13.17. Normal Shock Waves

Consider the movement of a piston in a cylinder, as shown in Figure 13.25. When we first move the piston, an infinitesimal (compression) pressure wave travels down the cylinder at the sonic speed. Behind the wave, the pressure, temperature, density, and increase slightly and the fluid has a small velocity following the wave (refer to Section 13.4).

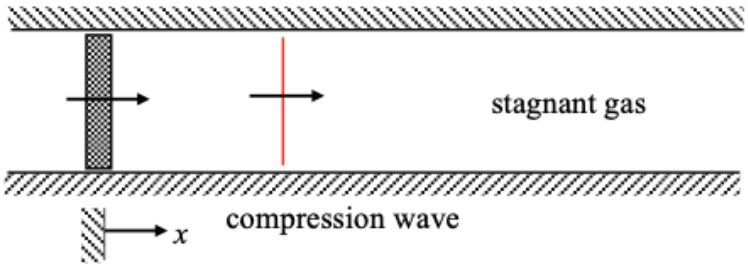


Figure 13.25. A single compression wave caused by a moving piston and traveling down the length of a cylinder into stagnant gas.

If we continue to increase the piston velocity, additional pressure waves will propagate down the cylinder (Figure 13.26). However, these waves travel at a slightly increased speed relative to a fixed observer due to the increased fluid temperature and fluid movement. The result is that the waves formed later catch up to the previous waves. When the waves catch up to the first wave, their effects add together so that the small changes across the individual waves now become a sudden and finite change called a shock wave (Figure 13.27).

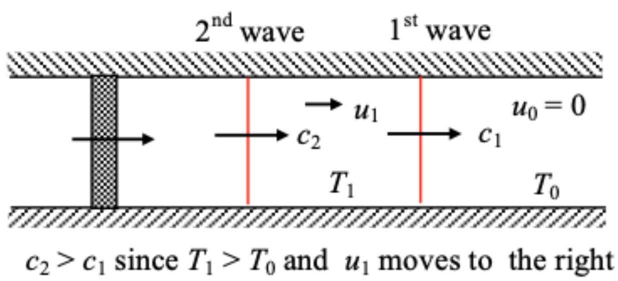


Figure 13.26. Multiple compression waves caused by an accelerating piston and traveling down the length of a cylinder.

Notes:

(1) The velocity of a shock wave is greater than the speed of sound. From the analysis used to determine the speed of a pressure wave Eq. (13.51),

$$c^2 = \frac{\Delta p}{\Delta \rho}\left(1 + \frac{\Delta \rho}{\rho}\right). \tag{13.138}$$

For a sound wave, $\Delta\rho \to d\rho \implies \Delta\rho/\rho \to 0$. For a shock wave, however, $\Delta\rho > 0$ so that $c_{\text{shock wave}} > c_{\text{sound wave}}$.

(2) A shock wave is a pressure wave across which there is a finite change in the flow properties.

(3) Shock waves only occur in supersonic flows. This fact is proven later in this section using the Second Law of Thermodynamics.

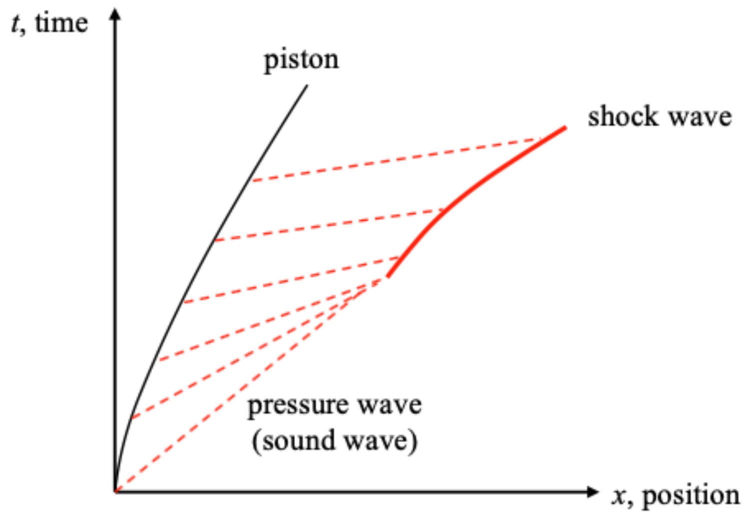


Figure 13.27. A plot illustrating the paths of an accelerating piston, weak compression waves, and the formation of a shock wave. The shock wave is defined as occurring where the compression waves first intersect.

(4) Shock waves are typically very thin, with thicknesses on the order of 1 μm. Thus, we consider the changes in the flow properties across the wave to be discontinuous.
(5) The sudden change in flow properties across the shock wave occurs non-isentropically since the thermal and velocity gradients are large within the shock wave itself.

To analyze a shock wave, we'll use an approach similar to that used to examine a sound wave. Let's consider a fixed shock wave across which flow properties change. A thin control volume of cross-sectional area A encompasses the wave as shown in Figure 13.28.

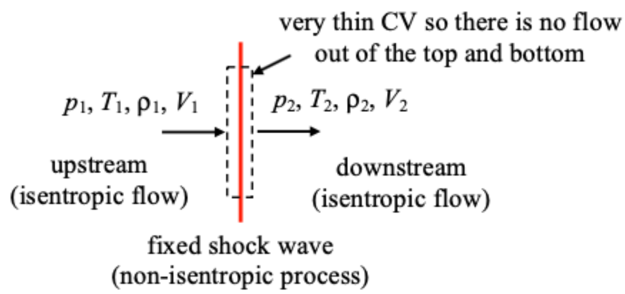


Figure 13.28. The control volume used to analyze changes in properties across a normal shock wave. Note that there is no heat transfer into the control volume.

From Conservation of Mass,

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2, \tag{13.139}$$

$$\rho_1 V_1 = \rho_2 V_2. \tag{13.140}$$

From the Linear Momentum Equation,

$$\dot{m} V_2 - \dot{m} V_1 = p_1 A - p_2 A, \tag{13.141}$$

$$\rho_1 V_1 (V_2 - V_1) = \rho_2 V_2 (V_2 - V_1) = p_1 - p_2. \tag{13.142}$$

From the First Law of Thermodynamics,

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2, \tag{13.143}$$

$$h_{01} = h_{02}. \tag{13.144}$$

Note that no heat is transferred into the control volume and, thus, the process is adiabatic.

From the Second Law of Thermodynamics,

$$\dot{m}s_2 - \dot{m}s_1 = \dot{\sigma}, \tag{13.145}$$

$$s_2 - s_1 = \sigma > 0. \tag{13.146}$$

The entropy production is greater than zero since *within the shock wave* there are internal irreversibilities. The thermal gradient and velocity gradient are enormous since the temperature and velocity have finite changes within the shock and the shock thickness is very small.

