13.4. Speed of Sound

The speed of sound, c, in a substance is the speed at which infinitesimal pressure disturbances propagate through the surrounding substance. To understand how the speed of sound depends on the substance properties, let's examine the following model.

Consider a wave moving at velocity, c, through a stagnant fluid. Across the wave, the fluid properties such as pressure, p, density, ρ, temperature, T, and the velocity, V, can all change as shown in Figure 13.7. Now let's

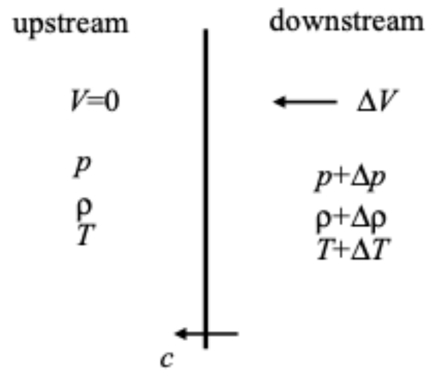


FIGURE 13.7. Flow across a pressure wave viewed from a frame of reference fixed to the ground.

change our frame of reference such that it moves with the wave, as shown in Figure 13.8. Apply Conservation

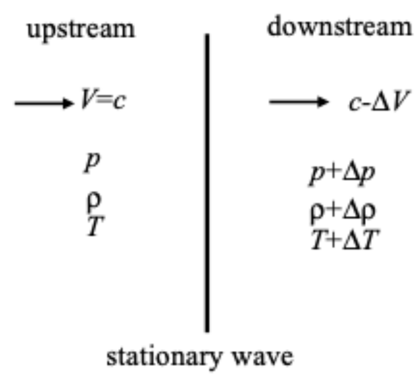


FIGURE 13.8. Flow across a pressure wave viewed from a frame of reference fixed to the wave.

of Mass and the Linear Momentum Equation to a thin control volume of cross-sectional area A surrounding the wave (Figure 13.9). From Conservation of Mass,

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0, \tag{13.37}$$

where,

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(the flow is steady in the frame of reference fixed to the wave),} \tag{13.38}$$

$$\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = -\rho c A + (\rho + \Delta\rho)(c - \Delta V)A. \tag{13.39}$$

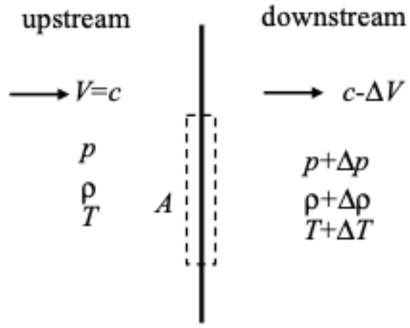


Figure 13.9. A thin control volume applied across the pressure wave.

Substitute and simplify,

$$\rho c A = (\rho + \Delta\rho)(c - \Delta V)A, \tag{13.40}$$

$$\rho c = \rho c - \rho\Delta V + \Delta\rho c - \Delta\rho\Delta V, \tag{13.41}$$

$$\Delta V = \frac{c\Delta\rho}{\rho + \Delta\rho}. \tag{13.42}$$

From the Linear Momentum Equation applied in the streamwise direction,

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}, \tag{13.43}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(the flow is steady in the frame of reference fixed to the wave)}, \tag{13.44}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = -\dot{m}c + \dot{m}(c - \Delta V) = -\dot{m}\Delta V = -\rho c A \Delta V, \tag{13.45}$$

$$F_{B,x} = 0, \tag{13.46}$$

$$F_{S,x} = pA - (p + \Delta p)A = -\Delta p A. \tag{13.47}$$

Substituting and simplifying,

$$-\rho c A \Delta V = -\Delta p A, \tag{13.48}$$

$$c = \frac{\Delta p}{\rho \Delta V}. \tag{13.49}$$

Making use of the relation derived from Conservation of Mass (Eq. (13.42)),

$$c = \frac{\Delta p(\rho + \Delta\rho)}{\rho c \Delta\rho}, \tag{13.50}$$

$$c^2 = \frac{\Delta p}{\Delta\rho}\left(1 + \frac{\Delta\rho}{\rho}\right). \tag{13.51}$$

