[\left(1+\frac{\gamma-1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]} \tag{8}$$

Thus, the error is a function only of the upstream Mach number and the specific heat ratio. Plotting the error as a function of upstream Mach number (for air, $\gamma = 1.4$) shows that, if we consider <1% error acceptable, the incompressibility assumption is valid for Ma_∞ <≈ 0.3.



An air blower takes air from the atmosphere (100 kPa (abs) and 293 K) and ingests it through a smooth entry duct so that the losses are negligible. The cross-sectional area of the entry duct just upstream of the blower and that of the exit duct are both 0.01 m^2.

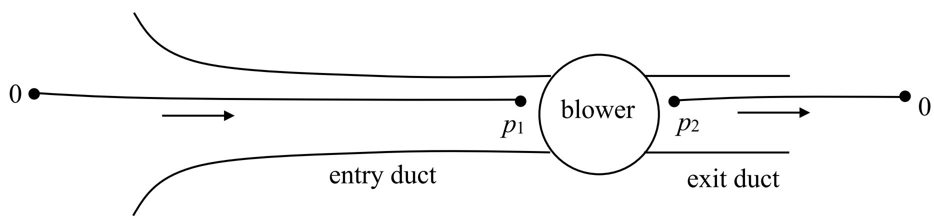


The pressure ratio, p_2/p_1, across the blower is 1.05 and the exit pressure is equal to atmospheric pressure. The air is assumed to behave isentropically upstream of the blower. Find:

a. the velocity of the air entering the blower, and
b. the mass flow rate of air through the system.

SOLUTION:

Apply the First Law between points 0 and 1 (refer to the figure below). Assume 1D, steady, isentropic (⇒ adiabatic) flow. Also neglect potential energy changes since a gas is the working fluid.



$$\dot{m}_1\left(h+\tfrac{1}{2}V^2\right)_1 - \dot{m}_0\left(h+\tfrac{1}{2}V^2\right)_0 = 0 \quad (1)$$

The velocity far upstream is negligible ($V_0 \approx 0$) and $\dot{m}_1 = \dot{m}_0$. Substitute and simplify.

$$h_0 = h_1 + \tfrac{1}{2}V_1^2 \quad (2)$$

Assume that the air behaves as a perfect gas ($\Delta h = c_P \Delta T$) and solve for V_1.

$$V_1 = \sqrt{2c_P\left(T_0 - T_1\right)} \quad (3)$$

Express the temperature at point 1 in terms of a pressure ratio making use of the fact that the flow is isentropic.

$$\frac{T_1}{T_0} = \left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}} \quad (4)$$

The pressure rise across the blower is specified in the problem statement. Furthermore, the pressure at point 2 is equal to atmospheric pressure, i.e., $p_2 = p_0$. Re-write Eq. (4) using this information.

$$\frac{T_1}{T_0} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{p_2}{p_1}\right)^{\frac{1-\gamma}{\gamma}} \quad (5)$$

Combine Eqs. (3) and (5).

$$V_1 = \sqrt{2c_P\left(T_0 - T_1\right)} = \sqrt{2c_P T_0\left(1 - \frac{T_1}{T_0}\right)}$$

$$\boxed{\therefore V_1 = \sqrt{2c_P T_0\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{1-\gamma}{\gamma}}\right]}} \qquad (6)$$

Using the given numerical data:

c_P = 1004.5 J/(kg·K) (for air)
γ = 1.4 (for air)
T_0 = 293 K
p_2/p_1 = 1.05
$\Rightarrow$ $\boxed{V_1 = 90\text{ m/s}}$

The mass flow rate can be found from the conditions at point 1:

$$\boxed{\dot{m} = \rho_1 V_1 A_1 = \left(\frac{p_1}{RT_1}\right)V_1 A_1} \qquad (7)$$

where the ideal gas law has been used. The pressure at point 1 can be found from the blower pressure ratio and the fact that p_2 is equal to atmospheric pressure.

