CHAPTER 13

Compressible Flow (aka Gas Dynamics)

13.1. Introduction to Gas Dynamics

13.1.1. What is Gas Dynamics?

Gas dynamics is a branch of fluid mechanics that examines the dynamics of compressible fluid flows and of gases in particular.

13.1.2. What is the motivation for studying compressible fluids?

Although topics regarding compressible fluid mechanics have been studied since the 1800s, few scientists and engineers were interested in the topic apart from those studying ballistics and steam turbine design. It wasn't until WW II with the development of high speed planes, rockets, and energetic explosives that the study of compressible flows became widespread. Ever since, the understanding of compressible fluid mechanics has been important in the development of not only the previously mentioned topics, but also of jet engines, rocketry, re-entry spacecraft, gas pipelines, combustion, and gas turbines.

13.1.3. What is special about compressible fluids?

Compressibility of a fluid results in several important phenomena that are not observed in incompressible fluids. Two of the most significant of these phenomena are shock waves and "choked" flow conditions. Both of these phenomena are the result of the fact that in compressible fluids, pressure disturbances propagate at a finite speed. For example, if one claps their hands, the pressure disturbance caused by the colliding hands propagates into the surrounding atmosphere with a finite speed (equal to the speed of sound). Thus, a finite amount of time passes before the surrounding air recognizes the effects of the clapping hands. In contrast, in a truly incompressible fluid, pressure disturbances propagate at an infinite speed. Thus, pressure disturbances are felt instantly everywhere in the fluid domain.

The fact that disturbances travel at finite speed raises the question of what happens if the cause of the pressure disturbance travels faster than the pressure disturbance itself? As an example, let's consider an aircraft flying in the atmosphere. When the aircraft moves slower than the disturbances propagate, pressure disturbances travel ahead of the aircraft and "inform" the air in front that the aircraft is about to arrive. Thus, the air can move smoothly out of the way as the aircraft approaches. However, if the aircraft travels faster than the speed of propagation, then the air in front of the aircraft can't move out of the way and begins to "pile up" in front of the aircraft. The result is the formation of a shock wave across which there is a rapid change in the air pressure, temperature, density and velocity.

Now let's consider a different situation. Imagine a large, pressurized tank with a converging nozzle that empties into another large tank (refer to the Figure 13.1). While holding the pressure in the left tank constant, let's begin to reduce the pressure in the right-hand tank.

When we lower the pressure in the right-hand tank, a pressure disturbance propagates upstream to the constant pressure tank and "informs" the fluid upstream that the pressure in the right-hand tank has dropped. As a result, the flow rate between tanks increases. As we continue to lower the pressure in the right-hand tank, the flow rate continues to increase until we reach a speed in the converging nozzle where the fluid speed is equal to the speed at which pressure disturbances propagate. Now if we continue to lower the pressure in the right hand tank, that pressure information can no longer propagate upstream since the fluid is flowing in

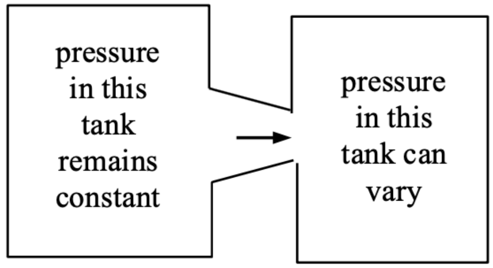

Figure 13.1. Discharge from a pressurized tank.

the opposite direction at the same speed. Thus, we have a "choked" flow condition where we can no longer increase the flow rate between the tanks.

In addition to these two phenomena, compressible flows have other counter-intuitive behaviors regarding how the fluid velocity varies with the area through which the fluid flows and how the speed is affected by frictional effects. We'll investigate all of these phenomena in this chapter.

13.1.4. What tools are required to study compressible fluid mechanics?

Several basic concepts are used in studying compressible fluid mechanics. These include:

- Conservation of Mass,
- The Linear Momentum Equations,
- The First Law of Thermodynamics,
- The Second Law of Thermodynamics,
- equations of state, e.g., the ideal gas law, and
- various concepts from thermodynamics.

In addition, we'll require knowledge of calculus, vector calculus, and differential equations (ODEs and PDEs).