For a sound wave, the changes across the wave are infinitesimally small (sound waves are defined as being infinitesimally weak pressure waves) so the previous equation becomes,

$$c^2 = \lim_{\Delta\to d} \frac{\Delta p}{\Delta\rho}\left(1 + \frac{\Delta\rho}{\rho}\right) = \frac{\partial p}{\partial\rho}. \tag{13.52}$$

We also need to specify the process by which these changes occur since pressure, in general, is a function of not only the density, but other properties as well, such as temperature. Since the changes across the wave are infinitesimally small and, thus, the velocity and temperature gradients are infinitesimally small, we can regard the wave as an internally reversible process. Additionally, the temperature gradient on either side of the wave is small so there is negligible heat transfer into the control volume. Hence, the process is adiabatic. As a result, the changes across a sound wave occur isentropically (an adiabatic, internally reversible process is isentropic),

$$\boxed{c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s} \quad \underline{\text{speed of sound in a continuous substance}} \tag{13.53}$$

Notes:

(1) Note that Eq. (13.53) is the speed of sound in *any* substance. It's not limited to just fluids.
(2) If the wave is not "weak", i.e., the changes in the flow properties across the wave are not infinitesimal, then viscous effects and temperature gradients within the wave will be significant and the process can no longer be considered isentropic. We will discuss this situation later when examining shock waves (Section 13.17).
(3) Note that according to Eq. (13.51) the stronger the wave, i.e., the greater $\Delta\rho$, the faster the wave will propagate. This effect will also be examined when discussing shock waves.
(4) For an ideal gas undergoing an isentropic process ($ds = 0$),

$$ds = 0 = c_p\frac{dT}{T} - R\frac{dp}{p} = c_v\frac{dT}{T} - R\frac{d\rho}{\rho}, \tag{13.54}$$

$$\frac{R}{c_v}\frac{d\rho}{\rho} = \frac{R}{c_p}\frac{dp}{p} \implies \left.\frac{\partial p}{\partial \rho}\right|_s = \frac{c_p}{c_v}\frac{p}{\rho} = k\frac{p}{\rho} = kRT. \tag{13.55}$$

Substituting into Eq. (13.53), the speed of sound for an ideal gas is,

$$\boxed{c = \sqrt{kRT}} \quad \underline{\text{speed of sound in an ideal gas}}. \tag{13.56}$$

(a) The absolute temperature must be used when calculating the speed of sound since the Ideal Gas Law was used in its derivation.
(b) The speed of sound in air ($k = 1.4, R = 287\,\mathrm{J\,kg^{-1}\,K^{-1}}$) at standard conditions ($T = 288\,\mathrm{K}$) is $340\,\mathrm{m\,s^{-1}}$ ($= 1115\,\mathrm{ft\,s^{-1}} \approx 1/5\,\mathrm{mi\,s^{-1}}$). This value helps explain the rule of thumb whereby the distance to a thunderstorm in miles is roughly equal to the number of seconds between a lightening flash and the corresponding thunder clap divided by five.
(c) It is not unexpected that the speed of sound is proportional to the square root of the temperature. Since disturbances travel through the gas as a result of molecular impacts, we should expect the speed of the disturbance to be proportional to the speed of the molecules. The temperature is equal to the random kinetic energy of the molecules and so the molecular speed is proportional to the square root of the temperature. Thus, the speed of sound is proportional to the square root of the temperature.

(5) Equation (13.53) can also be written in terms of the bulk modulus. The bulk modulus, E_v, of a substance is a measure of the compressibility of the substance. It is defined as the ratio of a differential applied pressure to the resulting differential change in volume of a substance at a given volume (refer to Figure 13.10),

$$E_v \coloneqq \frac{\partial p}{(-\partial V/V)} = \rho\frac{\partial p}{\partial \rho}. \tag{13.57}$$

Notes:

(a) $dp > 0 \implies dV < 0 \implies E_v > 0$.
(b) From Conservation of Mass, $dV/V = -d\rho/\rho$.
(c) $E_v \uparrow \implies$ compressibility $\downarrow$.