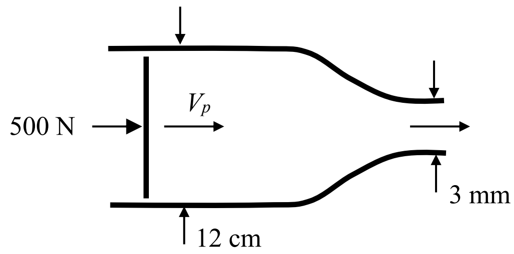
$$p_1 = \left(\frac{p_1}{p_2}\right)p_0 \qquad (8)$$

The temperature at point 1 can be found using Eq. (5). Using the given numerical data:

p_0 = 100 kPa (abs)
p_2/p_1 = 1.05
p_1 = 95.2 kPa (abs)
T_0 = 293 K
T_1 = 289 K
R = 287 J/(kg·K) (for air)
V_1 = 90 m/s
A_1 = 0.01 m^2
$\Rightarrow$ $\boxed{\dot{m} = 1.03\text{ kg/s}}$

A force of 500 N pushes a piston of diameter 12 cm through an insulated cylinder containing air at 20 °C. The exit diameter is 3 mm and the atmospheric pressure is 1 atm (abs). Estimate:

1. the exit velocity,
2. the velocity near the piston (V_p), and
3. mass flow rate out of the device.



SOLUTION:

The pressure at the piston face may be found from the piston force and piston diameter.

$$p_1 = \frac{F}{\frac{\pi}{4}d_1^2} + p_{\text{atm}} = \frac{(500\text{ N})}{\frac{\pi}{4}(0.12\text{ m})^2} + 101\text{ kPa} = 145\text{ kPa} \quad (1)$$

Assume the flow through the piston is isentropic. The velocity at the exit may be found by applying the First Law to the air inside the piston with 1 signifying the location adjacent to the piston face and 2 signifying the device's exit.

$$\left(h + \tfrac{1}{2}V^2\right)_2 - \left(h + \tfrac{1}{2}V^2\right)_1 = \dot{Q}_{\substack{\text{into}\\\text{air}}} + \dot{W}_{\substack{\text{on}\\\text{air}}} \quad (2)$$

Assuming perfect gas behavior, adiabatic conditions, and that $V_2 >> V_1$ (since the areas are so different):

$$c_P\left(T_2 - T_1\right) + \tfrac{1}{2}V_2^2 = 0 \quad (3)$$

$$V_2 = \sqrt{2c_P\left(T_1 - T_2\right)} \quad (4)$$

Also assume that the flow is isentropic so that:

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \Rightarrow T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \quad (5)$$

Using the given data:

c_P = 1004 J/(kg·K)
T_1 = (20 + 273) K = 293 K
p_1 = 145*10^3 Pa (from Eq. (1))
p_2 = 101*10^3 Pa (discharging into the atmosphere, assuming the exit Mach number is subsonic)
$\Rightarrow T_2$ = 264 K
$\therefore V_2$ = 241 m/s

Check that the exit Mach number is subsonic.

$$c_2 = \sqrt{\gamma R T_2} \Rightarrow c_2 = 326\text{ m/s} \quad (6)$$

Since $V_2 < c_2$, the exit flow is subsonic and the assumption that $p_2 = p_{\text{atm}}$ is a good one.

From conservation of mass applied to the same control volume:

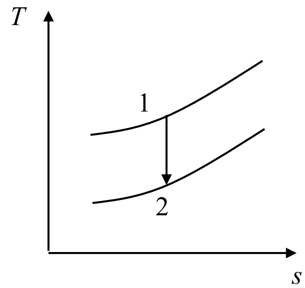
$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \Rightarrow V_1 = V_2\left(\frac{\rho_2}{\rho_1}\right)\left(\frac{D_2}{D_1}\right)^2 = V_2\left(\frac{p_2}{p_1}\right)\left(\frac{T_1}{T_2}\right)\left(\frac{D_2}{D_1}\right)^2 \quad (7)$$

$\therefore V_1 = 0.116$ m/s Clearly the assumption that $V_2 >> V_1$ was a good one.