13.2. Equations of State

Rather than duplicate what has been previously presented, the reader is encouraged to review Chapter 3 and, specifically, the sections on thermodynamic properties applied to ideal gases. Since compressibility effects become significant when the flow speed is larger than approximately one-third the speed of sound in the fluid, the compressibility of liquids is rarely considered. The speed of sound in water, for example, is nearly $1500\,\mathrm{m\,s^{-1}}$ and, thus, a flow speed of larger than approximately $500\,\mathrm{m\,s^{-1}}$ would be needed for compressibility to become a factor. Such high speed flows for liquids are uncommon. In contrast, the speed of sound in air at typical conditions is $340\,\mathrm{m\,s^{-1}}$ and so the compressibility of air becomes significant when the flow speed is larger than approximately $100\,\mathrm{m\,s^{-1}}$. A $100\,\mathrm{m\,s^{-1}}$ flow speed is easily achieved in normal engineering applications. Thus, compressible flow analyses typically focus on gases, which we generally model as ideal gases. Indeed, the study of compressible flow is sometimes referred to as “gas dynamics”, reflecting the emphasis on gases.

13.3. One-dimensional Flow

The reader is encouraged to review the section on flow dimensionality in Chapter 1. Much of our analysis of gas dynamics in conduits, e.g., pipes and converging and diverging ducts, will assume one-dimensional flow as an engineering approximation. Of course the flow of a real fluid through a pipe is not one-dimensional due to the no-slip condition at the pipe walls. If the Reynolds number of the flow is large enough, however, the flow may be approximated to be 1D with reasonable accuracy. As a flow’s Reynolds number increases, the velocity profile becomes more blunt-shaped and more closely approaches that of a uniform profile (Figure 13.2). Compressible flows are typically high speed so the Reynolds numbers are large and the 1D assumption is a good one.

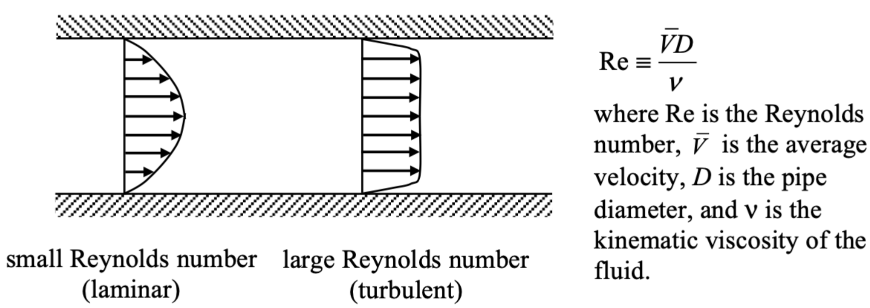

FIGURE 13.2. Example velocity profiles in a channel flow.

13.3.1. Governing Equations for 1D, Steady Flow

In this section we’ll write the governing equations (Conservation of Mass, Linear Momentum, and the First and Second Laws) for a 1D, steady flow. Most of what we’ll cover in this chapter will make these two assumptions, which are reasonable ones to make in many practical engineering situations.

Conservation of Mass: Apply Conservation of Mass to the control volume shown in Figure 13.3,

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} (\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}) = 0, \tag{13.1}$$

where,

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}, \tag{13.2}$$

$$\int_{CS} (\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}) = (-\rho V A) + [\rho V A + d(\rho V A)] = d(\rho V A). \tag{13.3}$$

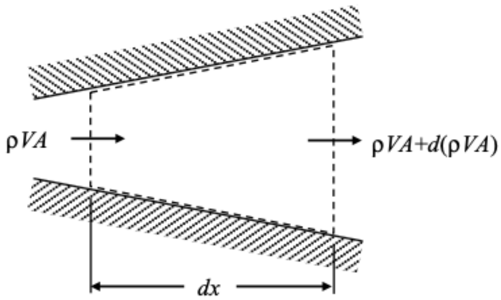

Figure 13.3. The control volume for applying Conservation of Mass.

Substituting and simplifying,

$$\boxed{d(\rho VA) = 0 \quad \text{or} \quad \dot{m} = \text{constant}}. \tag{13.4}$$

Linear Momentum Equations: Apply the Linear Momentum Equation in the x direction to the control volume

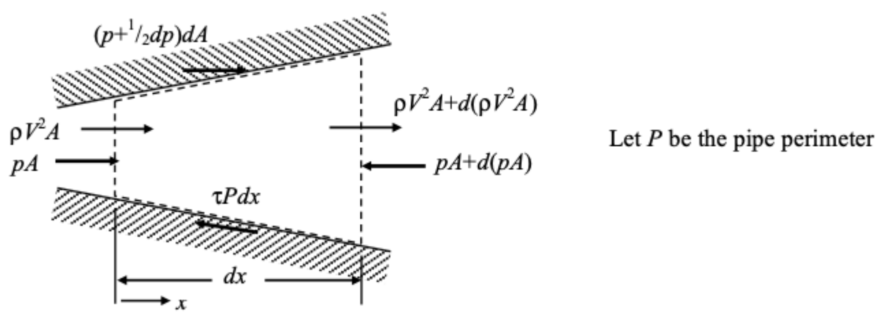

Figure 13.4. The control volume for applying the Linear Momentum Equation in the x direction.