Since we're assuming we're working with a perfect gas,

$$\frac{p_1}{\rho_1 T_1} = \frac{p_1}{\rho_1 T_1} = R \quad \text{(ideal gas law)}, \tag{13.147}$$

$$\Delta h = c_p \Delta T. \tag{13.148}$$

Combining Eqs. (13.140) and (13.142),

$$\frac{p_1}{\rho_1 V_1} - \frac{p_2}{\rho_2 V_2} = V_2 - V_1. \tag{13.149}$$

Substituting Eq. (13.147),

$$\frac{RT_1}{V_1} - \frac{RT_2}{V_2} = V_2 - V_1. \tag{13.150}$$

Substituting Eqs. (13.144) and (13.148),

$$\frac{R}{V_1}\left(T_0 - \frac{V_1^2}{2c_p}\right) - \frac{R}{V_2}\left(T_0 - \frac{V_2^2}{2c_p}\right) = V_2 - V_1, \tag{13.151}$$

$$RV_2T_0 - \frac{RV_2V_1^2}{2c_p} - RV_1T_0 - \frac{RV_1V_2^2}{2c_p} = V_1V_2(V_2 - V_1), \tag{13.152}$$

$$V_1V_2(V_2 - V_1) = R\left[(V_2 - V_1)T_0 + (V_2 - V_1)\frac{V_1V_2}{2c_p}\right], \tag{13.153}$$

$$V_1V_2(V_2 - V_1) = R(V_2 - V_1)\left(T_0 + \frac{V_1V_2}{2c_p}\right), \tag{13.154}$$

$$V_1V_2 = R\left(T_0 + \frac{V_1V_2}{2c_p}\right), \tag{13.155}$$

$$V_1V_2\left(1 - \frac{R}{2c_p}\right) = RT_0, \tag{13.156}$$

$$V_1V_2 = \frac{RT_0}{1 - \frac{R}{2c_p}}. \tag{13.157}$$

Finally, substituting the ideal gas relation,

$$\frac{R}{c_p} = \frac{c_p - c_v}{c_p} = \frac{k-1}{k}, \tag{13.158}$$

and re-arranging gives,

$$\boxed{V_1V_2 = \frac{2kRT_0}{k+1}} \quad \underline{\text{Prandtl's Equation}}. \tag{13.159}$$

Dividing both sides of Prandtl's equation by the sound speed on either side of the shock wave and utilizing the definition for the Mach number for an ideal gas,

$$\frac{V_1}{\sqrt{kRT_1}}\frac{V_2}{\sqrt{kRT_2}} = \frac{2}{k+1}\frac{\sqrt{kRT_0}}{\sqrt{kRT_1}}\frac{\sqrt{kRT_0}}{\sqrt{kRT_2}}, \tag{13.160}$$

$$\mathrm{Ma}_1\mathrm{Ma}_2 = \frac{2}{k+1}\sqrt{\frac{T_0}{T_1}}\sqrt{\frac{T_0}{T_2}}. \tag{13.161}$$

Recall that for the adiabatic flow of a perfect gas,

$$\frac{T}{T_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-1}, \tag{13.162}$$

so that,

$$\mathrm{Ma}_1\mathrm{Ma}_2 = \frac{2}{k+1}\sqrt{\frac{T_0}{T_1}}\sqrt{\frac{T_0}{T_2}}, \tag{13.163}$$

$$= \frac{2}{k+1}\left(1+\frac{k-1}{2}\mathrm{Ma}_1^2\right)^{\frac{1}{2}}\left(1+\frac{k-1}{2}\mathrm{Ma}_2^2\right)^{\frac{1}{2}}. \tag{13.164}$$

After additional algebra, we can reduce this equation to,

$$\boxed{\mathrm{Ma}_2^2 = \frac{(k-1)\mathrm{Ma}_1^2+2}{2k\mathrm{Ma}_1^2-(k-1)}}. \tag{13.165}$$

This equation relates the upstream and downstream Mach numbers across a normal shock wave.

Notes:

(1) When $\mathrm{Ma}_1 > 1$, then $\mathrm{Ma}_2 < 1$ (supersonic to subsonic flow) and when $\mathrm{Ma}_1 < 1$, then $\mathrm{Ma}_2 > 1$ (subsonic to supersonic flow).
(2) From experiments, we observe that shock waves never form in subsonic flows ($\mathrm{Ma}_1 < 1$) even though Eq. (13.165) doesn't give any indication that this would be the case. We will use the Second Law in a moment to show that shock waves can only form in supersonic flows ($\mathrm{Ma}_1 > 1$).

The temperature ratio across the shock wave can be determined using the adiabatic stagnation temperature relation for a perfect gas and noting that the stagnation temperature remains constant across a shock,

$$\frac{T_2/T_0}{T_1/T_0} = \frac{\left(1+\frac{k-1}{2}\mathrm{Ma}_2^2\right)^{-1}}{\left(1+\frac{k-1}{2}\mathrm{Ma}_1^2\right)^{-1}}, \tag{13.166}$$

$$\frac{T_2}{T_1} = \frac{\left(1+\frac{k-1}{2}\mathrm{Ma}_1^2\right)}{\left(1+\frac{k-1}{2}\mathrm{Ma}_2^2\right)}. \tag{13.167}$$

The pressure ratio across the shock can be determined by combining Eqs. (13.167), (13.147), and (13.140),

$$\frac{p_2}{p_1} = \frac{\rho_2 T_2}{\rho_1 T_1} = \frac{V_1 T_2}{V_2 T_1} = \frac{(\sqrt{kRT_1}\mathrm{Ma}_1)T_2}{(\sqrt{kRT_2}\mathrm{Ma}_2)T_1}, \tag{13.168}$$

$$= \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\sqrt{\frac{T_2}{T_1}}, \tag{13.169}$$

$$\frac{p_2}{p_1} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1}{2}}. \tag{13.170}$$

The density ratio across the shock is,

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{p_2 T_1}{p_1 T_2} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1}{2}}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right), \tag{13.171}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{3}{2}}. \tag{13.172}$$

We can also determine the ratio of the isentropic stagnation pressures and densities across the shock wave,

$$\frac{p_1/p_{01}}{p_2/p_{02}} = \left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k}{1-k}}, \tag{13.173}$$

$$\frac{p_{02}}{p_{01}} = \left(\frac{p_2}{p_1}\right)\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k}{1-k}}, \tag{13.174}$$

$$\frac{p_{02}}{p_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1}{2}}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k}{1-k}}, \tag{13.175}$$

$$\frac{p_{02}}{p_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1+k}{2(1-k)}}. \tag{13.176}$$

For the stagnation density,

$$\frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}}\frac{T_{01}}{T_{02}}, \tag{13.177}$$

but since $T_{01} = T_{02}$ (refer to Eq. (13.148)),

$$\frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1+k}{2(1-k)}}. \tag{13.178}$$

The sonic area ratio across the shock can be determined from the fact that the mass flow rate across the shock must remain constant,

$$\dot{m}_1 = \dot{m}_2, \tag{13.179}$$

$$\rho_1^* V_1^* A_1^* = \rho_2^* V_2^* A_2^*, \tag{13.180}$$

$$\frac{A_2^*}{A_1^*} = \frac{\rho_1^*}{\rho_2^*}\frac{V_1^*}{v_2^*}. \tag{13.181}$$

The sonic ratios can be determined from the following analyses,

$$\frac{\rho_1^*}{\rho_{01}} = \frac{\rho_2^*}{\rho_{02}} = \left(1+\frac{k-1}{2}\right)^{\frac{1}{1-k}}, \tag{13.182}$$

$$\frac{\rho_1^*}{\rho_2^*} = \frac{\rho_{01}}{\rho_{02}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1+k}{2(1-k)}}, \tag{13.183}$$

and,

$$\frac{V_1^*}{V_2^*} = \frac{c_1^*}{c_2^*} = \sqrt{\frac{T_1^*}{T_2^*}} = \sqrt{\frac{T_1^*/T_0}{T_2^*/T_0}} = 1. \tag{13.184}$$

Note that $T_{01} = T_{02}$ has been used in the previous equation. Substituting these two sonic ratios and simplifying gives,

$$\frac{A_2^*}{A_1^*} = \frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k+1}{2(k-1)}}. \tag{13.185}$$

Note that we could have also used the isentropic area ratios on either side of the shock wave to determine the sonic area ratio across the shock,

$$\frac{A_1}{A_1^*} = \frac{1}{\mathrm{Ma}_1}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{and} \quad \frac{A_2}{A_2^*} = \frac{1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_2^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}}, \tag{13.186}$$

so that,

$$\frac{A_2^*}{A_1^*} = \frac{A_1/A_1^*}{A_2/A_2^*} = \frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k+1}{2(k-1)}}, \tag{13.187}$$

where $A_1 = A_2$.