The mass flow rate is:

$$\dot{m} = \rho_2 V_2 A_2 = \frac{p_2}{RT_2} V_2 \tfrac{\pi}{4} D_2^2 \tag{8}$$

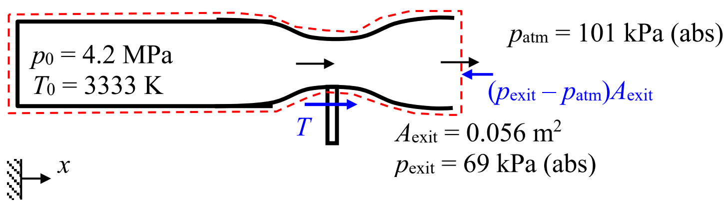
$$\boxed{\therefore \dot{m} = 2.27 * 10^{-3} \text{ kg/s}}$$



A small, solid fuel rocket motor is tested on a horizontal thrust stand at atmospheric conditions. The chamber (essentially a large tank) absolute pressure and temperature are maintained at 4.2 MPa (abs) and 3333 K, respectively. The rocket's converging-diverging nozzle is designed to expand the exhaust gas isentropically to an absolute pressure of 69 kPa. The nozzle exit area is 0.056 m^2. The gas may be treated as a perfect gas with a specific heat ratio of 1.2 and an ideal gas constant of 300 J/(kg·K). Determine, for design conditions:

a. the mass flow rate of propellant gas, and
b. the thrust force exerted on the test stand.

SOLUTION:



Determine the mass flow rate using the conditions at the exit. The Mach number at the exit may be found from the isentropic stagnation pressure ratio:

$$\frac{p_{exit}}{p_0} = \left(1 + \frac{k-1}{2}\text{Ma}_{exit}^2\right)^{\frac{k}{1-k}} \tag{1}$$

Using p_{exit} = 69e3 Pa (abs), p_0 = 4.2e6 Pa (abs), and k = 1.2:

$$\therefore \text{Ma}_{exit} = 3.136 \tag{2}$$

The exit temperature may be found using the stagnation temperature ratio:

$$\frac{T_{exit}}{T_0} = \left(1 + \frac{k-1}{2}\text{Ma}_{exit}^2\right)^{-1} \tag{3}$$

Using T_0 = 3333 K, Ma_{exit} = 3.136, and k = 1.2:

$$\therefore T_{exit} = 1680 \text{ K} \tag{4}$$

The exit density may be found using the ideal gas law:

$$p_{exit} = \rho_{exit} R T_{exit} \tag{5}$$

Using p_{exit} = 69e3 Pa, T_{exit} = 1680 K, and R = 300 J/(kg·K):

$$\therefore \rho_{exit} = 0.1369 \text{ kg/m}^3 \tag{6}$$

The exit velocity may be found using the speed of sound at the exit and the Mach number definition:

$$c_{exit} = \sqrt{kRT_{exit}} \tag{7}$$

$$V_{exit} = c_{exit}\text{Ma}_{exit} \tag{8}$$

Using k = 1.2, R = 300 J/(kg·K), T_{exit} = 1680 K, and Ma_{exit} = 3.136:

$$\therefore c_{exit} = 777.8 \text{ m/s} \tag{9}$$

$$\therefore V_{exit} = 2439 \text{ m/s} \tag{10}$$

The mass flow rate through the nozzle is:

$$\dot{m} = \rho_{exit} V_{exit} A_{exit} \tag{11}$$

Using ρ_{exit} = 0.1369 kg/m^3, V_{exit} = 2439 m/s, and A_{exit} = 0.056 m^2:

$$\boxed{\therefore \dot{m} = 18.69 \text{ kg/s}} \tag{12}$$

The thrust force, T, acting on the stand may be determined using the Linear Momentum Equation in the x-direction for the control volume shown in the figure.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{13}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)} \tag{14}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m} V_{\text{exit}} \tag{15}$$

$$F_{B,x} = 0 \tag{16}$$

$$F_{S,x} = T - \left(p_{\text{exit}} - p_{\text{atm}}\right) A_{\text{exit}} \tag{17}$$