The isentropic bulk modulus, $E_v|_s$, is defined as,

$$E_v|_s \coloneqq \left.\frac{\partial p}{-(dV/V)}\right|_s = \rho\left.\frac{\partial p}{\partial \rho}\right|_s. \tag{13.58}$$

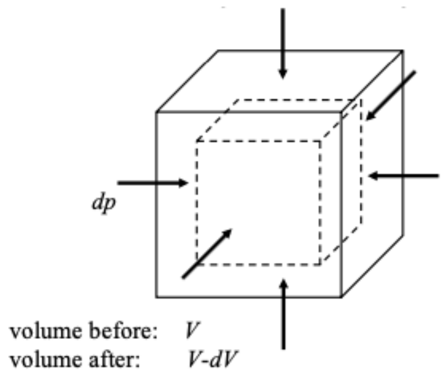


Figure 13.10. A schematic illustrating the concept of the bulk modulus.

Thus, the speed of sound can also be written as,

$$\boxed{c^2 = \frac{E_v|_s}{\rho}} \quad \underline{\text{alternate speed of sound expression}}. \tag{13.59}$$

Notes:

(a) The isentropic bulk modulus for air is $E_v|_s = k\rho RT$.
(b) The isentropic bulk modulus for water is 2.19 GPa. Thus, the speed of sound in water ($\rho =$ 1000 kg/m^3) is 1480 m s^{-1} (= 4900 ft s^{-1} $\approx$ 1 mi s^{-1} $\approx$ 5X faster than the speed of sound in air at standard conditions).
(c) For solids, the bulk modulus, E_v, is related to the modulus of elasticity, E, and Poisson's ratio, ν, by,

$$\frac{E_v}{E} = 3(1 - 2\nu). \tag{13.60}$$

For many metals, e.g., steel and aluminium, the Poisson's ratio is approximately $\nu \approx 1/3$ so that $E_v/E \approx 1$. The speed of sound in stainless steel ($E = 163\,\text{GPa}; \rho = 7800\,\text{kg/m}^3$) is 4570 m s^{-1} (= 15 000 ft s^{-1} $\approx$ 3 mi s^{-1} $\approx$ 3X faster than the speed of sound in water).

(6) The Mach number, Ma is a dimensionless parameter that is commonly used in the discussion of compressible flows. The Mach number is defined as,

$$\boxed{\text{Ma} := \frac{V}{c}}, \tag{13.61}$$

where V is the flow speed and c is the speed of sound in the flow.

Notes:

(a) Compressible flows are often classified by their Mach number:
$\text{Ma} < 1$ subsonic
$\text{Ma} = 1$ sonic
$\text{Ma} > 1$ supersonic
Additional sub-classifications include:
$\text{Ma} < 0.3$ incompressible
$\text{Ma} \approx 1$ transonic
$\text{Ma} > 5$ hypersonic
(b) The square of the Mach number, Ma^2, is a measure of a flow's macroscopic kinetic energy to its microscopic kinetic energy.

(7) The change in the properties across a sound wave can be found from the following analysis. From Conservation of Mass applied to the control volume shown in Figure 13.9 and Eq. (13.42), making

use of the fact that the property changes across the sound wave are infinitesimally small,

$$\frac{dV}{c} = \frac{d\rho}{\rho}. \tag{13.62}$$

For an ideal gas,

$$p = \rho RT \implies \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}. \tag{13.63}$$

Combining Eqs. (13.53), (13.56), and (13.62) and simplifying,

$$\frac{\rho}{p}c^2 = \frac{dp}{p}\frac{\rho}{d\rho} \implies \frac{kRT}{RT} = \frac{dp}{p}\frac{c}{dV}, \tag{13.64}$$

$$\boxed{\frac{dV}{c} = \frac{d\rho}{\rho} = \frac{1}{k}\frac{dp}{p}}. \tag{13.65}$$