Notes:

(1) The previous equations may be written only in terms of Ma_1 by substituting in Eq. (13.165). The resulting equations (after much algebra) are,

$$\boxed{\mathrm{Ma}_2^2 = \frac{(k-1)\mathrm{Ma}_1^2+2}{2k\mathrm{Ma}_1^2-(k-1)}}, \tag{13.188}$$

$$\boxed{\frac{T_2}{T_1} = \left[2+(k-1)\mathrm{Ma}_1^2\right]\left[\frac{2k\mathrm{Ma}_1^2-(k-1)}{(k+1)^2\mathrm{Ma}_1^2}\right]}, \tag{13.189}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(k+1)\mathrm{Ma}_1^2}{(k-1)\mathrm{Ma}_1^2+2}}, \tag{13.190}$$

$$\boxed{\frac{p_2}{p_1} = \frac{2k}{k+1}\mathrm{Ma}_1^2 - \frac{k-1}{k+1}}, \tag{13.191}$$

$$\boxed{\frac{T_{02}}{T_{01}} = 1}, \tag{13.192}$$

$$\boxed{\frac{p_{02}}{p_{01}} = \frac{\rho_{02}}{\rho_{01}} = \frac{A_1^*}{A_2^*} = \left(\frac{\frac{k+1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_1^2}\right)^{\frac{k}{k-1}}\left(\frac{2k}{k+1}\mathrm{Ma}_1^2-\frac{k-1}{k+1}\right)^{\frac{1}{1-k}}}. \tag{13.193}$$

(2) Note let's examine the change in specific entropy across the shock using the following expression for a perfect gas,

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right). \tag{13.194}$$

Substituting Eqs. (13.189) and (13.191) into this equation and plotting we obtain Figure 13.29. We observe that for $\mathrm{Ma}_1 < 1$ the entropy decreases across the shock. The Second Law, however, states that the entropy must increase across the shock (refer to Eq. (13.146)). Thus, shock waves can only form when $\mathrm{Ma}_1 > 1$.

Also note that as the upstream Mach number approaches one ($\mathrm{Ma}_1 \to 1$), the flow through the shock approaches an isentropic process. An infinitesimally weak shock wave, one occurring when $\mathrm{Ma}_1 = 1$, results in an isentropic process. This type of shock is, in fact, just a sound wave!

(3) Plots of the various property ratios are shown in Figure 13.30 as functions of the upstream Mach number. The temperature, pressure, density, and sonic area increase across the shock, with the ratios increasing as the Mach number increases. The stagnation pressure, stagnation density, and velocity decrease across the shock, with the ratios decreasing as the Mach number increases. The downstream Mach number also decreases across the shock and becomes smaller as the upstream Mach number increases. The stagnation temperature remains constant across the shock. Normal shock tables with the numerical values for these ratios can be found in the appendices of most compressible flow textbooks.

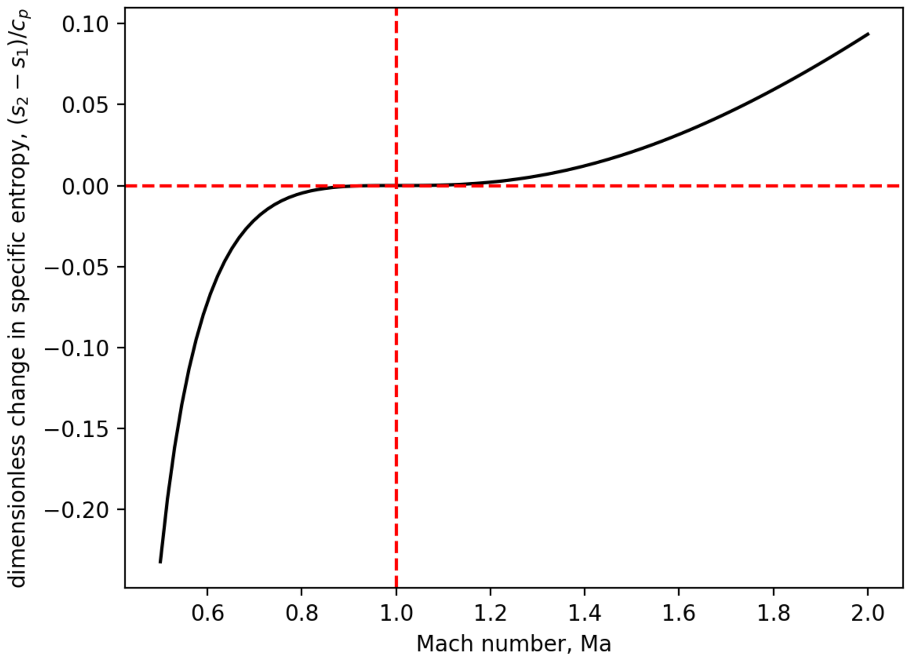


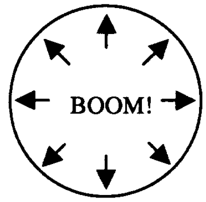
Figure 13.29. A plot of the dimensionless specific entropy change across a normal shock wave, $\Delta s/c_p$ as a function of the upstream Mach number, Ma_1, for $k = 1.4$.

(4) The shock strength is defined as the change in pressure across the shock wave relative to the upstream pressure, i.e., $\Delta p/p_1 = p_2/p_1 - 1$. Viewing the trends shown in Figure 13.30, the larger the incoming Mach number the stronger the shock wave.

(5) On a T-s diagram, the states across a shock wave correspond to the intersection of the Fanno and Rayleigh lines for the flow, as shown in Figure 13.31. The reason is because the flow across the shock satisfies the Fanno relations for Conservation of Mass (Eq. (13.140)), the First Law (Eq. (13.144)), and the ideal gas relations (Eqs. (13.147) and (13.148)). The shock also satisfies the Rayleigh relations for Conservation of Mass, the Linear Momentum Equation (Eq. (13.142)), and the ideal gas relations. The shock states must, therefore, occur at the intersection of the Fanno and Rayleigh lines in order for the shock to satisfy all of the basic relations simultaneously. Furthermore, state 2 lies to the right of state 1 in the T-s diagram since entropy must increase across the shock (from the Second Law).

An explosion creates a spherical shock wave propagating radially into still air at standard conditions. A recording instrument registers a maximum pressure of 200 psig as the shock wave passes by. Estimate:

a. **the speed of the shock wave with respect to a fixed observer in ft/sec**
b. **the wind speed following the shock with respect to a fixed observer in ft/sec**



SOLUTION:

- Note that although the shock wave is spherical, the flow across the shock can be considered as flow across a normal shock wave (assuming that the radius of curvature of the shock is large).