Substitute and simplify.

$$\dot{m} V_{\text{exit}} = T - \left(p_{\text{exit}} - p_{\text{atm}}\right) A_{\text{exit}} \tag{18}$$

$$\therefore T = \dot{m} V_{\text{exit}} + \left(p_{\text{exit}} - p_{\text{atm}}\right) A_{\text{exit}} \tag{19}$$

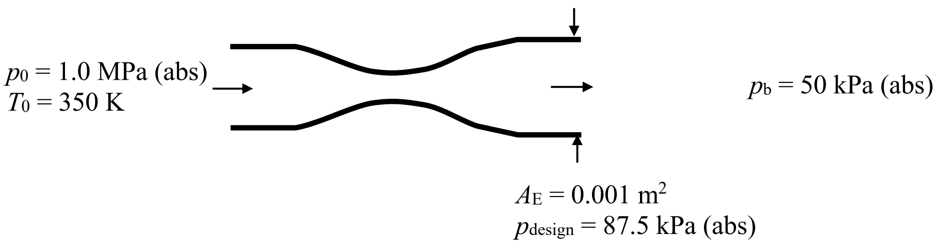
Using $\dot{m}$ = 18.69 kg/s, V_{exit} = 2439 m/s, p_{exit} = 69e3 Pa (abs), p_{atm} = 101e3 Pa (abs), and A_{exit} = 0.056 m^2:

$$\boxed{\therefore T = 4.380\text{e}4 \text{ N}} \tag{20}$$

Air flows isentropically in a converging-diverging nozzle, with exit area of 0.001 m^2. The nozzle is fed from a large plenum where the stagnation conditions are 350 K and 1.0 MPa (abs). The nozzle has a design back pressure of 87.5 kPa (abs) but is operating at a back pressure of 50.0 kPa (abs). Assuming the flow within the nozzle is isentropic, determine:

a. the exit Mach number, and
b. the mass flow rate through the nozzle.

SOLUTION:



The exit Mach number may be found using the isentropic pressure ratio at the exit. Since the back pressure is less than the design pressure (underexpanded conditions), the exit pressure will be equal to the design pressure.

$$\frac{p_E}{p_0} = \left(1 + \frac{k-1}{2}\text{Ma}_E^2\right)^{\frac{k}{1-k}} \tag{1}$$

Using p_E = 87.5 kPa, p_0 = 1.0 MPa, and k = 1.4, the exit Mach number is: $\boxed{\text{Ma}_E = 2.24}$.

The mass flow rate through the nozzle may be found using the exit conditions. First, determine the temperature at the exit using the adiabatic stagnation temperature ratio:

$$\frac{T_E}{T_0} = \left(1 + \frac{k-1}{2}\text{Ma}_E^2\right)^{-1} \tag{2}$$

Using T_0 = 350 K, k = 1.4, and Ma$_E$ = 2.24, the exit temperature is: $\underline{T_E = 174.5\text{ K}}$.

The speed of sound at the exit is:

$$c_E = \sqrt{kRT_E} \tag{3}$$

Using k = 1.4, R = 287 J/(kg·K), and T_E = 174.5 K, the speed of sound at the exit is: $\underline{c_E = 264.8\text{ m/s}}$.