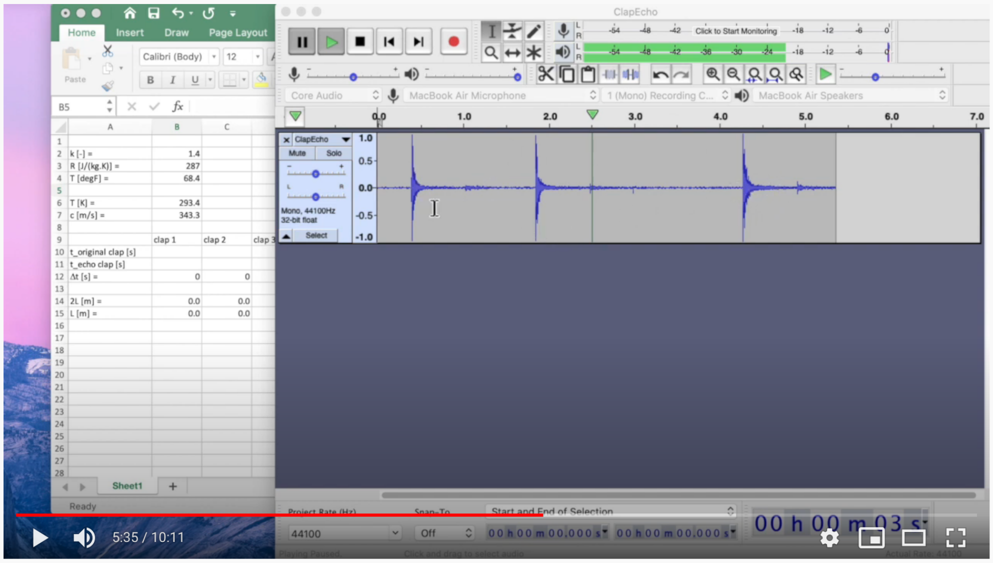
Now combine Eqs. (13.63) and (13.65),

$$\frac{dp}{p} = \frac{1}{k}\frac{dp}{p} + \frac{dT}{T}, \tag{13.66}$$

$$\boxed{\frac{dT}{T} = \left(\frac{k-1}{k}\right)\frac{dp}{p}}. \tag{13.67}$$

Thus, across a compression sound wave ($dp > 0$): $dV > 0, d\rho > 0$, and $dT > 0$. Across a rarefaction sound wave, also known as an expansion wave ($dp < 0$): $dV < 0, d\rho < 0$, and $dT < 0$.

Example:
Refer to https://www.youtube.com/watch?v=9R4xhCoBz9Y for a video demonstration that shows how a distance is calculated using a speed of sound analysis and temperature and transit time measurements.