$Ma_1, V_1, p_1, T_1, \rho_1 \longrightarrow$ | $Ma_2, V_2, p_2, T_2, \rho_2 \longrightarrow$

$p_1 = 14.7$ psia
$T_1 = 70°F = 530°R$

Change frame of reference to move w/ the shock wave

$\gamma = 1.4$
$R = 53.3 \frac{ft \cdot lb_f}{lb_m \cdot °R}$

$$\frac{p_2}{p_1} = \frac{(200 + 14.7)\text{ psia}}{14.7\text{ psia}} = 14.6$$

- Use normal shock relations to find:

$Ma_1 = 3.56$ using $\frac{p_2}{p_1} = 14.6$, $\gamma = 1.4$

$Ma_2 = 0.45$

- Upstream of the shock:

$$V_1 = Ma_1\sqrt{\gamma R T_1}$$

$$\therefore V_1 = 4020 \text{ ft/s}$$ using $Ma_1 = 3.56$, $\gamma = 1.4$, $R = 53.3 \frac{ft \cdot lb_f}{lb_m \cdot °R}$, $T_1 = 530°R$

∴ The shock propagates into the still air at a speed of 4020 ft/s

1 of 2

SOLUTION...

- Also from the normal shock relations:

$$\frac{T_2}{T_1} = 3.398 \quad \text{using} \quad Ma_1 = 3.56, \ \gamma = 1.4$$

$$\Rightarrow \underline{T_2 = 1800\,°R} \quad \text{using} \quad \frac{T_2}{T_1} = 3.398, \ T_1 = 530\,°R$$

NOTE: Could have also found T_2 using:

$$\frac{T_2}{T_{02}} = \left(1 + \frac{\gamma - 1}{2} Ma_2^2\right)^{-1}$$

and

$$\frac{T_1}{T_{01}} = \left(1 + \frac{\gamma - 1}{2} Ma_1^2\right)^{-1}$$

where $T_{02} = T_{01}$

$$\Rightarrow V_2 = Ma_2 \sqrt{\gamma R T_2}$$

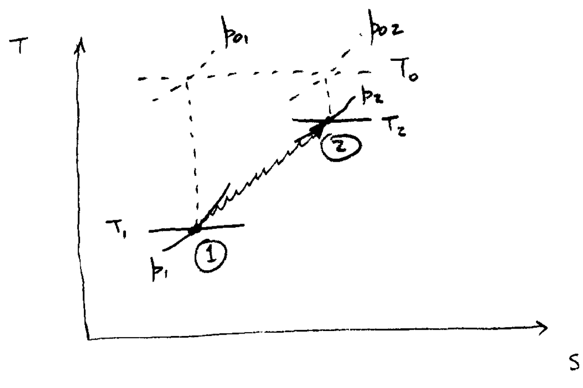
$$\therefore \underline{V_2 = 936 \ \tfrac{ft}{s}} \quad \text{using} \quad Ma_2 = 0.45, \ \gamma = 1.4, \ R = 53.3 \ \frac{ft \cdot lb_f}{lb_m \cdot °R}, \ T_2 = 1800\,°R$$

Note that this is the downstream air velocity relative to the shock.

NOTE: Could have also found V_2 using: $c_p T_1 + \frac{1}{2} V_1^2 = c_p T_2 + \frac{1}{2} V_2^2$

$\therefore$ air velocity relative to a fixed observer

$$\boxed{V_{\text{downstream air, relative to fixed observer}} = V_{shock} - V_2 = 4020 \ \tfrac{ft}{s} - 936 \ \tfrac{ft}{s} = 3080 \ \tfrac{ft}{s}}$$

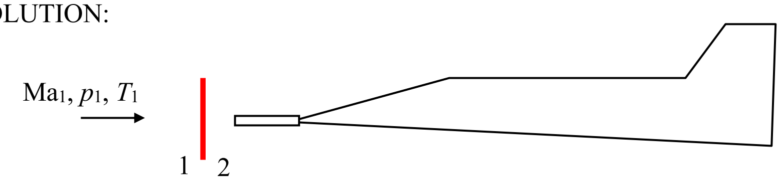


2 of 2

Stagnation pressure and temperature probes are located on the nose of a supersonic aircraft at 35,000 ft altitude. A normal shock stands in front of the probes. The temperature probe indicates $T_0 = 420$ °F behind the shock.

a. Calculate the Mach number and airspeed of the plane.
b. Find the static and stagnation pressures behind the shock.
c. Show the process and the static and stagnation points on a T-s diagram.

SOLUTION:



The pressure and temperature at an altitude of 35,000 ft using a U.S. Standard Atmospheric table (e.g., Table C.5 in Zucrow and Hoffman or using an online calculator such as http://www.digitaldutch.com/atmoscalc/) are:

$$p_1 = 3.458 \text{ psia} \tag{1}$$
$$T_1 = 393.9 \text{ °R} \tag{2}$$

The Mach number of the aircraft may be found by noting that the stagnation temperature remains constant across the shock wave ($T_{01} = T_{02} = 879$ °R) and using the adiabatic stagnation temperature ratio:

$$\frac{T_1}{T_{01}} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right)^{-1} \Rightarrow \boxed{\text{Ma}_1 = 2.48} \tag{3}$$

where, for air, $\gamma = 1.4$. The velocity is found from the Mach number and speed of sound:

$$V_1 = \text{Ma}_1 c_1 = \text{Ma}_1\sqrt{\gamma R T_1} \Rightarrow \boxed{V_1 = 2410 \text{ ft/s}} \tag{4}$$

where $R = 53.3$ (lbf·ft)/(lbm·°R).

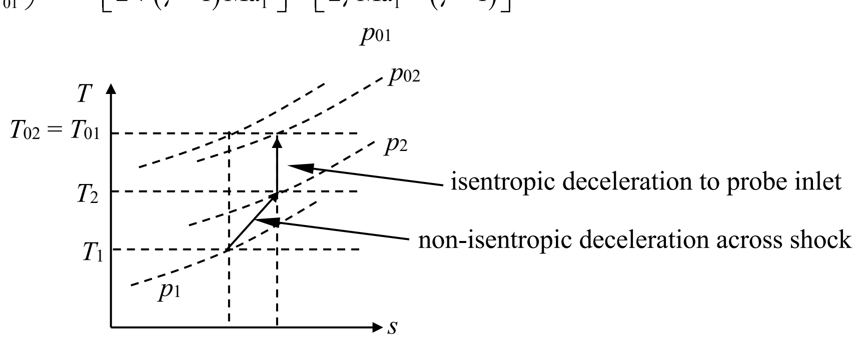
The static pressure downstream of the shock, p_2, may be found from the normal shock relations.

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\text{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \Rightarrow \boxed{p_2 = 24.2 \text{ psia}} \tag{5}$$

The stagnation pressure may be found by combining the stagnation pressure upstream of the shock with the stagnation pressure ratio across the shock.

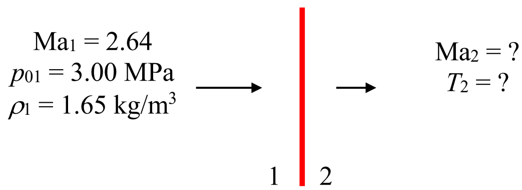
$$\frac{p_1}{p_{01}} = \left(1 + \frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_{01} = 57.4 \text{ psia} \tag{6}$$

$$p_{02} = p_{01}\left(\frac{p_{02}}{p_{01}}\right) = p_{01}\left[\frac{(\gamma+1)\text{Ma}_1^2}{2+(\gamma-1)\text{Ma}_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \Rightarrow \boxed{p_{02} = 29.1 \text{ psia}} \tag{7}$$



An air stream approaches a normal shock at $Ma_1 = 2.64$. Upstream, $p_{01} = 3.00$ MPa (abs) and $\rho_1 = 1.65$ kg/m^3. Determine the downstream Mach number and temperature.