The velocity of the air at the exit is:

$$V_E = c_E \text{Ma}_E \tag{4}$$

Using c_E = 264.8 m/s and Ma$_E$ = 2.24: $\underline{V_E = 593.8\text{ m/s}}$.

The density at the exit may be found using the ideal gas law:

$$\rho_E = \frac{p_E}{RT_E} \tag{5}$$

With p_E = 87.5 kPa, R = 287 J/(kg·K), and T_E = 174.5 K: $\underline{\rho_E = 1.747\text{ kg/m}^3}$.

The mass flow rate through the nozzle is:

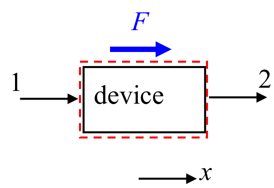
$$\dot{m} = \rho_E V_E A_E \tag{6}$$

Using the previous data, $\boxed{\dot{m} = 1.04\text{ kg/s}}$.

Oxygen (not air) enters a device with a cross-sectional area of 1 ft^2 (refer to this location as section 1) with a stagnation temperature of 1000 °R, stagnation pressure of 100 psia, and Mach number of 0.2. There is no heat transfer, work transfer, or losses as the gas passes through the device and expands to a pressure of 14.7 psia (section 2).

a. Determine the density, velocity, and mass flow rate at section 1.
b. Determine the Mach number, temperature, velocity, density, and area at section 2.
c. What force does the fluid exert on the device?

SOLUTION:



First determine the properties at section 1.

$$\rho_0 = \frac{p_0}{RT_0} \tag{1}$$

where

$$\rho_1 = \rho_0\left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{1}{1-\gamma}} \tag{2}$$

Using p_0 = 100 psia = 14400 $\text{lb}_\text{f}/\text{ft}^2$, T_0 = 1000 °R, R = 48.291 (ft·lb_f)/(lb_m·°R) = 1553.7 $\text{ft}^2/(\text{s}^2\cdot°\text{R})$, ρ_0 = 0.298 $\text{lb}_\text{m}/\text{ft}^3$. In addition, with γ = 1.395, and Ma_1 = 0.2, $\boxed{\rho_1 = 0.292\ \text{lb}_\text{m}/\text{ft}^3}$.

$$V_1 = c_1\text{Ma}_1 = \sqrt{\gamma RT_1}\text{Ma}_1 \tag{3}$$

where,

$$T_1 = T_0\left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{-1} \tag{4}$$

Using the given values, T_1 = 992.2 °R and $\boxed{V_1 = 293.3\ \text{ft/s}}$.

$$\dot{m} = \rho_1 V_1 A_1 \tag{5}$$

Using the given values, $\boxed{\dot{m} = 85.6\ \text{lb}_\text{m}/\text{s}}$.

Now use the isentropic relations to determine the properties at section 2.

$$p_2 = p_0\left(1+\frac{\gamma-1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \text{Ma}_2 = \left\{\frac{2}{\gamma-1}\left[\left(\frac{p_2}{p_0}\right)^{\frac{1-\gamma}{\gamma}}-1\right]\right\}^{\frac{1}{2}} \tag{6}$$

Using p_2 = 14.7 psia and p_0 = 100 psia, $\boxed{\text{Ma}_2 = 1.91}$.

$$T_2 = T_0\left(1+\frac{\gamma-1}{2}\text{Ma}_2^2\right)^{-1} \tag{7}$$

Using the given data, $\boxed{T_2 = 581.2\ °\text{R}}$.

$$V_2 = c_2\text{Ma}_2 = \sqrt{\gamma RT_2}\text{Ma}_2 \tag{8}$$

Using the given data, $\boxed{V_2 = 2144\ \text{ft/s}}$.

$$\rho_2 = \frac{p_2}{RT_2} \tag{9}$$

Using the given data, $\boxed{\rho_2 = 0.0754 \text{ lb}_\text{m}/\text{ft}^3}$.

$$\dot{m} = \rho_2 V_2 A_2 \Rightarrow A_2 = \frac{\dot{m}}{\rho_2 V_2} \tag{10}$$

Using the given data, $\boxed{A_2 = 0.53 \text{ ft}^2}$.