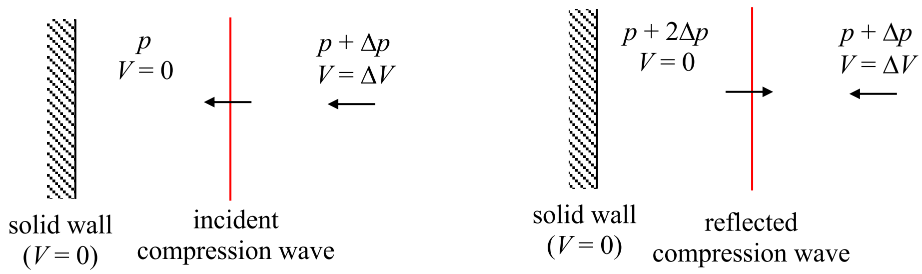


Speed of Sound Demonstration

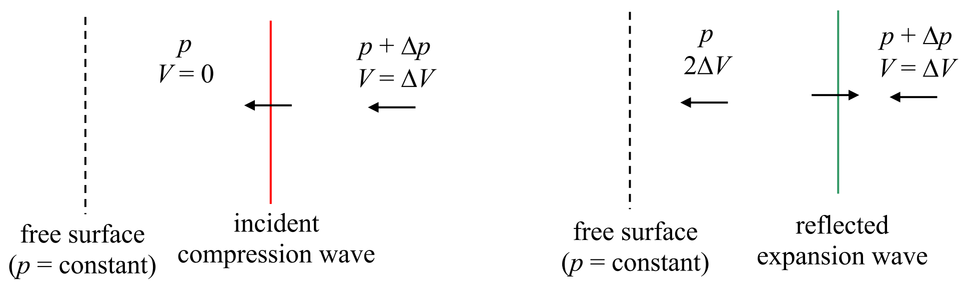
A weak compression pressure wave of magnitude Δp propagates through still air. Discuss the type of reflected wave that occurs (compression or expansion) and the boundary conditions that must be satisfied when the wave strikes normal to, and is reflected from:

a. a solid wall and
b. a free surface boundary (i.e., a surface where the pressure remains constant).

SOLUTION:



At a solid boundary the incident compression wave reflects as a compression wave in order to maintain zero air velocity at the wall. The pressure behind the reflected wave will be $p + 2\Delta p$ (two compressions).



At a free surface boundary the incident compression wave reflects as an expansion wave in order to maintain the free surface pressure. The velocity behind the reflected wave will be $2\Delta V$ in the direction opposite the wave.

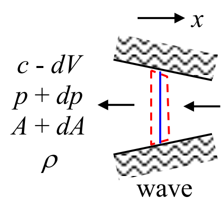
Consider a straight pipe filled with an incompressible liquid. The walls of the pipe are elastic so that the cross-sectional area, A, changes with the internal pressure, p, according to the relation:

$$A = A_0 + A_1 p$$

Thus, the pipe may have different cross-sectional areas at different axial positions depending on the internal pressure at each position. Find the speed of propagation, c, of a small pressure wave traveling along the pipe assuming A_0 and A_1 are known constants and that $A_1 p$ is always small compared with A_0. Give your answer in terms of A_0, A_1, and the density, ρ, of the liquid.

SOLUTION:

Apply conservation of mass and the linear momentum equation to the thin control volume shown below. Use a frame of reference that is fixed to the wave so that the flow appears steady.



Conservation of mass:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho c A + \rho\left(c - dV\right)\left(A + dA\right) \tag{1}$$

Note that the area is a function of the pressure.

$$A = A_0 + A_1 p \quad \text{and} \quad A + dA = A_0 + A_1\left(p + dp\right) \tag{2}$$

Substitute and simplify.

$$-\rho c A + \rho\left(c - dV\right)\left(A + dA\right) = 0$$

$$-\rho c\left(A_0 + A_1 p\right) + \rho\left(c - dV\right)\left[A_0 + A_1\left(p + dp\right)\right] = 0$$

$$-cA_0 - cA_1 p + cA_0 + cA_1\left(p + dp\right) - A_0 dV - dVA_1\left(p + dp\right) = 0$$

$$cA_1 dp - A_0 dV - dVA_1\left(p + dp\right) = 0$$

$$dV = \frac{cA_1 dp}{A_0 + A_1\left(p + dp\right)}$$

$$dV = \frac{cA_1 dp}{A_0 + A_1 p} \quad \text{(Note that } dp \ll p\text{.)} \tag{3}$$

Now apply the linear momentum equation in the x-direction to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m}c - \dot{m}\left(c - dV\right) = \dot{m}dV = \rho c A dV = \rho c\left(A_0 + A_1 p\right)dV \tag{4}$$

$$F_{B,x} = 0 \quad \text{(no body forces since the control volume is infinitesimally thin)}$$

$$\begin{aligned} F_{S,x} &= -pA + \left(p + dp\right)\left(A + dA\right) - \left(p + \tfrac{1}{2}dp\right)dA = -p\left(A_0 + A_1 p\right) + \left(p + dp\right)\left[A_0 + A_1\left(p + dp\right)\right] - pA_1 dp \\ &= pA_1 dp + dpA_0 + pA_1 dp - pA_1 dp \\ &= pA_1 dp + dpA_0 \end{aligned} \tag{5}$$

Substitute and simplify.