SOLUTION:



The downstream Mach number may be found from the normal shock relations:

$$\mathrm{Ma}_2^2 = \frac{(\gamma-1)\mathrm{Ma}_1^2 + 2}{2\gamma \mathrm{Ma}_1^2 - (\gamma-1)} \tag{1}$$

where $Ma_1 = 2.64$ and $\gamma = 1.4$.

$\boxed{\therefore \mathrm{Ma}_2 = 0.50}$

One method of finding the downstream temperature is to determine the upstream stagnation temperature and then use the downstream Mach number and the adiabatic stagnation temperature ratio along with the fact that the stagnation temperature remains constant across the shock wave to determine the downstream static temperature.

$$\frac{\rho_1}{\rho_{01}} = \left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{1}{1-\gamma}} \Rightarrow \rho_{01} = 14.6 \text{ kg/m}^3 \text{ (where } \rho_1 = 1.65 \text{ kg/m}^3\text{)} \tag{2}$$

$$T_{01} = \frac{p_{01}}{\rho_{01} R} \Rightarrow T_{01} = 714 \text{ K (where } R = 287 \text{ J/(kg·K))} \tag{3}$$

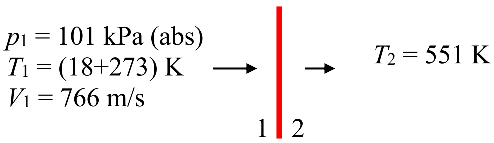
$$T_{02} = T_{01} \Rightarrow T_{02} = 714 \text{ K} \tag{4}$$

$$\frac{T_2}{T_{02}} = \left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{-1} \Rightarrow \boxed{T_2 = 680 \text{ K}} \text{ (where } \mathrm{Ma}_2 = 0.50\text{)} \tag{5}$$

Air approaches a normal shock with $T_1 = 18$ °C, $p_1 = 101$ kPa (abs), and $V_1 = 766$ m/s. The temperature immediately downstream from the shock is $T_2 = 551$ K.

1. Determine the velocity immediately downstream from the shock.
2. Determine the pressure change across the shock.
3. Calculate the corresponding pressure change for a frictionless, shockless deceleration between the same speeds and temperatures.

SOLUTION:



The velocity downstream of the shock may be found from conservation of energy.

$$c_P T_1 + \tfrac{1}{2}V_1^2 = c_P T_2 + \tfrac{1}{2}V_2^2 \tag{1}$$

$$V_2 = \sqrt{V_1^2 + 2c_P\left(T_1 - T_2\right)} \Rightarrow \boxed{V_2 = 254.3 \text{ m/s}} \tag{2}$$

using $c_P = 1004$ J/(kg·K).

The pressure change across the shock may be found using the normal shock relations.

$$\Delta p = p_2 - p_1 = p_1\left(\frac{p_2}{p_1} - 1\right) \Rightarrow \boxed{\Delta p = 4.73*10^5 \text{ Pa}} \tag{3}$$

where

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\text{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \Rightarrow p_2/p_1 = 5.6880 \tag{4}$$

and

$$\text{Ma}_1 = \frac{V_1}{c_1} = \frac{V_1}{\sqrt{\gamma R T_1}} \Rightarrow \text{Ma}_1 = 2.24 \tag{5}$$

In addition,

$$\text{Ma}_2^2 = \frac{\left(\gamma-1\right)\text{Ma}_1^2 + 2}{2\gamma\text{Ma}_1^2 - \left(\gamma-1\right)} \Rightarrow \text{Ma}_2 = 0.54 \tag{6}$$

Note that we could have also simply used:

$$\text{Ma}_2 = \frac{V_2}{c_2} = \frac{V_2}{\sqrt{\gamma R T_2}} \Rightarrow \text{Ma}_2 = 0.54 \text{ (Same result as the previous one!)} \tag{7}$$

The corresponding pressure change for an isentropic deceleration between the same speeds may be found by combining isentropic stagnation pressure ratios,

$$\Delta p = p_2 - p_1 = p_1\left(\frac{p_2}{p_1} - 1\right) \Rightarrow \boxed{\Delta p_{\text{isentropic}} = 8.41*10^5 \text{ Pa}} \tag{8}$$

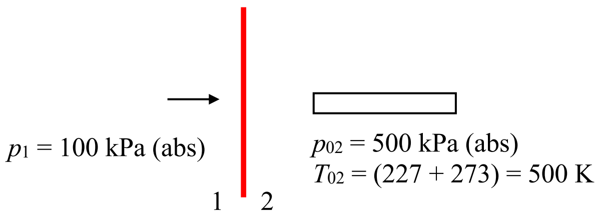
where,

$$\frac{p_2}{p_1} = \frac{\left(p_2/p_0\right)}{\left(p_1/p_0\right)} = \frac{\left(1 + \frac{\gamma-1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}}}{\left(1 + \frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}}} \Rightarrow p_2/p_1 = 9.3253 \tag{9}$$

A total pressure probe is inserted into a supersonic air flow. A shock wave forms just upstream of the impact hole. The probe measures a total pressure of 500 kPa (abs) and the stagnation temperature at the probe head is 227 °C. The static pressure upstream of the shock is measured with a wall tap to be 100 kPa (abs).

a. Determine the Mach number of the incoming flow.
b. Determine the velocity of the incoming flow.
c. Sketch the process on a *T-s* diagram.

SOLUTION:



Determine the upstream Mach number by combining the isentropic pressure ratio and the stagnation pressure ratio across a normal shock.

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \tag{1}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\text{Ma}_1^2}{2+(\gamma-1)\text{Ma}_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \tag{2}$$

$$\frac{p_1}{p_{02}} = \left(\frac{p_1}{p_{01}}\right)\left(\frac{p_{01}}{p_{02}}\right) = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}}\left[\frac{(\gamma+1)\text{Ma}_1^2}{2+(\gamma-1)\text{Ma}_1^2}\right]^{\frac{\gamma}{1-\gamma}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{1-\gamma}} \tag{3}$$

Solve Eqn. (3) numerically for Ma_1 given that p_1 = 100 kPa and p_{02} = 500 kPa (and γ = 1.4).

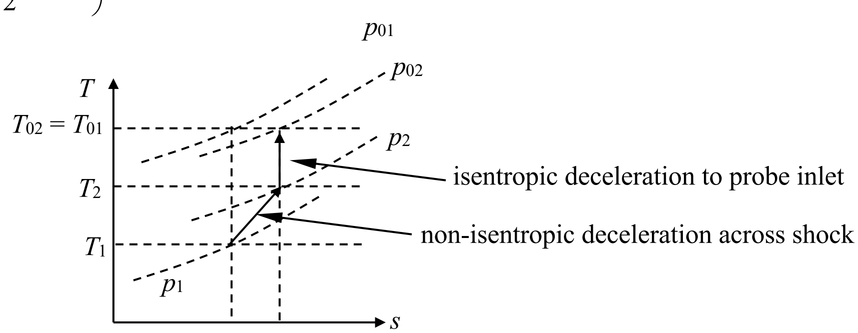
$$\boxed{\text{Ma}_1 = 1.87} \tag{4}$$

The velocity may be found from the Mach number and speed of sound on the upstream side of the shock wave.

$$V_1 = c_1\text{Ma}_1 = \sqrt{\gamma R T_1}\,\text{Ma}_1 \Rightarrow \boxed{V_1 = 643.1\text{ m/s}} \tag{5}$$

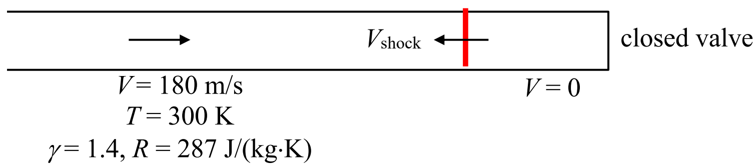
where the upstream static temperature is found from the adiabatic stagnation temperature ratio and noting that $T_{01} = T_{02}$.

$$\frac{T_1}{T_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{-1} \Rightarrow T_1 = 294.1\text{ K} \tag{6}$$

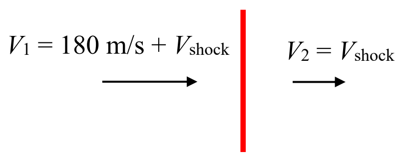


Air, with a temperature of 300 K, is flowing at 180 m/s through a constant-area, 30 m long pipe. A valve at the end of the pipe is suddenly closed and a normal shock wave propagates back into the pipe starting from the valve. How long will it be before the effect of closing the valve is felt at the pipe inlet?

