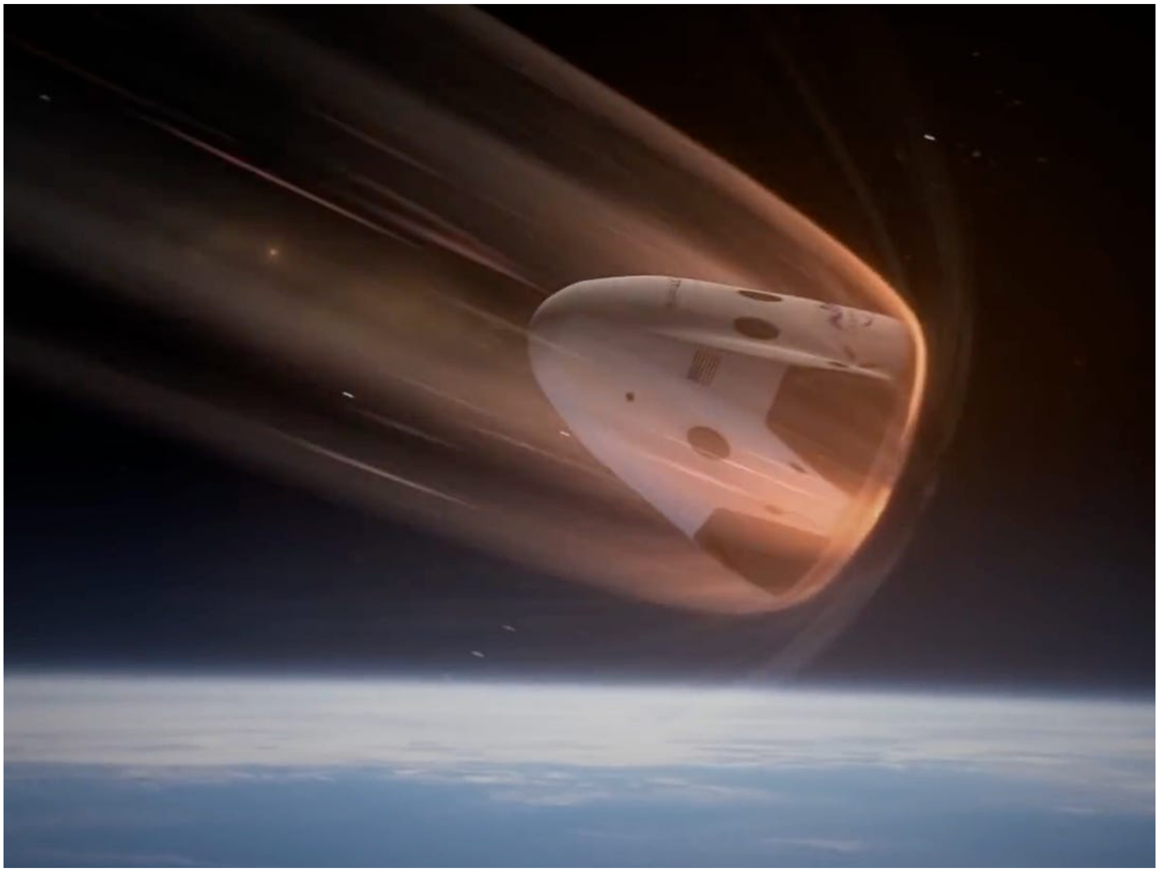
Compressible Flow – Isentropic Flow; Stagnation and Sonic Conditions


Artist rendition of the Space X Dragon re-entry

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1st Law of Thermodynamics:

$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS}\left(h + \frac{1}{2}V^2 + gz\right)(\rho \boldsymbol{u}_{rel}\cdot d\boldsymbol{A}) = \dot{Q}_{into\ CV} + \dot{W}_{other,on\ CV}$$

Assume **steady flow**:

$$\frac{d}{dt}\int_{CV} e\rho dV = 0$$

Assume **adiabatic conditions**:

$$\dot{Q}_{into\ CV} = 0$$

Assume **no work other than pressure work**:

$$\dot{W}_{other,on\ CV} = 0$$

Assume **1D flow with uniform properties over the inlet and outlet**:

$$\int_{CS}\left(h + \frac{1}{2}V^2 + gz\right)(\rho \boldsymbol{u}_{rel}\cdot d\boldsymbol{A}) = \dot{m}\left[\left(h + \frac{1}{2}V^2 + gz\right)_{out} - \left(h + \frac{1}{2}V^2 + gz\right)_{in}\right]$$

Assume **perfect gas behavior**:

$\Delta h = c_p \Delta T$ and $g\Delta z \ll \Delta h + \frac{1}{2}\Delta V^2$

Combining together:

$$\boxed{c_p T + \frac{1}{2}V^2 = \text{constant}}$$

Combine with the following:

definition of the Mach number: $\text{Ma} = \frac{V}{c}$

speed of sound for an ideal gas: $c = \sqrt{kRT}$

specific heat relations for an ideal gas: $c_p = c_v + R,\ c_p = \frac{kR}{k-1},\ c_v = \frac{R}{k-1}$

For a **1D, steady, <u>adiabatic</u> flow of a perfect gas with no work other than pressure work**

$$T\left(1 + \frac{k-1}{2}\text{Ma}^2\right) = \text{constant}$$

Consider two reference conditions:

stagnation conditions: (Ma, T) = (0, T_0)

$$\frac{T}{T_0} = \left(1 + \frac{k-1}{2}\text{Ma}^2\right)^{-1}$$

sonic conditions: (Ma, T) = (1, T^*)

$$\frac{T^*}{T_0} = \left(1 + \frac{k-1}{2}\right)^{-1}$$

Note that the speed of sound for an ideal gas is: $c = \sqrt{kRT}$

$$\frac{c}{c_0} = \left(1 + \frac{k-1}{2}\text{Ma}^2\right)^{-\frac{1}{2}} \quad \text{and} \quad \frac{c^*}{c_0} = \left(1 + \frac{k-1}{2}\right)^{-\frac{1}{2}}$$

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If the flow is adiabatic and internally reversible (=> an **isentropic** process!), then for a perfect gas:

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{k-1} \qquad \frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} \qquad \frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{k}$$

1D, steady, <u>isentropic</u> flow of a perfect gas with no work other than pressure work

$$\frac{p}{p_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{k}{1-k}} \qquad \frac{p^*}{p_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{1-k}} \qquad \left(\text{for air, } \frac{p^*}{p_0} = 0.5283\right)$$

$$\frac{\rho}{\rho_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{1}{1-k}} \qquad \frac{\rho^*}{\rho_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{1}{1-k}}$$