$$\rho c\left(A_0 + A_1 p\right)dV = pA_1 dp + dpA_0 = dp\left(A_0 + A_1 p\right)$$

$$\rho c dV = dp \tag{6}$$

Substitute in for dV using Eqn. (3).

$$\rho c\left(\frac{cA_1 dp}{A_0 + A_1 p}\right) = dp$$

$$c^2 = \frac{A_0 + A_1 p}{\rho A_1} \tag{7}$$

(8)

Since $A_1 p \ll A_0$ (given in the problem statement), Eqn. (8) becomes:

$$\boxed{c^2 = \frac{A_0}{\rho A_1}} \tag{9}$$

13.5. The Mach Cone

Consider the propagation of infinitesimal pressure waves, i.e., sound waves, emanating from an object at rest (Figure 13.11). The waves will travel at the speed of sound, c, and form circles (spheres in 3D) with radii depending on the time when the sound wave was emitted.

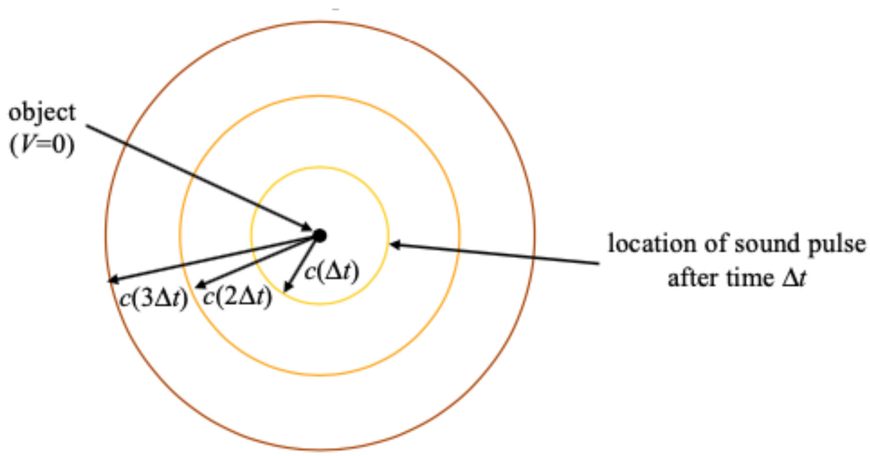


Figure 13.11. Sound waves emanating from an object at rest.

Now consider an object moving at a subsonic speed, $V < c \implies \text{Ma} < 1$ (Figure 13.12). For this case the pressure pulses are more closely spaced in the direction of the object's motion and more widely spaced behind the object. Thus, the frequency of the sound in front of the object increases, while the frequency behind the object decreases. This phenomenon is known as the Doppler Shift.

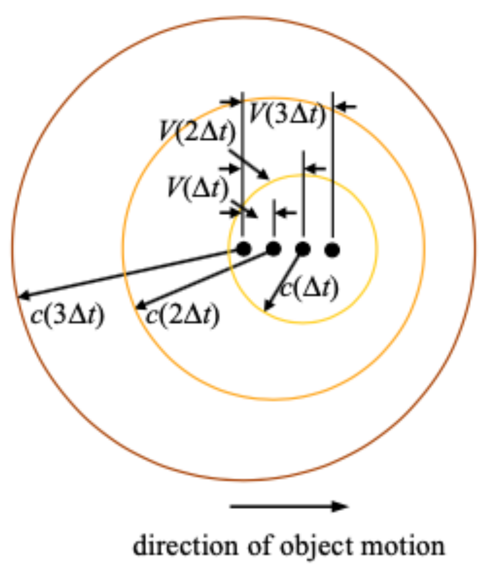


Figure 13.12. Sound waves emanating from an object moving to the right at a subsonic speed ($V < c \implies \text{Ma} < 1$).

Now consider an object traveling at a sonic speed, $V = c \implies \text{Ma} = 1$. Since no wave fronts propagate ahead of the object, an observer standing in front of the object won't hear it approaching until the object reaches the observer. Note that the infinitesimal pressure changes in front of the object begin to "pile up" on one another, producing a sudden, finite pressure change, also known as a shock wave.