SOLUTION:



Change the frame of reference shown in the figure (which is with respect to the ground), to a frame of reference that is fixed to the shock wave so that the normal shock relations may be used.



Use an iterative procedure for determining V_{shock}.

1. Assume a value for Ma_1.
2. Calculate Ma_2 using the normal shock relations.
$$\text{Ma}_2^2 = \frac{(\gamma-1)\text{Ma}_1^2 + 2}{2\gamma\text{Ma}_1^2 - (\gamma-1)} \tag{1}$$
3. Determine T_2 using the normal shock relations.
$$T_2 = \left(\frac{T_2}{T_1}\right)T_1 \tag{2}$$
where
$$\frac{T_2}{T_1} = \left[2+(\gamma-1)\text{Ma}_1^2\right]\left[\frac{2\gamma\text{Ma}_1^2-(\gamma-1)}{(\gamma+1)^2\text{Ma}_1^2}\right] \tag{3}$$
4. Determine V_{shock} using:
$$V_{\text{shock}} = V_2 = \text{Ma}_2\sqrt{\gamma R T_2} \tag{4}$$
5. Determine V_1.
$$V_1 = \text{Ma}_1\sqrt{\gamma R T_1} \tag{5}$$
6. Determine V_1'.
$$V_1' = 180 \text{ m/s} + V_{\text{shock}} \tag{6}$$
7. If V_1' is greater than V_1, then the assumed value for Ma_1 was too small and a larger value for Ma_1 should be used. If $V_1' < V_1$, then choose a smaller value for Ma_1. Repeat steps 2 – 7 until a converged solution is obtained.

Using the given iterative procedure:

$$\text{Ma}_1 = 1.36 \Rightarrow V_{\text{shock}} = 291 \text{ m/s} \tag{7}$$

The valve closing will be felt 30 m upstream in time:

$$\boxed{T = L/V_{\text{shock}} = 0.10 \text{ s}} \tag{8}$$

A stagnation tube is placed in a supersonic flow in which the static pressure and temperature far upstream are 60 kPa (abs) and -20 °C. The difference between the stagnation pressure measured by the stagnation tube and the upstream static pressure is 449 kPa. Determine the upstream Mach number and velocity of the flow.

SOLUTION:

Since there is no throat upstream of the stagnation tube, there must be a shock wave that forms in order to slow the flow from supersonic to subsonic conditions, and eventually stagnation conditions at the inlet to the stagnation tube.

p_1 = 60 kPa
T_1 = -20 °C = 253 K
γ = 1.4, R = 287 J/(kg·K)

$p_{02} - p_1$ = 449 kPa

Re-arrange the given conditions in order to solve for the upstream Mach number.

$$p_{02} - p_1 = \left(\frac{p_{02}}{p_1} - 1\right)p_1 = \left(\frac{p_{02}}{p_2}\frac{p_2}{p_1} - 1\right)p_1 = 449 \text{ kPa} \tag{1}$$

where

$$\text{Ma}_2^2 = \frac{(\gamma-1)\text{Ma}_1^2 + 2}{2\gamma\text{Ma}_1^2 - (\gamma-1)} \tag{2}$$

$$\frac{p_2}{p_{02}} = \left(1 + \frac{\gamma-1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \tag{3}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\text{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \tag{4}$$

Iterate to a converged solution using the following approach.

1. Assume a value for Ma_1.
2. Determine Ma_2 using Eq. (2).
3. Determine p_2/p_{02} using Eq. (3).
4. Determine p_2/p_1 using Eq. (4).
5. Substitute the values calculated in the previous steps into the left-hand side of Eq. (1), along with p_1 = 60 kPa.
6. Check to see if the calculation from step 5 equals the right-hand side of Eq. (1). If the calculation is smaller than the right-hand side of Eq. (1) then the assumed Ma_1 was too small and a larger Ma_1 should be chosen. If the calculation is larger than the right-hand side of Eq. (1) then the assumed Ma_1 was too large and a smaller Ma_1 should be chosen. Steps 2 through 6 should be repeated until a converged solution results.

Following the previous iterative procedure:

$$\boxed{\text{Ma}_1 = 2.493}$$

and

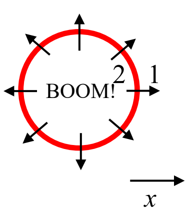
$$\boxed{V_1 = \text{Ma}_1\sqrt{\gamma R T_1} = 795 \text{ m/s}} \tag{5}$$

According to a newspaper article, at the center of a 12,600 lb_m "Daisy-Cutter" bomb explosion the overpressure in the air is approximately 1000 psi. Estimate:

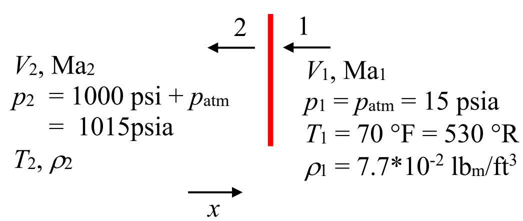
a. the speed of the resulting shock wave into the surrounding air,
b. the wind speed following the shock wave,
c. the temperature after the shock wave has passed, and
d. the air density after the shock wave has passed.

SOLUTION:

Change the frame of reference from one that is fixed to the ground to one that is fixed to the wave as shown in the schematic below. Treat the explosion shock wave as a normal shock.



Change the frame of reference so the wave appears stationary (substract the velocity of the wave to all velocities.)



The pressure ratio across the wave is:

$$\frac{p_2}{p_1} = \frac{1015 \text{ psia}}{15 \text{ psia}} = 69$$

Using the normal shock relations:

$$\begin{aligned} p_2/p_1 = 69 \Rightarrow \quad & Ma_1 = 7.7 \\ \Rightarrow \quad & Ma_2 = 0.4 \\ \Rightarrow \quad & T_2/T_1 = 12.5 \\ \Rightarrow \quad & \rho_2/\rho_1 = 5.5 \end{aligned}$$

Now determine the unknown quantities.

$$V_1 = \mathrm{Ma}_1\sqrt{\gamma R T_1} = (7.7)\sqrt{(1.4)\left(53.3\ \tfrac{\mathrm{ft\cdot lb_f}}{\mathrm{lb_m\cdot{}^\circ R}}\right)(530\ {}^\circ\mathrm{R})}$$

$$\therefore V_1 = 8700\ \mathrm{ft/s} \quad \text{(Note that is the velocity w/r/t the wave.)}$$

$$T_2 = \left(\frac{T_2}{T_1}\right)T_1 = (12.5)(530\ {}^\circ\mathrm{R})$$

$$\boxed{\therefore T_2 = 6600\ {}^\circ\mathrm{R} = 6100\ {}^\circ\mathrm{F}}$$

$$\rho_2 = \left(\frac{\rho_2}{\rho_1}\right)\rho_1 = (5.5)\left(7.7*10^{-2}\ \mathrm{lb_m/ft^3}\right)$$

$$\boxed{\therefore \rho_2 = 0.42\ \mathrm{lb_m/ft^3}}$$

$$V_2 = \mathrm{Ma}_2\sqrt{\gamma R T_2} = (0.4)\sqrt{(1.4)\left(53.3\ \tfrac{\mathrm{ft\cdot lb_f}}{\mathrm{lb_m\cdot{}^\circ R}}\right)(6600\ {}^\circ\mathrm{R})}$$

$$\therefore V_2 = 1600\ \mathrm{ft/s} \quad \text{(Note that is the velocity w/r/t the wave.)}$$