To determine the force the fluid exerts on the device, apply the Linear Momentum Equation in the x-direction to the control volume shown in the figure,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{11}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)} \tag{12}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}V_1 + \dot{m}V_2 = \dot{m}\left(V_2 - V_1\right) \tag{13}$$

$$F_{B,x} = 0 \tag{14}$$

$$F_{S,x} = F + p_1 A_1 - p_2 A_2 \tag{15}$$

Substitute and simplify.

$$\dot{m}\left(V_2 - V_1\right) = F + p_1 A_1 - p_2 A_2$$

$$\boxed{F = \dot{m}\left(V_2 - V_1\right) - p_1 A_1 + p_2 A} \tag{16}$$

Substitute the given values to find F = -7950 lb_f. Note that this is the force that the device exerts on the fluid. Hence, the $\boxed{\text{force the fluid exerts on the device is 7950 lb}_\text{f} \text{ acting in the } +x\text{-direction}}$.

The Concorde aircraft flies at Ma≈2.3 at 11 km standard altitude. Estimate the temperature in °C at the front stagnation point. At what Mach number would it have a front stagnation point temperature of 450 °C?

SOLUTION:

The temperature at the stagnation point is determined using:

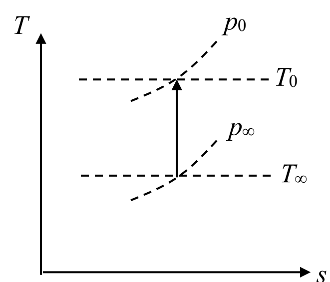
$$\frac{T}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_\infty^2\right)^{-1} \tag{1}$$

where

γ = 1.4
Ma_∞ = 2.3
T_∞ = 217 K (from standard atmosphere tables at an altitude of 11 km)
$\Rightarrow$ $\boxed{T_0 = 447\ \mathrm{K} = 174\ °\mathrm{C}}$

For the next part of the problem, re-arrange Eqn. (1):

$$\mathrm{Ma}_\infty = \sqrt{\frac{2}{\gamma - 1}\left(\frac{T_0}{T} - 1\right)} \tag{2}$$



Using the given data:

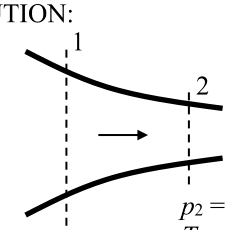
γ = 1.4
T_∞ = 217 K
T_0 = 450 °C = 723 K
$\Rightarrow$ $\boxed{\mathrm{Ma}_\infty = 3.4}$

A steady flow of air passes through a converging nozzle. At the nozzle inlet, the static pressure and temperature are p_1 = 150 kPa (abs), T_1 = 500 K, and V_1 = 150 m/s. At the nozzle exit, p_2 = 98.32 kPa (abs), T_2 = 453.2 K, and V_2 = 341.4 m/s. Assume steady, uniform flow, and that the air behaves as a perfect gas with $\gamma = 1.4$, R = 287 J/(kg·K), and c_p = 1005 J/(kg·K).

a. Is the flow through the nozzle adiabatic?
b. Is the flow through the nozzle isentropic?
c. Is the flow through the nozzle frictionless?

Support all of your answers.

SOLUTION:



p_2 = 98.32 kPa (abs)
T_2 = 453.2 K
V_2 = 341.4 m/s

p_1 = 150 kPa (abs)
T_1 = 500 K
V_1 = 150 m/s

If the flow is adiabatic then the stagnation temperature will remain constant, i.e., $T_{02} = T_{01}$, where:

$$T_0 = T + \frac{V^2}{2c_p} \tag{1}$$

Using the given data:

T_1 = 500 K
V_1 = 150 m/s
T_2 = 453.2 K
V_2 = 341.4 m/s
c_p = 1005 J/(kg·K)
$\Rightarrow$ T_{01} = 511.2 K and T_{02} = 511.2 K

Since the stagnation temperatures are equal, the flow must be adiabatic.