shown in Figure 13.4,

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}, \tag{13.5}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}, \tag{13.6}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \left(-\rho V^2 A\right) + \left[\rho V^2 A + d\left(\rho V^2 A\right)\right] = d\left(\rho V^2 A\right), \tag{13.7}$$

$$F_{B,x} = 0 \quad \text{(most compressible flows involve gases and, thus, the body forces are negligible)}, \tag{13.8}$$

$$F_{S,x} = (pA) - \left[pA + d(pA)\right] + \left[\left(p + \frac{1}{2}dp\right)dA\right] - \tau P dx = -d(pA) + pdA - \tau P dx = -Adp - \tau P dx. \tag{13.9}$$

Note that higher-order terms have been neglected in the previous expression and that the friction force acts only in the x-direction since the boundaries vary smoothly (the slope is small, no discontinuities).

Re-write the friction force term using a hydraulic diameter, D_H, defined as,

$$D_H := \frac{4A}{P}, \tag{13.10}$$

and friction factor,

$$\tau = f_F \left(\frac{1}{2}\rho V^2\right) \quad f_F \text{ is known as a Fanning friction factor,} \tag{13.11}$$

$$\tau = \frac{1}{4} f_D \left(\frac{1}{2}\rho V^2\right) \quad f_D \text{ is known as a Darcy friction factor } (f_F = 4f_D), \tag{13.12}$$

so that the Linear Momentum Equation becomes,

$$dp + \rho V dV + \left(\frac{1}{2}\rho V^2\right)\left(\frac{4f_F}{D_H}\right) dx = 0, \tag{13.13}$$

or,

$$\boxed{\frac{dp}{\rho} + V dV + \left(\frac{1}{2}V^2\right)\left(\frac{4f_F}{D_H}\right) dx = 0}. \tag{13.14}$$

Notes:

(1) Gravitational effects have been neglected in the previous analysis since, when dealing with gases, gravitational effects are typically very small compared to other terms in the Momentum Equation.
(2) The Darcy friction factor, f_D, is the friction factor used in the Moody diagram for pipe flows.
(3) For a frictionless flow, Eq. (13.14) integrates to,

$$\int \frac{dp}{\rho} + \frac{1}{2}V^2 = \text{constant}. \tag{13.15}$$

The integral appears because the pressure can, in general, be a function of the density.
(4) For an incompressible fluid, $\rho =$ constant so that after integration along a streamline (recall this is 1D flow),

$$\int \frac{dp}{\rho} = \frac{p}{\rho} + \text{constant}. \tag{13.16}$$

(5) For an ideal gas ($p = \rho RT$) undergoing an isothermal process ($T =$ constant),

$$\int \frac{dp}{\rho} = \int \frac{d(\rho RT)}{\rho} = RT_0 \int \frac{d\rho}{\rho}, \tag{13.17}$$

$$\therefore \int \frac{dp}{\rho} = RT_0 \ln\left(\frac{\rho}{\rho_0}\right), \tag{13.18}$$

where T_0 and ρ_0 are a reference temperature and density, respectively.
(6) For an ideal gas undergoing an isentropic process ($s =$ constant),

$$ds = 0 = c_p(T)\frac{dT}{T} - R\frac{dp}{p}, \tag{13.19}$$

$$dp = \frac{c_p(T)}{R}(\rho RT)\frac{dT}{T}, \tag{13.20}$$

$$dp = \rho c_p(T) dT. \tag{13.21}$$

Thus,

$$\int \frac{dp}{\rho} = \int c_p(T) dT = \int dh = h + \text{constant}, \tag{13.22}$$

where h is the specific enthalpy. Note that if the ideal gas has constant specific heats, i.e., is a "perfect" gas, then $\Delta h = c_p \Delta T$.