To determine the shock and downstream wind speed with respect to the ground, we must change back to our original frame of reference.

$$V_{\substack{\text{shock,}\\ \text{w/r/t ground}}} = V_1$$

$$\boxed{\therefore V_{\substack{\text{shock,}\\ \text{w/r/t ground}}} = 8700\ \mathrm{ft/s}}$$

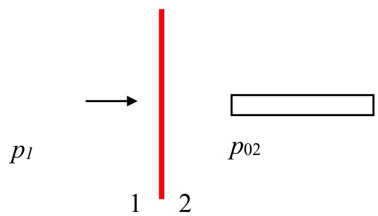
$$V_{\substack{\text{downstream wind,}\\ \text{w/r/t ground}}} = \underbrace{V_{\substack{\text{downstream wind,}\\ \text{w/r/t shock}}}}_{=-V_2} + V_{\substack{\text{shock,}\\ \text{w/r/t ground}}}$$

$$\boxed{\therefore V_{\substack{\text{downstream wind,}\\ \text{w/r/t ground}}} = 7100\ \mathrm{ft/s}}$$

A Pitot tube which senses the stagnation pressure at its mouth, p_0, is often used to measure the speed of an airplane. Such a device is incorporated into the nose of a supersonic airplane for the purpose of measuring the Mach number, Ma, at which the airplane is traveling (Ma > 1).

Assume that the ambient pressure of the air, p_1, through which the airplane is traveling, is known. If a bow shock forms ahead of the Pitot tube, find the relation between the measured quantity, p_1/p_0, and the required quantity, Ma. The relation also involves the specific heat ratio. Note that the answer cannot be written explicitly as Ma = fcn(p_1/p_0), but can be written as p_1/p_0 = fcn(Ma).

SOLUTION:



Determine the upstream Mach number by combining the isentropic pressure ratio and the stagnation pressure ratio across a normal shock.

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \tag{1}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\mathrm{Ma}_1^2}{2+(\gamma-1)\mathrm{Ma}_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \tag{2}$$

$$\boxed{\frac{p_1}{p_{02}} = \left(\frac{p_1}{p_{01}}\right)\left(\frac{p_{01}}{p_{02}}\right) = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}}\left[\frac{(\gamma+1)\mathrm{Ma}_1^2}{2+(\gamma-1)\mathrm{Ma}_1^2}\right]^{\frac{\gamma}{1-\gamma}}\left[\frac{\gamma+1}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{1-\gamma}}} \tag{3}$$

Solve Eq. (3) numerically for Ma given that p_1 and p_{02} (and γ).

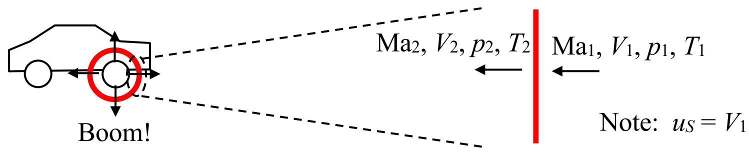
An automobile tire bursts sending a shock wave (assume this is a normal shock wave) propagating into the ambient air that has a pressure of p_1, sonic speed, c_1, and specific heat ratio, γ. If the pressure behind the shock is p_2 (roughly the inflated tire pressure), show that the speed of propagation of the shock, u_S, is given by:

$$u_S = c_1\sqrt{\frac{\gamma-1}{2\gamma}+\frac{p_2}{p_1}\frac{\gamma+1}{2\gamma}}$$

Calculate this speed if the temperature of the ambient air is 30 °C and the pressure ratio is $p_2/p_1 = 3.0$ (*e.g.* $p_1 = 14.7$ psia and $p_2 = 44.1$ psia).

SOLUTION:

Put our frame of reference on the shock wave.



Write the normal shock relation for the pressure rise across the shock.

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \quad (1)$$

Re-arrange and express the Mach number in terms of the velocity and speed of sound.

$$\boxed{V_1 = u_S = c_1\sqrt{\frac{\gamma-1}{2\gamma}+\frac{p_2}{p_1}\frac{\gamma+1}{2\gamma}}} \quad \text{where } \mathrm{Ma}_1 = V_1/c_1 \quad (2)$$

For $T_1 = 30 + 273 = 303$ K, $\gamma = 1.4$, $R = 287$ J/(kg.K), and $p_2/p_1 = 3.0$, $\underline{c_1 = 348.9\text{ m/s}}$, and $\boxed{u_S = 574.8\text{ m/s}}$.

The Mach number and temperature upstream of a shock wave are 2 and 7 °C, respectively. What is the air speed, relative to the shock wave, downstream of the shock wave?

SOLUTION:

Use the normal shock relations to determine the downstream Mach number.

$$\mathrm{Ma}_2^2 = \frac{(k-1)\mathrm{Ma}_1^2 + 2}{2k\mathrm{Ma}_1^2 - (k-1)} \Rightarrow \underline{\mathrm{Ma}_2 = 0.58} \quad (1)$$

where $k = 1.4$ and $\mathrm{Ma}_1 = 2$.

Determine the stagnation temperature upstream of the shock wave.

$$\frac{T_1}{T_{01}} = \left(1 + \frac{k-1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow \underline{T_{01} = 504\text{ K}} \quad (2)$$

where $T_1 = (273 + 7)$ K = 280 K.

Note that the stagnation temperature remains constant across a shock wave, so $\underline{T_{02} = T_{01}}$. Use the downstream stagnation temperature and downstream Mach number to determine the downstream static temperature:

$$\frac{T_2}{T_{02}} = \left(1 + \frac{k-1}{2}\mathrm{Ma}_2^2\right)^{-1} \Rightarrow \underline{T_2 = 473\text{ K}} \quad (3)$$

Use the definition of the Mach number and the speed of sound for an ideal gas to determine the air speed downstream of the shock wave:

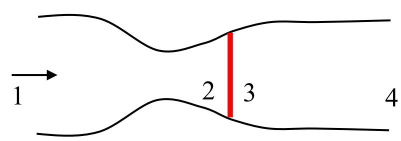
$$V_2 = \mathrm{Ma}_2 c_2 \Rightarrow V_2 = \mathrm{Ma}_2\sqrt{kRT_2} \Rightarrow \boxed{V_2 = 252\text{ m/s}} \quad (4)$$

where $R_{air} = 287$ J/(kg.K).

A supersonic aircraft flies at a Mach number of 2.7 at an altitude of 20 km. Air enters the engine inlet and is slowed isentropically to a Mach number of 1.3. A normal shock occurs at that location. The resulting flow is decelerated adiabatically, but not isentropically, further to a Mach number of 0.4. The final static pressure is 104 kPa (abs). Evaluate:

a. the stagnation temperature for the flow,
b. the pressure change, Δp, across the shock,
c. the final stagnation pressure, and
d. the total entropy change throughout the entire process.
e. Sketch the process on a Ts diagram.