If the flow is isentropic, then the stagnation pressure will remain constant, i.e., $p_{02} = p_{01}$ (the stagnation density will also remain constant, i.e., $\rho_{02} = \rho_{01}$).

$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}^2\right)^{\gamma/(1-\gamma)} \tag{2}$$

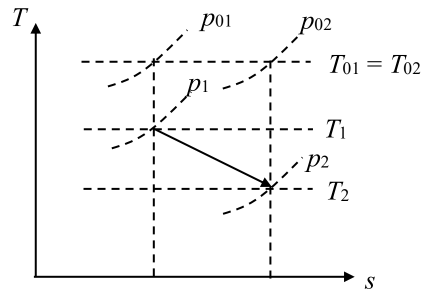
Using the given data:

p_1 = 150 kPa (abs)
V_1 = 150 m/s
T_1 = 500 K
$\Rightarrow$ Ma_1 = 0.34 where $\text{Ma} = \frac{V}{c} = \frac{V}{\sqrt{\gamma RT}}$

p_2 = 98.32 kPa (abs)
V_2 = 341.4 m/s
T_2 = 453.2 K
$\Rightarrow$ Ma_2 = 0.80
$\Rightarrow$ p_{01} = 162.1 kPa (abs) and p_{02} = 149.9 kPa (abs)

Since the stagnation pressures are not equal, the flow is not isentropic.

Since the flow is adiabatic but non-isentropic, then some other irreversible process must take place. Two common irreversible processes that occur in gas dynamics are frictional effects and shock waves. Shock waves cannot be the source of the entropy since shock waves only occur in supersonic flows and the flow in this converging nozzle remains subsonic throughout. Hence, we can conclude that
the flow in this nozzle is not frictionless.



A supersonic wind tunnel test section is designed to have a Mach number of 2.5 at a temperature of 60 °F and 5 psia. The fluid is air.

a. Determine the required inlet stagnation temperature and pressure.
b. Calculate the required mass flow rate for a test section area of 2.0 ft^2.

SOLUTION:

The stagnation properties may be found using the isentropic relations:

$$\frac{p_{TS}}{p_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_{\mathrm{TS}}^2\right)^{\frac{\gamma}{1-\gamma}} \quad (1)$$

$$\frac{T_{TS}}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_{\mathrm{TS}}^2\right)^{-1} \quad (2)$$

where

p_{TS} = 5 psia = 720 $\mathrm{lb_f/ft^2}$
T_{TS} = (60 + 459) °R = 519 °R
Ma_{TS} = 2.5
γ_{air} = 1.4
$\therefore$ $\boxed{p_0 = 85.4\text{ psia}}$ and $\boxed{T_0 = 1170\text{ °R}}$

The mass flow rate may be found using:

$$\dot{m}_{TS} = \rho_{TS} V_{TS} A_{TS} = \left(\frac{p_{TS}}{RT_{TS}}\right)\left(c_{TS}\mathrm{Ma}_{TS}\right)A_{TS} \quad (3)$$

where the speed of sound in the test section, c_{TS}, is:

$$c_{TS} = \sqrt{\gamma R T_{TS}} \quad (4)$$

Using the given data:

R_{air} = 53.3 $(\mathrm{ft\cdot lb_f})/(\mathrm{lb_m\cdot °R})$
A_{TS} = 2 ft^2
$\Rightarrow c_{TS}$ = 1120 ft/s
ρ_{TS} = 0.0260 $\mathrm{lb_m/ft^3}$
$\boxed{\dot{m}_{TS} = 145\ \mathrm{lb_m/s}}$