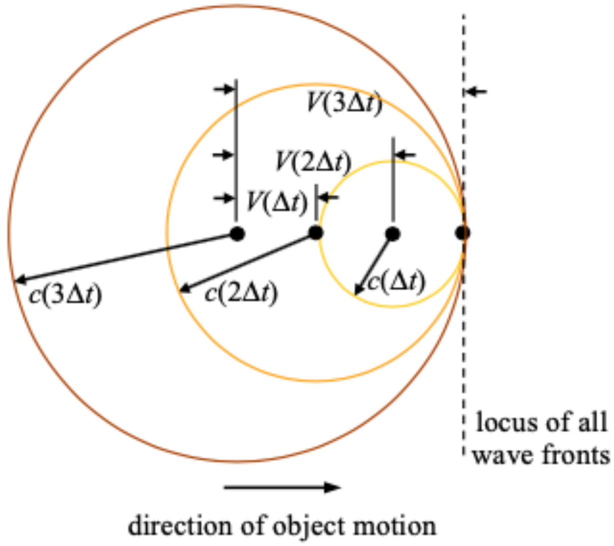


FIGURE 13.13. Sound waves emanating from an object moving to the right at the sonic speed ($V = c \implies \text{Ma} = 1$).

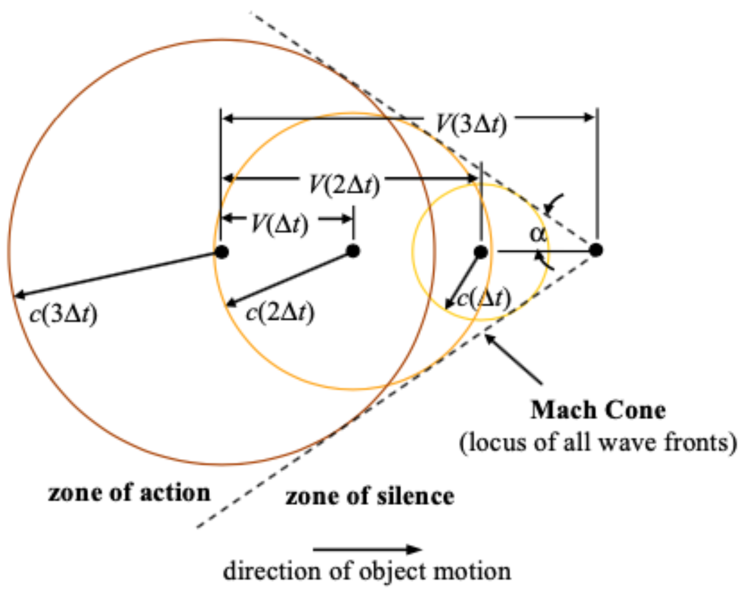


FIGURE 13.14. Sound waves emanating from an object moving to the right at a supersonic speed ($V > c \implies \text{Ma} > 1$).

Lastly, consider an object travelling at supersonic speeds, $V > c \implies \text{Ma} > 1$ (Figure 13.14). For this case, the object outruns the pressure pulses it generates. The locus of wave fronts forms a cone, which is known as the Mach Cone. The object cannot be heard outside of the Mach Cone and, thus, this region is termed the zone of silence. Inside the cone, which is known as the zone of action, the object can be heard. The angle of the cone, known as the Mach angle, α, is given by,

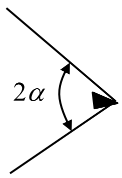
$$\sin\alpha = \frac{c\Delta t}{V\Delta t} = \frac{c}{V} = \frac{1}{\mathrm{Ma}}. \tag{13.68}$$

A projectile in flight carries with it a more or less conical-shaped shock front. From physical reasoning it appears that at great distances from the projectile this shock wave becomes truly conical and changes in velocity and density across the shock become vanishingly small.

Photographs of a bullet in flight show that at a great distance from the bullet the total included angle of the cone is 50.3°. The pressure and temperature of the undisturbed air are 14.62 psia and 73 °F, respectively. Calculate:

a. the velocity of the bullet, in ft/sec, and
b. the Mach number of the bullet relative to the undisturbed air.