SOLUTION:



The static temperature and pressure at an altitude of 20 km is, using the U.S. Standard Atmosphere, are T_1 = 216.65 K and p_1 = 5474.9 Pa (abs) (using http://www.digitaldutch.com/atmoscalc/ for example). The stagnation temperature is then:

$$\frac{T_1}{T_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow T_1/T_0 = 0.4068 \Rightarrow \boxed{T_0 = 533\text{ K}} \tag{1}$$

Note that the stagnation temperature will remain constant throughout the entire process since there is no heat transfer.

The pressure ratio across the shock wave is may be found using the normal shock relations and noting that $\mathrm{Ma}_1 = 2.7$ and $\mathrm{Ma}_2 = 1.3$.

$$\frac{p_3}{p_2} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_2^2 - \frac{\gamma-1}{\gamma+1} \Rightarrow p_3/p_2 = 1.8050 \tag{2}$$

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1/p_{01} = 0.0430 \Rightarrow p_{01} = 127\text{ kPa (abs)} \tag{3}$$

$$\frac{p_2}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_2/p_{01} = 0.3609 \Rightarrow p_2 = 46.0\text{ kPa (abs)} \tag{4}$$

$$\Delta p = p_3 - p_2 = p_2\left(\frac{p_3}{p_2}-1\right) \Rightarrow \boxed{\Delta p = 37.0\text{ kPa}} \tag{5}$$

The stagnation pressure at station 4 may be found from the isentropic stagnation pressure ratio and knowing that $\mathrm{Ma}_4 = 0.4$ and $p_4 = 104$ kPa (abs).

$$\frac{p_4}{p_{04}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_4^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_4/p_{04} = 0.8956 \Rightarrow \boxed{p_{04} = 116\text{ kPa (abs)}} \tag{6}$$

The total entropy change throughout the process may be found using the perfect gas, entropy relation:

$$\Delta s = s_4 - s_1 = c_P \ln\frac{T_4}{T_1} - R\ln\frac{p_4}{p_1} \Rightarrow \boxed{\Delta s = 26.3 \text{ J/(kg·K)}} \tag{7}$$

where p_1 and T_1 are 5474.9 Pa and 216.65 K, respectively (from a U.S. Standard Atmosphere), p_4 = 104 kPa (given), c_P = 1004 J/(kg·K), and R = 287 J/(kg·K). The temperature T_4 may be found from:

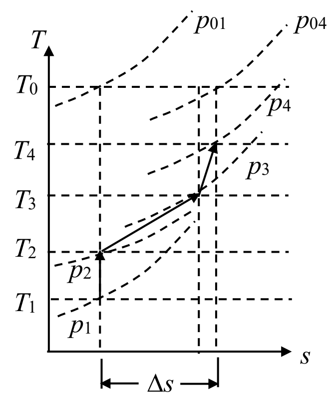
$$\frac{T_4}{T_{04}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_4^2\right)^{-1} \Rightarrow T_4/T_{04} = 0.9690 \Rightarrow T_4 = 516 \text{ K} \tag{8}$$

where $T_{04} = T_{01}$ = 533 K (from Eq. (1)).

Note that we could have also found Δs using stagnation conditions (refer to the Ts diagram below).

$$\Delta s = s_{04} - s_{01} = c_P \ln\frac{T_{04}}{T_{01}} - R\ln\frac{p_{04}}{p_{01}} \Rightarrow \Delta s = 26.4 \text{ J/(kg·K)} \quad \text{(Same as before, within error!)} \tag{9}$$

where $T_{04} = T_{01}$, p_{04} = 116 kPa, and p_{01} = 127 kPa.



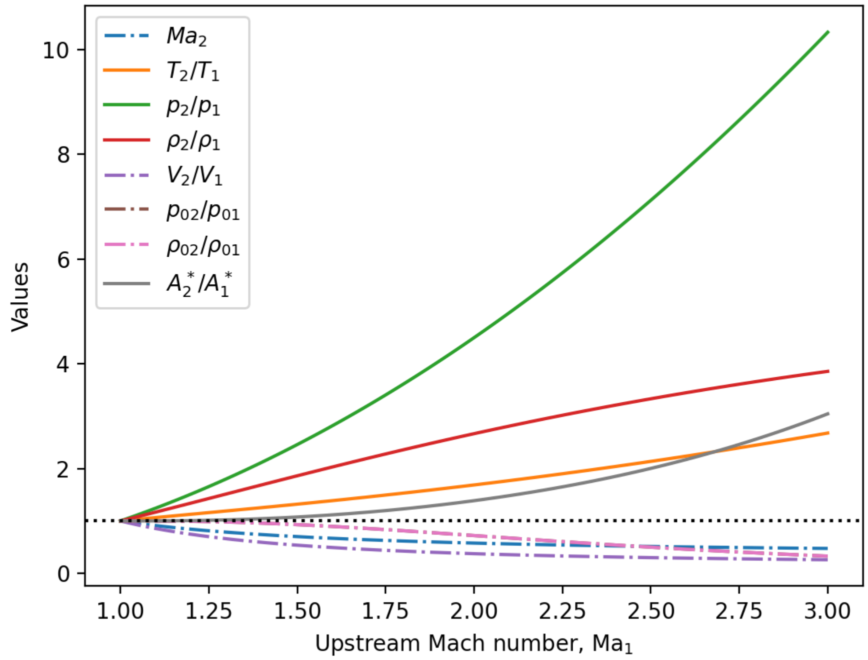


Figure 13.30. Plots of the various property ratios as functions of the upstream Mach number, Ma_1, for $k = 1.4$. Note that $T_{02}/T_{01} = 1$ and $p_{02}/p_{01} = \rho_{02}/\rho_{01}$.

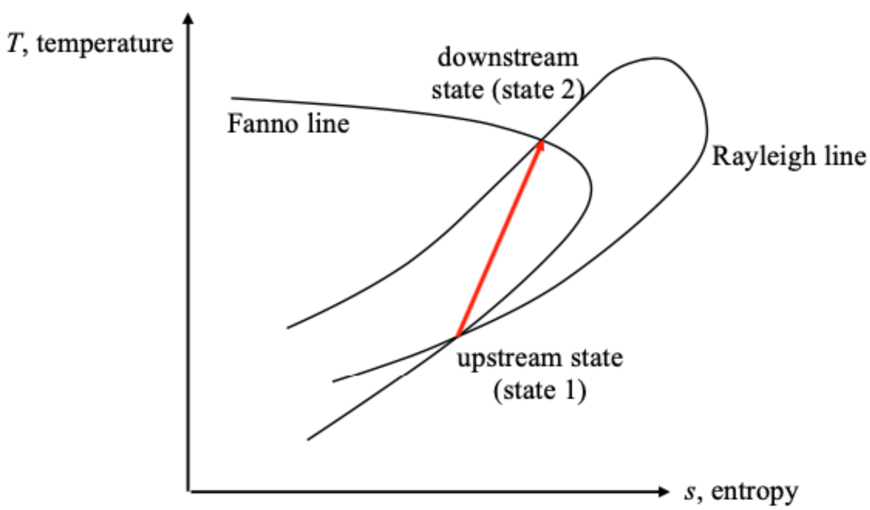


Figure 13.31. On a T-s plot, the states across a normal shock occur at the intersection of the Fanno and Rayleigh lines.

13.18. Flow through Converging-Diverging Nozzles

Consider flow through a converging-diverging nozzle (aka a de Laval nozzle) as shown in Figure 13.32. Let's hold the stagnation pressure, p_0, fixed and vary the back pressure, p_B. The plot shown in Figure 13.33 shows