First Law of Thermodynamics Apply the First Law of Thermodynamics to the control volume shown in Figure 13.5,

$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS}\left(h + \frac{1}{2}V^2 + gz\right)(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \delta\dot{Q}_{\text{into CV}} + \dot{W}_{\text{other,on CV}}, \tag{13.23}$$

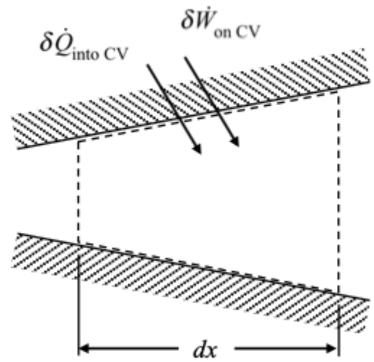

Figure 13.5. The control volume for applying the First Law of Thermodynamics.

where,

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady flow)}, \tag{13.24}$$

$$\begin{aligned}\int_{CS}\left(h+\frac{1}{2}V^2+gz\right)(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}) &= \left\{-\left(h+\frac{1}{2}V^2\right)+\left[\left(h+\frac{1}{2}V^2\right)+d\left(h+\frac{1}{2}V^2\right)\right]\right\}\dot{m}\\ &= d\left(h+\frac{1}{2}V^2\right)\dot{m} \quad (gz \text{ is negligible for gases}),\end{aligned} \tag{13.25}$$

$$\int_{CS}\delta\dot{Q}_{\text{into CV}} = \delta\dot{q}_{\text{into CV}} \quad (\delta\dot{q}_{\text{into CV}} \text{ is the rate of energy addition via heat transfer per unit volume}), \tag{13.26}$$

$$\dot{W}_{\text{other,on CV}} = 0 \quad \text{(assuming no work other than pressure work)}. \tag{13.27}$$

Substitute and simplify,

$$d\left(h+\frac{1}{2}V^2\right)\dot{m} = \delta\dot{q}_{\text{into CV}} \quad \text{or} \quad d\left(h+\frac{1}{2}V^2\right) = \delta q_{\text{into CV}}, \tag{13.28}$$

where $\delta q_{\text{into CV}}$ is the rate of energy transfer via heat transfer into the control volume per unit mass of the fluid.

Notes:

(1) For an adiabatic flow ($\delta q_{\text{into CV}} = 0$), the First Law becomes,

$$h+\frac{1}{2}V^2 = \text{constant}, \tag{13.29}$$

which is the same expression as what is obtained from the Linear Momentum Equation for an ideal gas undergoing an isentropic process (refer to Eqs. (13.15) and (13.22)).

Second Law of Thermodynamics Apply the Second Law of Thermodynamics to the control volume shown in Figure 13.6,

$$\frac{d}{dt}\int_{CV} s\rho dV + \int_{CS} s\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \int_{CS}\frac{\delta\dot{Q}_{\text{into CV}}}{T} + \dot{\sigma}, \tag{13.30}$$

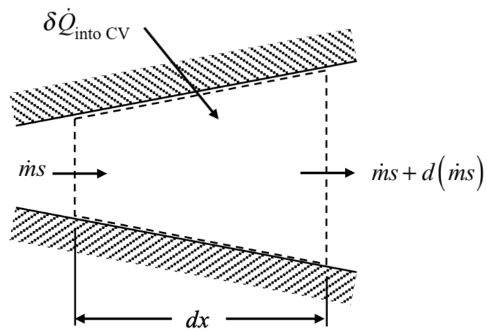

FIGURE 13.6. The control volume for applying the Second Law of Thermodynamics.

where,

$$\frac{d}{dt}\int_{CV} s\rho dV = 0 \quad \text{(steady flow)}, \tag{13.31}$$

$$\int_{CS} s\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \left[-s + (s+ds)\right]\dot{m} = (ds)\dot{m}, \tag{13.32}$$

$$\int_{CS} \frac{\delta\dot{Q}_{\text{into CV}}}{T} = \frac{\delta\dot{q}_{\text{into CV}}}{T}, \tag{13.33}$$

$$\dot{\sigma} = \dot{\sigma}. \tag{13.34}$$

Substitute and simplify,

$$(ds)\dot{m} = \frac{\delta\dot{q}_{\text{into CV}}}{T} + \dot{\sigma} \quad \text{or} \quad ds = \frac{\delta q_{\text{into CV}}}{T} + \sigma, \tag{13.35}$$

where $\delta q_{\text{into CV}}$ is the heat added to the control volume per unit mass of the flowing gas and σ is the rate of entropy generation per unit mass of the flowing gas.

Notes:

(1) For an adiabatic flow, $\delta\dot{q}_{\text{into CV}} = 0$, and so,

$$ds = \sigma \geq 0. \tag{13.36}$$

The equality in this equation only holds if the flow is internally reversible.