SOLUTION:

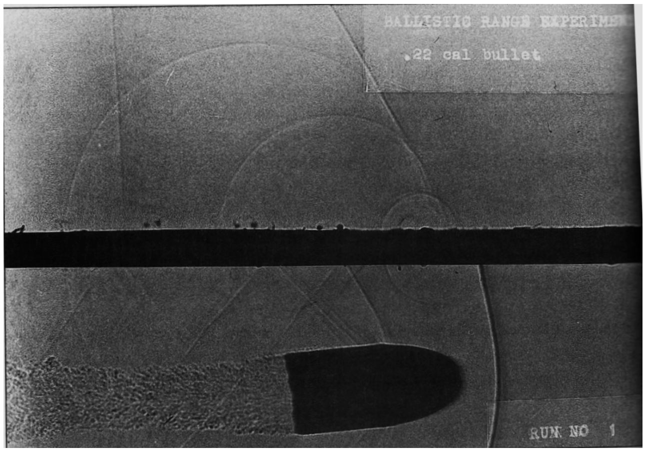


$2\alpha = 50.3°$
$p_{atm} = 14.62$ psia
$T_{atm} = 73$ °F = 533 °R
$\gamma_{air} = 1.4$
$R_{air} = 53.3$ (ft·lb$_f$)/(lb$_m$·°R)

$$\boxed{\sin\alpha = \frac{1}{\text{Ma}}} \Rightarrow \boxed{\text{Ma} = 2.35}$$

$$\text{Ma} = \frac{V}{c} = \frac{V}{\sqrt{\gamma R T}} \Rightarrow \boxed{V = \text{Ma}\sqrt{\gamma R T_{atm}}} \Rightarrow \boxed{V = 2660 \text{ ft/s}}$$

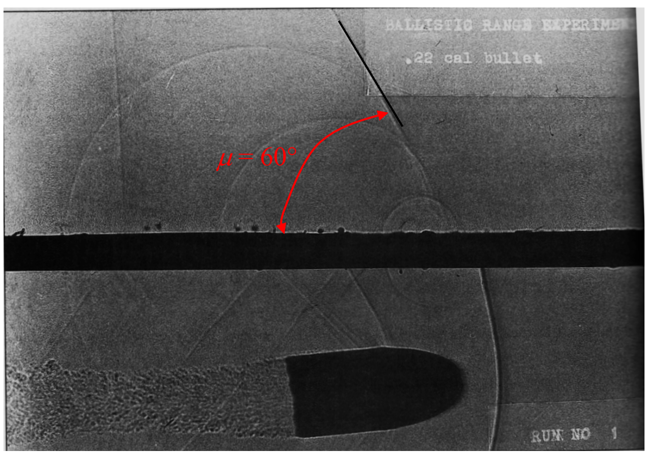
Determine the Mach number of the .22 caliber bullet shown below. Note that the plate in the figure has holes through which weak pressure disturbances can propagate.



If the temperature of the air at which the test is conducted is 70 °F, determine the speed of the bullet.

SOLUTION:

Determine the Mach angle from the photograph. Note that since the waves above the plate are very weak, they will be Mach waves.



The angle of the Mach waves is related to the Mach number via:

$$\boxed{\sin\mu = \frac{1}{\mathrm{Ma}}} \text{ and } \mu = 60° \Rightarrow \boxed{\mathrm{Ma} = 1.2} \quad (1)$$

Now determine the speed of the bullet from the definition of the Mach number.

$$\mathrm{Ma} = \frac{V}{c} \Rightarrow V = c\mathrm{Ma} \quad (2)$$

$$\boxed{\therefore V = \mathrm{Ma}\sqrt{\gamma R T}} \quad (3)$$

For air at 70 °F (530 °R) and Ma = 1.2, $\gamma = 1.4$, $R = 53.3$ ($\mathrm{lb_f}$·ft)/($\mathrm{lb_m}$·°R) $\Rightarrow$ $\boxed{V = 1350 \text{ ft/s}}$